International Journal of Algebra, Vol. 4, 2010, no. 14, 669 - 675

A Note on Irreducible Elements in a Finite Poset

S. Parameshwara Bhatta and H. S. Ramananda

Department of Mathematics
Mangalore University
Mangalagangothri 574 199
Karnataka State, India
ramanandahs@gmail.com

Abstract

Let P be a finite poset. In this paper a Boolean poset B is constructed such that there is a cover preserving embedding of P in B. In addition it is proved that the number of maximal chains in P is at most equal to the number of linear extensions of join-irreducibles of P.

Mathematics Subject Classification(2000): 06A06, 06E75, 06B05

Keywords: Boolean poset, distributive poset, join-irreducible element, linear extension

1 Introduction

The theories of Boolean algebras, distributive lattices and uniquely complemented lattices are extended to partially ordered sets(posets) by Josef Niederle[5], B.N Waphare and Vinayak Joshi [7] and Ivan Chajda [2]. Also properties of posets, generated by irreducible elements, are studied by Marcel Erné[4], linear extensions of irreducible elements in a finite lattice is studied by Ivan Rival [3]. In continuation of their work we prove the following.

In the first section , we construct a Boolean poset B, for a given finite poset P, with a cover preserving embedding $f : P \to B$. This shows that there is no characterization of a Boolean poset by finitely many forbidden subposets. In other words this shows that, Birkhoff's characterization theorem for distributive lattices cannot be extended to distributive posets.

In the second section, we prove that the number of maximal chains in a finite poset P is less than or equal to the number of linear extensions of join-irreducible elements of P.

Notation and Definitions:

Let P be a poset and X be a subset of P. The set
$X^l := \{y \in P|\ y \leq x$ for every $x \in X\}$
is called the *lower cone* of X in P. The set
$X^u := \{y \in P|\ x \leq y$ for every $x \in X\}$
is called the *upper cone* of X in P.

A map $f : K \to L$ between two posets K and L is an *order embedding* if
(1) f is injective;
(2) for all $a, b \in K$ we have $a \leq b \Leftrightarrow f(a) \leq f(b)$.
An order embedding is *cover preserving* if $a \prec b$ implies $f(a) \prec f(b)$; see M. Stern[6].

An element j of a poset P is said to be *join-irreducible* if it satisfies the condition that, whenever $j = \vee A$, where $A \subseteq P$, then $j \in A$. The set of all join-irreducible elements of a poset P is denoted by $J(P)$. For an element a of a poset P, $r(a) := \{x \in J(P)|\ x \leq a\} = J(P) \cap (a]$ and $R(P) := \{r(a)|\ a \in P\}$.

A poset P is said to be *distributive* if
$\{x, (y, z)^u\}^l = \{(x, y)^l, (x, z)^l\}^{ul}$ holds for every x, y, z in P. An element $y \in P$ is said to be a *complement* of x in P if $(x, y)^l = P^l$ and $(x, y)^u = P^u$. A poset P is said to be *Boolean* if it is distributive and complemented .

A collection S of subsets of a set X is said to be a *pre-ring* of sets if $Y, S_1, ..., S_n \in S$, for some $n \geq 2$, then following conditions are met:

1. $S_1 \cap ... \cap S_n \not\subseteq Y \Rightarrow (\exists S' \in S)(S' \subseteq S_1 \cap ... \cap S_n \ \&\ S' \not\subseteq Y)$;

2. $S_1 \cup ... \cup S_n \not\supseteq Y \Rightarrow (\exists S' \in S)(S' \supseteq S_1 \cup ... \cup S_n \ \&\ S' \not\supseteq Y)$.

A collection S of subsets of a set X is said to be a *weak pre-field* of sets if the following conditions are met:

1. $Y \in S \Rightarrow X - Y \in S$;

2. $Y, Z \in S$ and $Y \cap Z \neq \phi \Rightarrow \exists U \in S$ such that $\phi \neq U \subseteq Y \cap Z$; see Joseph Niederle[5].

Let $\overline{P}$ denote the Dedekind-Mac Neille completion by cuts of a poset P; see [1]. It is known that the map $\varphi : P \to \overline{P}$ defined by $\varphi(a) = (a]$ is an order embedding.

Let P be an n element poset. A *linear extension* of P is a one-to -one, order preserving map from P into $\{1, 2, ..., n\}$. Let E_P denote the set of all linear extensions of P and C_P denote the set of all maximal chains in P.

2 Cover preserving embeddings

First we prove some equivalent definitions of join-irreducible elements in a poset and we use them throught this paper.

Lemma 2.1. *Following statements are equivalent for an element j of a poset P:*

1. $j \in J(P)$;
2. *j is not the least upper bound for the set $X = \{x \in P | x < j\}$;*
3. *$j \neq 0$ and there exists an element $r \in P$ such that $j \not\leq r$ but $a \leq r$ whenever $a < j$.*

Proof. Clearly(1) $\Leftrightarrow$(2).
(2) $\Rightarrow$(3): Suppose an element j of P satisfies (2). As the least element 0 is the join of empty set, $j \neq 0$. Suppose there exists no r in P as in (3), then j will be the least upper bound for the set $X = \{x \in P|\ x < j\}$ a contradiction.
(3) $\Rightarrow$ (2): Clearly r will be an upper bound for the set X. As $j \not\leq r$, j cannot be the least upper bound for the set X. □

The following proposition extends a well-known theorem of lattices to posets.

Proposition 2.2. *Let P be a poset satisfying the descending chain condition. Then every element $a \in P$ can be written as $a = \vee r(a)$.*

Proof. Assume the contrary that $a \neq \vee r(a)$ for some $a \in P$. Let

$$S = \{x \in P|\ x \neq \vee r(x)\}. \tag{1}$$

Then S is non-empty and as P satisfies the descending chain condition S has a minimal element, say b. Since $b \neq \vee r(b)$, it follows that $b \notin J(P)$, so that

$$b = \vee\{x \in P|\ x < b\}. \tag{2}$$

Let $X = \cup\{r(x)|\ x < b\}$. We assert that $b = \vee X$. Clearly b is an upper bound for X. If u is any other upper bound for X, then u is an upper bound for r(x), for each $x < b$. As b is minimal in S, $x = \vee r(x)$ holds for each $x < b$. Hence $x \leq u$ for each $x < b$. In view of 2, $b \leq u$ as desired.
Now observe that

$$\begin{aligned} X &= \cup\{r(x)|\ x < b\} \\ &= \cup\{(x] \cap J(P)|\ x < b\} \\ &= J(P) \cap (\cup\{(x]|\ x < b\}) \\ &= J(P) \cap (b] = r(b). \end{aligned}$$

This implies that $b = \vee r(b)$, a contradiction to the choice of b. □

In what follows, P will be a finite poset. The next theorem shows that $(P, \leq)$ is order isomorphic with $(R(P), \subseteq)$.

Theorem 2.3. *Let P be a poset. The map $f : P \to R(P)$ defined by $f(a) = r(a)$ is an order isomorphism.*

Proof. From Proposition 2.2, every element a in P can be written as $a = \vee r(a)$ and hence f is well defined and bijective. Now $a \leq b \Leftrightarrow r(a) \subseteq r(b)$. □

Example 2.4. *In Figure 1, the poset $R(P_1)$ isomorphic to the poset P_1 is given.*

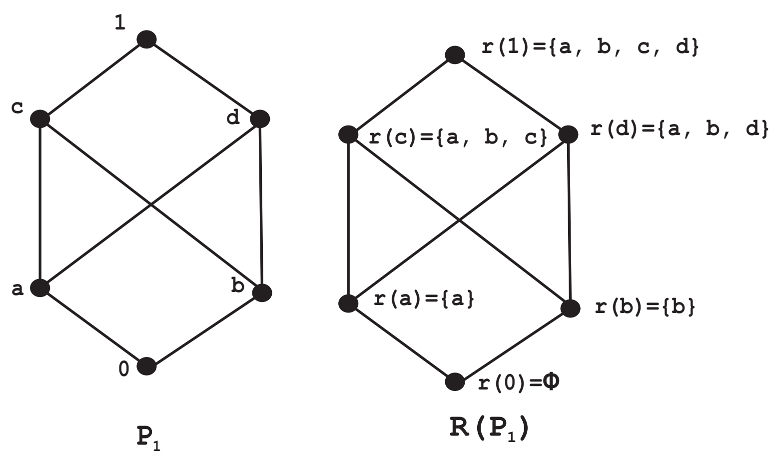

Figure 1:

To prove the main theorem of this section, we need the following result.

Theorem 2.5 ([5]). *Every pre-ring of sets is a distributive poset and every weak pre-field of sets is a Boolean poset with respect to inclusion.*

Lemma 2.6. *Let P be a poset. Then there exists a distributive poset D with a cover preserving embedding $f : P \to D$.*

Proof. Let $|J(P)| = n$. Consider a set X with $n + 1$ elements such that $J(P) \subseteq X$. Let D be the following collection of subsets of X:
(1) ϕ, $X \in D$;
(2) $a \in X \Rightarrow \{a\}$, $X - \{a\} \in D$;
(3) $a \in P \Rightarrow r(a) \in D$.

We claim that D is a pre-ring of sets.
Suppose that Y, S_1, ..., $S_i \in D$ and $S_1 \cap ... \cap S_i \not\subseteq Y$ for some $i \geq 2$. Choose $a \in S_1 \cap ... \cap S_i$ such that $a \notin Y$. Then $\{a\} \in D$, $\{a\} \subseteq S_1 \cap ... \cap S_i$ and

$\{a\} \not\subseteq Y$.
Suppose that $Y, S_1, ..., S_j \in D$ and $Y \not\subseteq S_1 \cup ... \cup S_j$ for some $j \geq 2$. Choose $a \in Y$ such that $a \notin S_1 \cup ... \cup S_j$. Then $X - \{a\} \in D$, $S_1 \cup ... \cup S_j \subseteq X - \{a\} \in S$ and $X - \{a\} \not\subseteq Y$. Thus the Claim holds. Hence D is a distributive poset with respect to inclusion by Theorem 2.5.

Let $f : P \to D$ be a map defined by $f(a) = r(a)$. In view of Theorem 2.3, f is an order embedding. It remains to show that f is cover preserving. Let $a, b \in P$ such that $a \prec b$. If $a = 0$, then clearly $r(a) = \phi \prec r(b) = \{b\}$ in D. Assume $a \neq 0$. Suppose $r(a) \subseteq Y \subseteq r(b)$ for some $Y \in D$. Then

$$1 \leq |r(a)| \leq |Y| \leq |r(b)| \leq n. \tag{3}$$

Note that the set Y must be one of the types:
(i) $Y = \{x\}$ for some $x \in X$;
(ii) $Y = X - \{x\}$ for some $x \in X$;
(iii) $Y = r(c)$ for some $c \in P$.
In view of 3, if (i) holds, then $r(a) = \{a\} = \{x\} = Y$; if (ii) holds, then $r(b) = Y$.
If (iii) holds, then using the fact that f is an order embedding, it follows that $a \leq c \leq b$. Since $a \prec b$, either $a = c$ or $c = b$. That is, either $r(a) = r(c)$ or $r(b) = r(c)$. Hence f is a cover preserving embedding. □

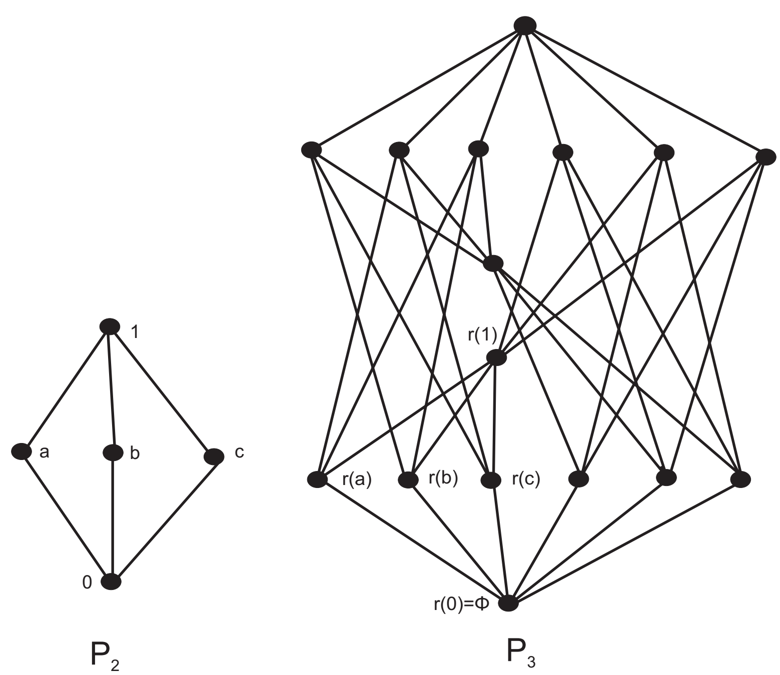

Figure 2:

Theorem 2.7. *Let P be a poset. Then there exists a Boolean poset B with a cover preserving embedding $f : P \to B$.*

Proof. Let $|J(P)| = n$. Consider a set X with $2n$ elements such that $J(P) \subseteq X$. Let B be the following collection of subsets of X :
(1)$\phi \in B$;
(2) $Y \in B \Rightarrow X - Y \in B$;
(3) $a \in P \Rightarrow r(a) \in B$;
(4) $a \in X \Rightarrow \{a\} \in B$.
It is easy to see that B is a weak pre-field of sets. Hence B is a Boolean poset with respect to inclusion by Theorem 2.5.

Let $f : P \to B$ be a map defined by $f(a) = r(a)$. As in the proof of Lemma 2.6, it follows that f is a cover preserving embedding. □

Example 2.8. *In Figure 2, the poset P_2 is embedded in a Boolean poset P_3 with the embedding defined in Theorem 2.7.*

3 Linear extensions of join-irreducibles

In this section, we extend the following result of Ivan Rival [3] to posets.

Theorem 3.1 ([3]). *Let L be a finite lattice. Then $|C_L| \leq |E_{J(L)}|$ and the equality occurs if and only if L is distributive.*

First we prove two lemmas.

Lemma 3.2. *Let P be a poset. Then $X \in J(\overline{P})$ if and only if $X = (j]$ for some $j \in J(P)$.*

Proof. ($\Rightarrow$) Let Y be the unique lower cover of X in $\overline{P}$. Then $X = Y \cup Z$, for some $\phi \neq Z \subseteq P$.
Claim: $Z = (j] = X$ for a unique $j \in P$.
Choose an element $j \in Z$ such that $j \notin Y$. Clearly X is an upper bound for Y and $(j]$ in $\overline{P}$. As $Y \prec X$, $X = Y \vee (j]$. Using the fact that X is join-irreducible in $\overline{P}$ we get $(j] = X$. That is $Z = (j] = X$. The uniqueness of j is obvious.Thus the claim holds.
Hence $X = Y \cup \{j\} = (j]$ and $Y = \{y \in P|\ y < j\}$. Now as $j \notin Y$, j is not the least upper bound for the set Y. Hence by Lemma 2.1, $j \in J(P)$.
($\Leftarrow$) Let $X = (j]$ for some $j \in J(P)$. By Lemma 2.1 there exists $r \in P$ such that $j \not\leq r$ but $a \leq r$ whenever $a < j$. Let $Y = \{y \in P|\ y < j\}$. We assert that $Y \in \overline{P}$. Clearly, $Y \subseteq Y^{ul}$. On the other hand, let $y \in Y^{ul}$. We have $\{j, r\} \subseteq Y^u$, hence $\{j,\ r\}^l \supseteq Y^{ul}$. As $j \not\leq r$, $y < j$ and $y < r$. Therefore $y \in Y = \{y \in P|\ y < j\}$. Hence $Y \in \overline{P}$ as desired. It is easy to see that Y is the unique lower cover of $(j]$ in $\overline{P}$. Hence $(j] \in J(\overline{P})$. □

Lemma 3.3. *Let P be a poset. Then $J(P)$ is order isomorphic to $J(\overline{P})$.*

Proof. We know that the map $\varphi : P \to \overline{P}$ defined by $\varphi(a) = (a]$ is an order embedding. In view of Lemma 3.2, clearly the restriction map $\varphi|_{J(P)} : J(P) \to J(\overline{P})$ is an order isomorphism. □

Theorem 3.4. *Let P be a poset. Then $|C_P| \leq |E_{J(P)}|$.*

Proof. Since the map $\varphi : P \to \overline{P}$ defined by $\varphi(a) = (a]$ is an order embedding, clearly $|C_P| \leq |C_{\overline{P}}|$. From Theorem 3.1, $|C_{\overline{P}}| \leq |E_{J(\overline{P})}|$ and by the Lemma 3.3, $|E_{J(P)}| = |E_{J(\overline{P})}|$. Hence $|C_P| \leq |E_{J(P)}|$. □

Corollary 3.5. *Let P be a poset. Then $|C_P| = |E_{J(P)}|$ if and only if P and $\overline{P}$ have same number of maximal chains and $\overline{P}$ is a distributive lattice.*

Proof. Follows from Theorem 3.1 and Theorem 3.4. □

References

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Pub. Vol. **25**, Third Edition 1940.

[2] I. Chajda, Complemented ordered sets, *Arch. Math. (Brno)* **28** (1992), 25-34.

[3] Ivan Rival, A note on linear extensions of irreducible elements in a finite lattice, *Algebra Univers.* **6** (1976), 99-103.

[4] M. Erné, Posets generated by irreducible elements, *order* **20**(2003), 79-89.

[5] J. Niederle, Boolean and distributive ordered sets: Characterization and representation by sets, *Order* **12** (1995), 189-210.

[6] M. Stern, *Semimodular lattices , Theory and applications*, Cambridge University Press, U.K. 1999.

[7] B. N. Waphare and V. V. Joshi, On uniquely complemented posets, *Order* **22**(2005), 11-20.

Received: January, 2010