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The Factor Sets of Gr-Categories of the Type (Π, A)

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Abstract

Each $\Gamma-$monoidal extension of a $Gr-$category is determined by a factor set on a group Γ. In this paper, we shall show that the such factor set can be reduced and induces a 3-cocycle of $\Gamma-$groups. Moreover, this correspondence determines a bijection between the set of equivalence classes of $\Gamma-$monoidal extensions of $Gr-$categories of the type (Π, A) and the cohomology group $H^3_\Gamma(\Pi, A)$.

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1 Introduction

The notion of graded monoidal category was introduced by Fröhlich and Wall [4] by a generalization of some manifolds of categories with an action of a group Γ. Γ will be also regarded as a category with exactly one object, say $*$, where the morphisms are the elements of Γ and the composition is the group operation. By a $\Gamma-$*grading* on a category $\mathcal{D}$, we shall mean a functor $gr : \mathcal{D} \to \Gamma$. The grading is called *stable* if for any $C \in ob\mathcal{D}$, and any $\sigma \in \Gamma$, there exists a morphism u in $\mathcal{D}$ with domain C such that $gr(u) = \sigma$. We refer to σ as the *grade* of u. If $(\mathcal{D}, gr)$ is a $\Gamma-$graded category, we define $Ker\mathcal{D}$ to be the subcategory of all morphisms of grade 1.

For any $\Gamma-$graded category $(\mathcal{D}, gr)$, authors of [4] have considered the category $Rep(\mathcal{D}, gr)$ of $\Gamma-$functors $F : \Gamma \to \mathcal{D}$, together with natural transformations. An object of $Rep(\mathcal{D}, gr)$ consists an object X of $\mathcal{D}$ with a homomorphism $\Gamma \to Aut_{\mathcal{D}}(C)$. Now if I is the unit object of $\mathcal{D}$, then the homomorphism $\Gamma \to Aut_{\mathcal{D}}(I)$, which is also denoted by I, is the right inverse of the graded homomorphism $Aut_{\mathcal{D}}(I) \to \Gamma$. Let U denote the normal subgroup of automorphisms of grade 1. We obtain that $Aut_{\mathcal{D}}(I)$ is a split extension of U by $I(\Gamma)$, where $I(\Gamma)$ is isomorphic to Γ. This extension determines an action of Γ on U by

$$\sigma u = I(\sigma) \circ u \circ I(\sigma^{-1}).$$

In [1], authors have considered the $\Gamma-$monoidal extensions of a monoidal category as a categorification of the group extension problem. A $\Gamma-$monoidal extension of the monoidal category $\mathcal{C}$ is a $\Gamma-$monoidal category $\mathcal{D}$ together with a monoidal isomorphism $j : \mathcal{C} \to Ker\mathcal{D}$.

Construction and classification problems of $\Gamma-$monoidal extensions have been solved by raising the main results of Schreier-Eilenberg-MacLane on group extensions to categorical level. With notions of a *factor set* and a *crossed product extension,* authors proved that there exists a bijection

$$\Delta : H^2(\Gamma, \mathcal{C}) \leftrightarrow Ext(\Gamma, \mathcal{C})$$

between the set of cohomology classes of factor sets on Γ with coefficients in the monoidal category $\mathcal{C}$ and the set of equivalence classes of $\Gamma-$monoidal extensions of $\mathcal{C}$.

The case where $\mathcal{C}$ is a Gr-category (also called a *categorical group*) has been considered in [2]. Then, the $\Gamma-$equivariant structure appears on the $\Pi-$module A, where $\Pi = \Pi_0(\mathcal{C}), A = Aut_{\mathcal{C}}(1) = \Pi_1(C)$. The category ${}_\Gamma\mathcal{CG}$ of $\Gamma-$graded $Gr-$categories is classified by the functors

$$cl :_\Gamma \mathcal{CG} \to \mathcal{H}^3_\Gamma \ ; \ \int_\Gamma : \mathcal{Z}^3_\Gamma \to_\Gamma \mathcal{CG},$$

where $\mathcal{Z}^3_\Gamma$ is the category in which any object is a triple (Π, A, h), where (Π, A) is a $\Gamma-$pair and $h \in Z^3_\Gamma(\Pi, A)$; and $\mathcal{H}^3_\Gamma$ is the category obtained from $\mathcal{Z}^3_\Gamma$ when $h \in Z^3_\Gamma(\Pi, A)$ is replaced with $h \in H^3_\Gamma(\Pi, A)$.

In our opinion, the most interesting as well as the most complicated part of the proof of Classification Theorem (Theorem 3.3) of [2] is the construction of 3-cocycle induced by a $\Gamma-$extension $\mathcal{G}$ via a *skeleton* category. In this paper, we expect to introduce another proof of the Classification Theorem based on the notions of a *factor set* and a *crossed product extension* when we use thorough knowledge of $Gr-$functors. It is known that each $Gr-$category is equivalent to a $Gr-$category of the type (Π, A) with strict unitivity constraints in [7]. So we assert that we just have to prove the Theorem for manifolds of graded $Gr-$categories $\mathcal{G}$ whose kernel $Ker\mathcal{G}$ is a $Gr-$category of the type (Π, A) where (Π, A) is fixed. Then, the 3-cocycle h can be determined from a factor set analogously as the determination of *the obstruction* in the group extension problem.

The main detail of the proof of Classification Theorem can be summarized as follows: Thanks to the description of $Gr-$functors of $Gr-$categories of the type (Π, A) in [6], we can describe explicitly a factor set and show that the $\Gamma-$equivariant structure of A is a necessary condition of a factor set. This gives us an interpretation to the existence of the functor $I : \Gamma \to \mathcal{D}$. Moreover, the notion of a factor set can be reduced if the condition $F^1 = id$ is omitted. Then we shall show that each factor set is homotopic (natural equivalent) to an *almost strict* factor set. Each such factor set induces a 3-cocycle of $\Gamma-$groups h. This construction of h is rather simple compared to the way done in explicitly in [2]. Therefore, we obtain a bijection

$$\Omega :_{\Gamma} \mathbf{Gr}(\Pi, A) \leftrightarrow H^3_{\Gamma}(\Pi, A),$$

where ${}_{\Gamma}\mathbf{Gr}(\Pi, A)$ is the set of equivalence classes of $\Gamma-$extensions of Gr-categories of the type (Π, A).

2 Some basic notions

A monoidal category is called a *Gr−category* (or a *categorical group*) if every object is invertible and every morphism is an isomorphism. Each $Gr-$category $\mathcal{G}$ is equivalent to a $Gr-$category $\mathcal{S}$ of the type (Π, A), where $\Pi = \Pi_0(\mathcal{G})$ is the group of iso-classes of objects of $\mathcal{G}$, $A = \Pi_1(\mathcal{G}) = Aut(1)$ is a left $\Pi-$module. Objects of the category $\mathcal{S}$ are all elements $x \in \Pi$, morphisms are automorphisms

$$Aut(x) = \{x\} \times A,$$

where $x \in \Pi$. The composition of two morphisms is defined by

$$(x, u) \circ (x, v) = (x, u + v).$$

The operation $\otimes$ is defined by

$$x \otimes y = xy,$$
$$(x, u) \otimes (y, v) = (xy, u + xv).$$

The associativity constraint a is associated to a $3-$cocycle (in the sense of group cohomology) $\xi \in Z^3(\Pi, A)$, and the unitivity constraints are strict (in the sense $l_x = r_x = id_x$).

If $(F, \widetilde{F}, \widehat{F})$ is a monoidal functor between $Gr-$categories, then the isomorphism $\widehat{F}$ can be deduced from the pair $(F, \widetilde{F})$. Moreover, in [6], the author has described monoidal functor between $Gr-$categories of the type (Π, A). Thanks to this description, we can prove a necessary condition of a factor set (Theorem 3.2).

Definition 2.1 (See [6]). Let $\mathcal{S} = (\Pi, A, \xi)$, $\mathcal{S}' = (\Pi', A', \xi')$ be Gr-categories. A functor $F : \mathcal{S} \to \mathcal{S}'$ is called a *functor of the type* (φ, f) if

$$F(x) = \varphi(x), F(x, u) = (\varphi(x), f(u)),$$

and ($\varphi : \Pi \to \Pi'$, $f : A \to A'$) is a pair of group homomorphisms satisfying $f(xa) = \varphi(x)f(a)$ for $x \in \Pi, a \in A$.

We have

Theorem 2.2 (See [6]). *Let* $\mathcal{S} = (\Pi, A, \xi)$, $\mathcal{S}' = (\Pi', A', \xi')$ *be Gr-categories and* $F = (F, \widetilde{F}, \widehat{F})$ *be a monoidal functor from* $\mathcal{S}$ *to* $\mathcal{S}'$. *Then,* F *is a functor of the type* (φ, f).

Let us recall from [1], [2], [4] some basic notions about graded extension of a $Gr-$category.

Let $(\mathcal{G}, gr)$ and $(\mathcal{H}, gr)$ be stable $\Gamma-$graded categories. A *graded functor* $F : (\mathcal{G}, gr) \to (\mathcal{H}, gr)$ is a functor $F : \mathcal{G} \to \mathcal{H}$ preserving grades of morphisms. Suppose that $F' : (\mathcal{G}, gr) \to (\mathcal{H}, gr)$ is also a graded functor. A *graded natural equivalence* $\theta : F \to F'$ is a natural equivalence of functors such that all isomorphisms $\theta_X : FX \to F'X$ are of grade 1.

For $\Gamma-$graded category $(\mathcal{G}, gr)$, let $\mathcal{G} \times_\Gamma \mathcal{G}$ denote the subcategory of product category $\mathcal{G} \times \mathcal{G}$ consisting of isomorphisms which are pairs of morphisms of the same grade of $\mathcal{G}$. A $\Gamma-$*monoidal category* consists of a stable $\Gamma-$graded category $(\mathcal{D}, gr)$ together with $\Gamma-$functors

$$\otimes : \mathcal{D} \times_\Gamma \mathcal{D} \to \mathcal{D}; \quad I : \Gamma \to \mathcal{D},$$

and natural isomorphisms of grade 1:

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z),$$
$$l_X : I \otimes X \xrightarrow{\sim} X, \quad r_X : X \otimes I \xrightarrow{\sim} X,$$

where $I = I(*)$ such that for all object $X, Y, Z, T \in \mathcal{D}$, two coherence conditions hold:

$$(a_{X,Y,Z} \otimes id_T) \cdot a_{X,Y\otimes Z,T} \cdot (id_X \otimes a_{Y,Z,T}) = a_{X\otimes Y,Z,T} \cdot a_{X,Y,Z\otimes T},$$
$$id_X \otimes l_y = (r_X \otimes id_Y) \cdot a_{X,I,Y}.$$

If $(\mathcal{D}, gr, \otimes)$ and $(\mathcal{D}', gr', \otimes^{'})$ are $\Gamma-$monoidal categories, then a $\Gamma-$*monoidal functor* from $(\mathcal{D}, gr, \otimes)$ to $(\mathcal{D}', gr', \otimes^{'})$, $(F, \widetilde{F}, \widehat{F})$, consists of a $\Gamma-$functor $F : (\mathcal{D}, gr) \to (\mathcal{D}', gr')$, natural isomorphism of grade 1

$$\widetilde{F}_{X,Y} : F(X \otimes Y) \xrightarrow{\sim} FX \otimes^{'} FY,$$

and an isomorphism of grade 1: $\widehat{F} : FI \xrightarrow{\sim} I'$, such that coherence conditions hold. A *morphism* between two $\Gamma-$monoidal functors $F = (F, \widetilde{F}, \widehat{F})$ and $F' = (F', \widetilde{F'}, \widehat{F'})$ is a monoidal morphism $\theta : F \to F'$.

A $\Gamma-$*monoidal extension* of a monoidal categroy $\mathcal{C}$ consists of a $\Gamma-$monoidal category $\mathcal{D} = (\mathcal{D}, gr)$, and a monoidal isomorphism $J = (J, \widetilde{J}, \widehat{J}) : \mathcal{C} \to Ker\mathcal{D}$.

If $(\mathcal{D}, J)$ and $(\mathcal{D}', J')$ are two $\Gamma-$monoidal extensions of a monoidal category $\mathcal{C}$, by a *morphism* of extensions we mean a $\Gamma-$monoidal functor $F : \mathcal{D} \to \mathcal{D}'$ such that $F.J = J'$. We usually say that $\Gamma-$monoidal extensions $(\mathcal{D}, J)$ and $(\mathcal{D}', J')$ are equivalent, thanks to the following proposition.

Proposition 2.3. *Every morphism of* $\Gamma-$*monoidal extensions of the monoidal category* $\mathcal{C}$ *is an isomorphism.*

Definition 2.4 (See [1]). Let Γ be a group and $\mathcal{C}$ be any monoidal category. By a *factor set* on Γ with coefficients in a monoidal category $\mathcal{C}$, we shall mean a pair (θ, F) consisting of:

• A family of monoidal autoequivalences

$$F^\sigma = (F^\sigma, \widetilde{F^\sigma}, \widehat{F^\sigma}) : \mathcal{C} \longrightarrow \mathcal{C}, \sigma \in \Gamma.$$

• A family of isomorphisms of monoidal functors

$$\theta^{\sigma,\tau} : F^\sigma F^\tau \xrightarrow{\sim} F^{\sigma\tau}, \sigma, \tau \in \Gamma$$

satisfying the conditions:

i) $F^1 = id_{\mathcal{C}}$

ii) $\theta^{1,\sigma} = id_{F^\sigma} = \theta^{\sigma,1}$, $\sigma \in \Gamma$

iii) for all $\sigma, \tau, \gamma \in \Gamma$, the following diagrams are commutative:

$$\begin{array}{ccc} F^\sigma F^\tau F^\gamma & \xrightarrow{\theta^{\sigma,\tau} F^\gamma} & F^{\sigma\tau} F^\gamma \\ {\scriptstyle F^\sigma \theta^{\tau,\gamma}} \downarrow & & \downarrow {\scriptstyle \theta^{\sigma\tau,\gamma}} \\ F^\sigma F^{\tau\gamma} & \xrightarrow{\theta^{\sigma,\tau\gamma}} & F^{\sigma\tau\gamma}, \end{array}$$

According to Theorem 2.2, any monoidal autoequivalence F^σ is of the form $F^\sigma = (\varphi^\sigma, f^\sigma)$. This remark is used frequently throughout the paper.

Definition 2.5 (See [2]). Let Γ be a group, Π be a $\Gamma-$group. A $\Gamma-$module A is called an *equivariant module* on the $\Gamma-$group Π if A is a $\Pi-$module satisfying

$$\sigma(xa) = (\sigma x)(\sigma a),$$

for all $\sigma \in \Gamma, x \in \Pi$ and $a \in A$. Then, we shall say that (Π, A) is a $\Gamma-$*pair.*

3 $\Gamma-$graded extension of a $Gr-$category of the type (Π, A)

According to [2], each factor set (θ, F) on Γ with coefficients in a monoidal category $\mathcal{C}$ determines a *crossed product extension* $\Delta(\theta, F)$. It has the same objects as category $\mathcal{C}$ and its morphisms are pairs $(u,\sigma): A \to B$ consisting of an element $\sigma \in \Gamma$ and a morphism in $\mathcal{C}$, $u: F^{\sigma}(A) \to B$. The composition of two morphisms

$$A \xrightarrow{(u,\sigma)} B \xrightarrow{(v,\tau)} C$$

is defined by

$$(v,\tau)\cdot(u,\sigma) = (v\cdot F^{\sigma}(u)\cdot(\theta_A^{\tau,\sigma})^{-1}, \tau\sigma).$$

The stable $\Gamma-$grading on $\Delta(\theta,F)$ is given by $gr(u,\sigma)=\sigma$. If $(u,\sigma): X \to X'$ and $(v,\sigma): Y \to Y'$ then

$$(u,\sigma)\otimes(v,\sigma) = ((u\otimes v).\widetilde{F}^{\sigma}_{X,Y}, \sigma)$$

and the unit $\Gamma-$functor $I: \Gamma \to \Delta(\theta,F)$ is defined by $I(\sigma) = (\widehat{F}^{\sigma}, \sigma)$, where I is the unit object of $\mathcal{C}$.

We shall begin by proving that any $\Gamma-$extension of a $Gr-$category is equivalent to a $\Gamma-$extension of a $Gr-$category of the type (Π, A). This is a consequence of the following proposition.

Proposition 3.1. *If $G: \mathcal{C} \to \mathcal{C}'$ is a monoidal equivalence, then each factor set (θ,F) of $\mathcal{C}$ induces a factor set (θ',F') of $\mathcal{C}'$. Moreover, respective crossed product extensions are $\Gamma-$equivalent.*

Proof. Let $H: \mathcal{C}' \to \mathcal{C}$ be a monoidal equivalence such that $\beta: H\circ G \cong id_{\mathcal{C}}$. Let ${F'}^{\sigma}$ be the composition $H\circ F^{\sigma}\circ G$ and

$${\theta'}^{\sigma,\tau}_X = G(\theta^{\sigma,\tau}_{HX}\circ F^{\sigma}(\beta_{F^{\tau}HX})).$$

One can verify that (θ',F') is a factor set of $\mathcal{C}'$. A monoidal functor $G: \mathcal{C} \to \mathcal{C}'$ can be extended to a $\Gamma-$functor

$$\Delta_G : \Delta(\theta,F) \leftrightarrow \Delta(\theta',F')$$

as follows: for an object X of $\mathcal{C}$, set $\Delta_G X = GX$; for a morphism $(u,\sigma): X \to Y$, where $u: F^{\sigma}X \to Y$, set

$$\Delta_G(u,\sigma) = (G(u\circ F^{\sigma}(\beta_X)),\sigma),\ \widetilde{\Delta}_G = \widetilde{G},\ \widehat{\Delta}_G = \widehat{G}.$$

One can verify that $(\Delta_G, \widetilde{\Delta}_G, \widehat{\Delta}_G)$ is a $\Gamma-$equivalence. □

We now prove a necessary condition of a factor set of a $Gr-$category.

Theorem 3.2. *Let Γ be a group and $\mathcal{S} = \mathcal{S}(\Pi, A, \xi)$ be a Gr-category. If (θ,F) is a factor set on Γ, with coefficients in $\mathcal{S}$, then*

i) There exists a pair of group homomorphisms

$$\varphi: \Gamma \to Aut\Pi\ ;\ f: \Gamma \to AutA,$$

and A is equipped with a $\Gamma-$equivariant $\Pi-$module structure, induced by φ, f.

ii) The condition i) in definition of the factor set can be deduced from the rest conditions.

Proof. i) According to Theorem 1.1, any autoequivalence $F^\sigma, \sigma \in \Gamma$, of a factor set is of the form:

$$(\varphi^\sigma, f^\sigma) : \mathcal{S} \to \mathcal{S}.$$

Since $\widetilde{F}^\sigma_{x,y} : F^\sigma(xy) \to F^\sigma x.F^\sigma y, \sigma \in \Gamma$ is a morphism in (Π, A), we have

$$F^\sigma(xy) = F^\sigma x.F^\sigma y, \forall x, y \in \Pi.$$

This shows that $\varphi^\sigma = F^\sigma$ is an endomorphism of the group Π. Furthermore, since F^σ is an equivalence, φ^σ is an automorphism of the group Π, i.e., $\varphi^\sigma \in Aut\Pi$. On the other hand, since $\theta^{\sigma,\tau}_x : F^\sigma F^\tau x \to F^{\sigma\tau} x$ is a morphism in (Π, A), we have

$$(F^\sigma F^\tau)(x) = F^{\sigma\tau}(x), \forall x \in \Pi.$$

Thus, $\varphi^\sigma \varphi^\tau = \varphi^{\sigma\tau}$. This proves that

$$\varphi : \Gamma \to Aut\Pi, \quad \sigma \mapsto \varphi^\sigma$$

is a group homomorphism. Then $\varphi^1 = \varphi(1) = id_\Pi$.
Now, for convenience, for all $\sigma \in \Gamma, x \in \Pi, a \in A$, let us denote

$$\sigma x = \varphi^\sigma x, \quad \sigma a = f^\sigma a \tag{1}$$

Let $\widetilde{F}^\sigma_{x,y} = (\sigma(xy), \widetilde{f}^\sigma(x, y))$, where $\widetilde{f}^\sigma : \Pi^2 \to A$ and $\widehat{F}^\sigma = (1, c^\sigma) : F^\sigma 1 \to 1$ are maps. Since F^σ is a monoidal functor, we have

$$(\sigma x)\widetilde{f}^\sigma(y, z) - \widetilde{f}^\sigma(xy, z) + \widetilde{f}^\sigma(x, yz) - \widetilde{f}^\sigma(x, y) = \sigma(\xi(x, y, z)) - \xi(\sigma x, \sigma y, \sigma z)\,, \tag{2}$$

$$\sigma(x)c^\sigma + \widetilde{f}^\sigma(x, 1) = 0\,, \tag{3}$$

$$c^\sigma + \widetilde{f}^\sigma(1, x) = 0\,. \tag{4}$$

We now consider isomorphisms of monoidal functors $\theta^{\sigma,\tau} = (\theta^{\sigma,\tau}_x)$, where

$$(\theta^{\sigma,\tau}_x) = (\varphi^\sigma x, t^{\sigma,\tau}(x)) : F^\sigma F^\tau(x) \to F^{\sigma\tau}(x),$$

where $t^{\sigma,\tau} : \Pi \to A$ are maps. According to the notion of natural transformation of a monoidal functor, we have the following equations:

$$f^\sigma f^\tau = f^{\sigma\tau}\,, \tag{5}$$

$$(\sigma\tau x)t^{\sigma,\tau}(y) - t^{\sigma,\tau}(xy) + t^{\sigma,\tau}(x) = \widetilde{f}^{\sigma\tau}(x, y) - \widetilde{f}^\sigma(F^\tau x, F^\tau y) - \sigma(\widetilde{f}^\tau(x, y))\,, \tag{6}$$

$$\sigma(c^{\tau}) + c^{\sigma} - c^{\sigma\tau} = t^{\sigma,\tau}(1). \tag{7}$$

From the equality (5), we are led to a group homomorphism

$$f : \Gamma \to AutA,$$

given by $f(\sigma) = f^{\sigma}$ and therefore $f^1 = f(1) = id_A$. Then, sine F^{σ} is a functor of the type $(\varphi^{\sigma}, f^{\sigma})$, $\varphi^{\sigma}(xb) = \varphi^{\sigma}(x)f^{\sigma}(b)$, i.e., A is a $\Gamma-$equivariant $\Pi-$module with $\Gamma-$actions (1).

ii) From the condition ii) in the definition of a factor set, we are led to $t^{\sigma,1} = t^{1,\sigma} = 0$. From the equality (6), for $\tau = 1$, we obtain $\widetilde{f}^1_{x,y} = 0$, i.e., $\widetilde{F}^1_{x,y} = id$. From the equation (4), for $\sigma = 1$, we have $c^1 = 0$, i.e., $\widehat{F}^1 = id$. Thus, F^1 is an identity monoidal functor. This completes the proof. □

Definition 3.3. Let Γ be a group and $\mathcal{C}$ be a $Gr-$category of the type (Π, A). Two factor sets (θ, F) and (μ, G) on Γ with coefficients in $\mathcal{C}$ are *cohomologous* if there exists a family of isomorphisms of monoidal functors

$$u^{\sigma} : (F^{\sigma}, \widetilde{F}^{\sigma}, \widehat{F}^{\sigma}) \xrightarrow{\sim} (G^{\sigma}, \widetilde{G}^{\sigma}, \widehat{G}^{\sigma}), \quad \sigma \in \Gamma$$

satisfying

$$u^1 = id_{(\Pi,A)},$$

$$u^{\sigma\tau}.\theta^{\sigma\tau} = \mu^{\sigma,\tau}.u^{\sigma}G^{\tau}.F^{\sigma}u^{\tau}, \quad \sigma, \tau \in \Gamma.$$

Remark 3.4. *If two factor sets* $(\theta, F), (\mu, G)$ *are cohomologous, then* $F^{\sigma} = G^{\sigma}, \sigma \in \Gamma$.

Indeed, from the definition of cohomologous factor sets, there exists a family of isomorphisms

$$u^{\sigma} : (F^{\sigma}, \widetilde{F}^{\sigma}, \widehat{F}^{\sigma}) \to (G^{\sigma}, \widetilde{G}^{\sigma}\widehat{G}^{\sigma}), \quad \sigma \in \Gamma.$$

Since $u^{\sigma}_x : F^{\sigma}x \to G^{\sigma}x$ is a morphism in (Π, A), we have $G^{\sigma}x = F^{\sigma}x$. Furthermore, for any $a \in A$, by the commutative diagram

$$\begin{array}{ccc} F^{\sigma}x & \xrightarrow{u^{\sigma}_x} & G^{\sigma}x \\ {\scriptstyle F^{\sigma}(x,a)}\downarrow & & \downarrow{\scriptstyle G^{\sigma}(x,a)} \\ F^{\sigma}x & \xrightarrow{u^{\sigma}_x} & G^{\sigma}x\,, \end{array}$$

we have $F^{\sigma}(x, a) = G^{\sigma}(x, a)$.

We call a factor set (θ, F) *almost strict* if $\widehat{F}^{\sigma} = id_I$ for all $\sigma \in \Gamma$.

Lemma 3.5. *Let* $\mathcal{S}$ *be a* $Gr-$*category of the type* (Π, A)*. Any factor set* (θ, F) *on* Γ *with cofficients in* $\mathcal{S}$ *is cohomologous to an almost strict factor set* (μ, G).

Proof. For each $\sigma \in \Gamma$, consider a family of isomorphisms in $\mathcal{S}$:

$$u_x^\sigma = \begin{cases} id_{F^\sigma x} & \text{if } x \neq 1, \\ (\widehat{F^\sigma})^{-1} & \text{if } x = 1, \end{cases}$$

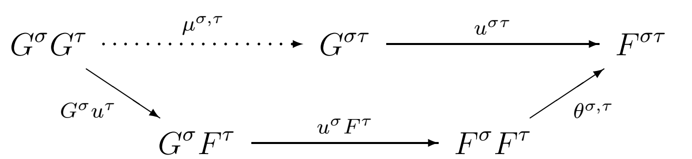
where $1 \neq \sigma \in \Gamma$, and $u^1 = id$. Then, we define G^σ uniquely such that $u^\sigma : G^\sigma \to F^\sigma$ is a natural transformation by setting $G^\sigma = F^\sigma$ and

$$\widetilde{G}^\sigma_{x,y} = (u_x^\sigma \otimes u_y^\sigma)^{-1} \widetilde{F}^\sigma_{x,y}(u^\sigma_{xy}); \quad \widehat{G}^\sigma = id_I.$$

For such setting, clearly we have

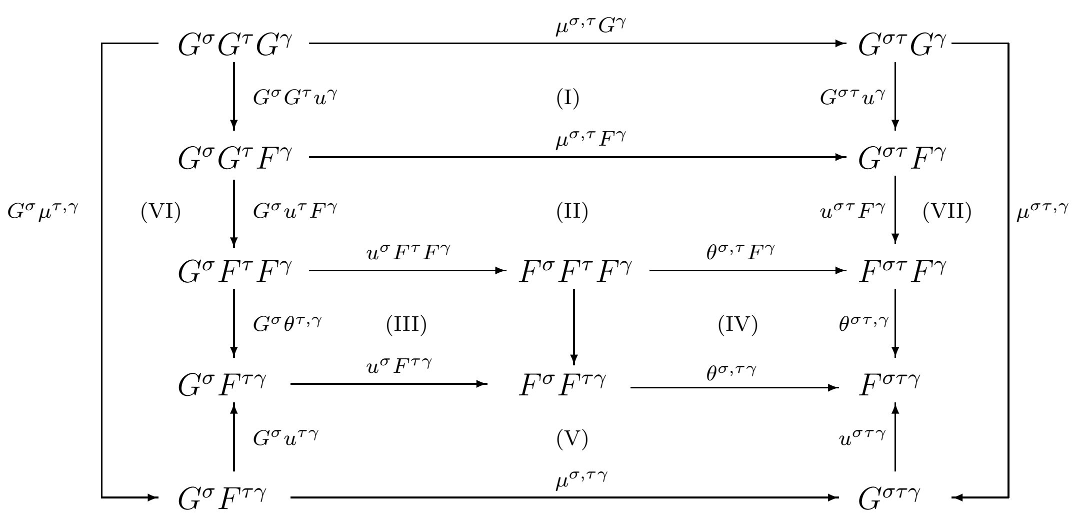
$$G^\sigma = (G^\sigma, \widetilde{G}^\sigma, \widehat{G}^\sigma) : \mathcal{S} \to \mathcal{S}$$

is a monoidal equivalence. Since $\widehat{G}^\sigma = id_I$, the factor set (μ, G) is almost strict. Now, we can choose $\mu^{\sigma,\tau} : G^\sigma G^\tau \to G^{\sigma\tau}$ the natural transformation which makes the following diagram



(8)

commute, for all σ, $\tau \in \Gamma$. Clearly, $\mu^{\sigma,\tau}$ is a isomorphism of monoidal functors. One can verify that the family of $\mu^{\sigma,\tau}$ satisfies the condition ii) of the definition of a factor set. We now prove that they satisfy the condition iii). Consider the following diagram



In this diagram, the region (I) commutes thanks to the naturality of $\mu^{\sigma,\tau}$; the regions (II), (V), (VI), (VII) commute thanks to (8); the region (III) commutes thanks to the naturality of u^σ; the region (IV) commutes thanks to the definition of factor set (θ, F). So the perimeter commutes. This completes the proof. □

4 Classification theorem

$\Gamma-$pair (Π, A) is given. In [3], cohomology groups $H^n_\Gamma(\Pi, A)$, with $n \leq 3$, can be regarded as group cohomologies of the struncated complex:

$$\widetilde{C}_\Gamma\,(\Pi, A) : 0 \longrightarrow C^1_\Gamma(\Pi, A) \xrightarrow{\partial} C^2_\Gamma(\Pi, A) \xrightarrow{\partial} Z^3_\Gamma(\Pi, A) \longrightarrow 0,$$

where $C^1_\Gamma(\Pi, A)$ consists of all normalized maps $f : \Pi \to A$, $C^2_\Gamma(\Pi, A)$ consists of all normalized maps $g : \Pi^2 \cup (\Pi \times \Gamma) \to A$ and $Z^3_\Gamma(\Pi, A)$ consists of all normalized maps $h : \Pi^3 \cup (\Pi^2 \times \Gamma) \cup (\Pi \times \Gamma^2) \to A$ satisfying conditions of a $3-$cocycle:

$$h(x, y, zt) + h(xy, z, t) = x(h(y, z, t)) + h(x, yz, t) + h(x, y, z)\,, \tag{9}$$

$$\sigma h(x, y, z) + h(xy, z, \sigma) + h(x, y, \sigma) = h(\sigma x, \sigma y, \sigma z) + (\sigma x)h(y, z, \sigma) + h(x, yz, \sigma)\,, \tag{10}$$

$$\sigma h(x, y, \tau) + h(\tau x, \tau y, \sigma) + h(x, \sigma, \tau) + (\sigma\tau x)h(y, \sigma, \tau) = h(x, y, \sigma\tau) + h(xy, \sigma, \tau)\,, \tag{11}$$

$$\sigma h(x, \tau, \gamma) + h(x, \sigma, \tau\gamma) = h(x, \sigma\tau, \gamma) + h(\gamma x, \sigma, \tau)\,, \tag{12}$$

for all $x, y, z, t \in \Pi;\ \sigma, \tau, \gamma \in \Gamma$.
For each $g \in C^2_\Gamma(\Pi, A), \partial g$ is given by

$$(\partial g)(x, y, z) = xg(y, z) - g(xy, z) + g(x, yz) - g(x, y), \tag{13}$$

$$(\partial g)(x, y, \sigma) = \sigma g(x, y) - g(\sigma x, \sigma y) - (\sigma x)g(y, \sigma) + g(xy, \sigma) - g(x, \sigma), \tag{14}$$

$$(\partial g)(x, \sigma, \tau) = \sigma g(x, \tau) - g(x, \sigma\tau) + g(\tau x, \sigma). \tag{15}$$

We now show that each factor set on Γ induces a $3-$cocycle of $\Gamma-$groups.

Proposition 4.1. *Each almost strict factor set* (θ, F) *on* Γ *with coefficients in a Gr−category* $\mathcal{S} = (\Pi, A, \xi)$ *induces an element* $h \in Z^3_\Gamma(\Pi, A)$.

Proof. Suppose $F^\sigma = (F^\sigma, \widetilde{F}^\sigma, id)$. Then, $\widetilde{F}^\sigma_{x,y}$ is associated to $\widetilde{f} : \Pi^2 \times \Gamma \to A$. The family of isomorphisms of monoidal functors $\theta^{\sigma,\tau}$ is associated to a function $t : \Pi \times \Gamma^2 \to A$. From functions $\xi, \widetilde{f}, t$, we determine a function h as follows:

$$h : \Pi^3 \cup (\Pi^2 \times \Gamma) \cup (\Pi \times \Gamma^2) \to A,$$

where $h = \xi \cup \widetilde{f} \cup t$, in the sense

$$h\mid_{\Pi^3} = \xi; h\mid_{\Pi^2\times\Gamma} = \widetilde{f}; \text{ and } h\mid_{\Pi\times\Gamma^2} = t. \tag{16}$$

The above-determined h is a 3−cocycle of Γ−groups. Indeed, 3−cocycle of ξ leads to (9). Equations (3), (6) turn into (10), (11). The cocycle condition

$$\theta^{\sigma\tau,\gamma}.\theta^{\sigma,\tau}F^\gamma = \theta^{\sigma,\tau\gamma}.F^\sigma\theta^{\tau,\gamma}$$

leads to equation (12). However, we need prove the normalized property of h. First, since the unitivity constraints of (Π, A) are strict and the factor set (θ, F) is almost strict, equations (3), (4), (7) turn into:

$$h(x, 1, \sigma) = \widetilde{f}^\sigma(x, 1) = 0 = \widetilde{f}^\sigma(1, x) = h(1, x, \sigma),$$

$$h(1, \sigma, \tau) = t^{\sigma,\tau}(1) = 0.$$

Since $\widehat{F}^1 = id$, we have $h(x, y, 1_\Gamma) = \widetilde{f}^1(x, y) = 0$. Thanks to the normalized property of associativity constraint ξ, we have

$$h(1, y, z) = h(x, 1, z) = h(x, y, 1) = 0.$$

Thanks to ii) in the definition of a factor set, we have $h(x, 1_\Gamma, \tau) = h(x, \sigma, 1_\Gamma) = 0$. Let $h^{(\theta,F)} = h$, we have $h^{(\theta,F)} \in Z^3_\Gamma(\Pi, A)$. This completes the proof. □

Proposition 4.2. *Each element* $h^{(\mu,G)} \in H^3_\Gamma(\Pi, A)$ *determines a* Γ−*monoidal extension of a Gr−category of the type* (Π, A).

Proof. According to the definition of cocycle of Γ−groups, we have functions (16). Hence, we may determine the factor set (θ, F) :

$$F^\sigma x = \sigma x; \quad F^\sigma(x, c) = (\sigma x, \sigma c),$$
$$\widehat{F}^\sigma = id_1, \quad \widetilde{F}^\sigma_{x,y} = (\sigma(xy), \widetilde{f}(x, y, \sigma)), \quad \theta^{\sigma\tau}x = (\sigma\tau x, t(x, \sigma, \tau)),$$

for all $\sigma, \tau \in \Gamma, x \in \Pi, c \in A$. Clearly, the these factor set induces h. □

In order to prove the Classification Theorem, we need the following lemma.

Lemma 4.3. *Let (μ, G), (θ, F) be two almost strict factor sets on Γ with coefficients in a $Gr-$category $\mathcal{S}$ of the type (Π, A) and be cohomologous. Then, they determine the same $\Pi-$module $\Gamma-$equivariant structure on A and $3-$cocyles inducing $h^{(\theta,F)}$, $h^{(\mu,G)}$ are cohomologous.*

Proof. Since (θ, F), (μ, G) are cohomologous, there exists a family of isomorphisms of functors $u^\sigma : F^\sigma \to G^\sigma, \sigma \in \Gamma$. According to Remark 3.4, $F^\sigma = G^\sigma, \sigma \in \Gamma$. Then, they induce the same Π-module Γ-equivariant structure on A thanks to relation (1). Now, we prove that $h^{(\theta,F)}$ and $h^{(\mu,G)}$ are cohomologous. Denote $h^{(\mu,G)} = h'$. Hence, by the determination of $h^{(\mu,G)}$ referred in Proposition 4.1, for all $x, y \in \Pi, \sigma, \tau \in \Gamma$, we have

$$\widetilde{G}^\sigma_{x,y} = (\sigma(xy), h'(x, y, \sigma)) \ ; \quad \mu^{\sigma,\tau}_x = (\sigma\tau x, h'(x, \sigma, \tau)),$$

Let $u : \Pi \times \Gamma \to A$ be the function defined by $u(x, \sigma) = u^\sigma_x$. It determines an extending $2-$cochain g of u, where $g|_{\Pi^2} : \Pi^2 \to A$ is the null map.

Since $(u^{\sigma\tau}.\theta^{\sigma,\tau})x = (\mu^{\sigma,\tau}.u^\sigma G^\tau.F^\sigma u^\tau)x$, we have

$$g(x, \sigma\tau) + h(x, \sigma, \tau) = h'(x, \sigma, \tau) + g(\tau x, \sigma) + \sigma g(x, \tau). \tag{17}$$

Since $\widetilde{G}^\sigma_{x,y} \cdot u^\sigma_{x\otimes y} = (u^\sigma_x \otimes u^\sigma_y) \cdot \widetilde{F}^\sigma_{x,y}$, we have

$$h'(x, y, \sigma) - h(x, y, \sigma) = g(x, \sigma) + (\sigma x)g(y, \sigma) - g(xy, \sigma). \tag{18}$$

Since $\widehat{F}^\sigma = \widehat{H}^\sigma = id$, we have $u^\sigma_1 = id_1$. Hence, $g(1, \sigma) = 0$, for all $\sigma \in \Gamma$. Since $u^1 = id_{((\Pi,A),\otimes)}$, we have $g(x, 1_\Gamma) = 0$, for all $x \in G$. From the determination of g and relations (17) - (18), we have $g \in C^2_\Gamma(\Pi, A)$ and $h^{(\theta,F)} - h^{(\mu,G)} = \partial g$. This completes the proof. □

We can simply the equivalence classification problem of $\Gamma-$extensions of $Gr-$categories by classifying $\Gamma-$extensions of $Gr-$categories with the first two invariants as the following definition.

Definition 4.4. Let Π be a $\Gamma-$group, A be a $\Gamma-$equivariant $\Pi-$module. We say that $\Gamma-$monoidal extension $\mathcal{G}$ of a $Gr-$category has a *pre-stick* of the type (Π, A) if there exists a pair

$$(p, q) : (\Pi, A) \to (\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G})),$$

where $p : \Pi \to \Pi_0(\mathcal{G})$ is an equivariant isomorphism (to make $\Pi_1(\mathcal{G})$ be a $\Pi-$module) and $q : A \to \Pi_1(\mathcal{G})$ is an isomorphism of $\Gamma-$equivariant $\Pi-$modules.

Obviously, each $\Gamma-$functor between two $\Gamma-$extensions whose pre-sticks are of the type (Π, A) is a $\Gamma-$equivalence.

Theorem 4.5. *There exists a bijection*

$$\Omega :_{\Gamma} \mathbf{Gr}(\Pi, A) \to H^3_{\Gamma}(\Pi, A),$$

where ${}_{\Gamma}\mathbf{Gr}(\Pi, A)$ *is the set of equivalence classes of* $\Gamma-$*monoidal extensions whose pre-sticks are of the type* (Π, A).

Proof. Each element of ${}_{\Gamma}\mathbf{Gr}(\Pi, A)$ may be represented by a crossed product extension $\Delta(\theta, F)$ of a $Gr-$category $S = (\Pi, A, \xi)$. According to Lemma 3.5, it is possible to assume that (θ, F) is almost strict. Then, (θ, F) induces a $3-$cocycle $h = h^{(\theta,F)}$ (Proposition 4.1). According to Lemma 4.3, the correspondence $cl(\theta, F) \to cl(h^{(\theta,F)})$ is a map. According to Proposition 4.2, this correspondence is a surjection. We now prove that it is an injection.

Let Δ and Δ' be two crossed product extensions of $Gr-$categories $\mathcal{S} = \mathcal{S}(\Pi, A, \xi), \mathcal{S}' = \mathcal{S}'(\Pi, A, \xi')$ by factor sets (θ, F), (θ', F'). Moreover, $3-$cocycles inducing h, h' are cohomologous. We shall prove that two $\Gamma-$extensions Δ and Δ' are equivalent. According to the determination of h, h', we have

$$h\mid_{\Pi^3} = \xi, h'\mid_{\Pi^3} = \xi' \text{ and } \xi' = \xi + \delta g,$$

where $g : \Pi^2 \to A$ is a function. Then, there exists a $Gr-$equivalence of two $Gr-$categories, $(K, \widetilde{K}, \widehat{K}) : \mathcal{S} \to \mathcal{S}'$, where $\widetilde{K}_{x,y} = (\bullet, g(x, y))$. According to Proposition 3.1, we may extend the $Gr-$functor $(K, \widetilde{K}, \widehat{K})$ to a $\Gamma-$equivalence

$$(\Delta_K, \widetilde{\Delta}_K, \widehat{\Delta}_K) : \Delta \to \Delta'$$

So Ω is an injection. □

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