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Ideals and Direct Products of Zero-Square Nearrings

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Abstract

We consider a zero symmetric right nearring N. The concepts, zero-square nearring of type-1/type-2, zero-square ideal of type-1/type-2, and zero square dimension of a nearring were introduced and obtained several important results. Finally, some relations between the zero-square dimension of the direct sum of finite number of nearrings, and the sum of the zero-square dimension of individual nearrings are obtained. Necessary examples are provided.

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1 Introduction

Throughout the paper N stands for nearring. A nearring N a zero-square if $x^2 = 0$ for all $x \in N$.

A nearring N is said to be *monogenic* if there exists $0 \neq a \in N$ such that $Na = N$. A left (respectively, right) ideal I of N is said to be *monogenic* if there exists $0 \neq a \in I$ such that $Na = I$ (respectively, $aN = I$). A monogenic nearring N is *primitive* if and only if N is faithful and simple (which is same as $0-$*primitive* as in 4.2 of Pilz [2]).

A proper ideal P of N is said to be a *prime* ideal if it satisfies the condition: A, B are ideals of N such that $AB \subseteq P$ imply $A \subseteq P$ or $B \subseteq P$. A proper ideal S of N is said to be a *semiprime* ideal if it satisfies the condition: A is an ideal of N with $A^2 \subseteq S$ implies $A \subseteq S$. A nearring N is a *subdirect product* of family of nearrings $\{S_i : i \in I\}$ if there is a monomorphism $k : N \to S$ $= \prod S_i, i \in I$ such that $\pi_i ok$ is epimorphism for all $i \in I$, where $\pi_i : S \to S_i$ canonical epimorphism.

An element $x \in N$ is said to be *nilpotent* if there exists a positive integer n such that $x^n = 0$. If every element of an ideal I of N is nilpotent, then we say that I is a *nil ideal*. For an ideal I of N, the quotient nearring of N with respect to I is denoted by N/I. N is said to be *nil* if every element of N is nilpotent.

Let I, J be two ideals of N such that $I \subseteq J$. We say that I is essential in J (denoted by $I \leq_e J$) if it satisfies the condition: K is an ideal of N, $K \subseteq J$, $I \cap K = (0)$ imply $K = (0)$. If I is essential in J and $I \neq J$, then we say that J is a proper essential extension of I. A non-zero ideal I of N is said to be *uniform* if B is a non zero ideal of N, and $B \subseteq I$ implies $B \leq_e I$.

We say that N has finite dimension on ideals (denoted by FDI) if N do not contain infinite number of non zero ideals whose sum is direct.

Theorem 1.1 *N is a subdirect product of the nearrings $\{S_i : i \in I\}$, if and only if $S_i \cong N/K_i$, K_i an ideal of N and $\bigcap K_i, i \in I = 0$.*

Theorem 1.2 *(Satyanarayana et.al. [5]): Suppose N is a nearring with FDI. Then*

(i) (existence) there exist uniform (two sided) ideals $U_1, U_2, ..., U_n$ in N whose sum is direct and essential in N;

(ii) (uniqueness) if $V_i, 1 \leq i \leq k$, are uniform ideals of N whose sum is direct and essential in N, then $k = n$.

The number n of the above Theorem is independent of the choice of the uniform ideals, and this number n is called the dimension of N (it is denoted by $dimN$).

Theorem 1.3 *(Satyanarayana et.al. [5]): Suppose $I_i, 1 \leq i \leq k$ are ideals of the nearrings $N_i, 1 \leq i \leq k$ respectively. Then the following two conditions are equivalent:*

(i) $I_i \leq_e N_i, 1 \leq i \leq k$;

(ii) $I_1 \oplus I_2 \oplus ... \oplus I_k \leq_e N_1 \oplus N_2 \oplus ... \oplus N_k$.

From Theorems 1.2 and 1.3, we get the following theorem.

Theorem 1.4 *(Satyanarayana et.al. [5]): If $N_i, 1 \leq i \leq k$ are nearrings with FDI, then $dim(N_1 \oplus N_2 \oplus ... \oplus N_k) = dimN_1 + dimN_2 + ... + dimN_k$.*

The ideal generated by an element $x \in N$ is denoted by $< x >$. We do not present the proofs of some results in this paper when they are simple or parallel to those results in the literature on nearring theory.

2 Zero-square Nearrings

In this section we define and study the concepts zero-square nearring of type-1/type-2. Zero-square nearring of type-2 is same as the zero-square nearring studied by the earlier authors. We prove that every zero-square nearring of type-1 is a zero-square nearring of type-2, but the converse need not be true, in general.

Definition 2.1 *(i) A nearring N is said to be a* zero-square nearring of type-1 *if $x^2 = 0$ for all $x \in N$, and there exists two elements $a, b \in N$ such that $ab \neq 0$.*

(ii) A nearring N is said to be a zero-square nearring of type-2 *if $x^2 = 0$ for all $x \in N$.*

Zero-square nearrings of type-2 are same as the zero-square nearrings studied by the earlier authors like Stanley. Every zero-square nearring of type-1 is a zero-square nearring of type-2.

Example 2.2 *(i) Every null nearring (that is $N^2 = 0$) is a zero-square nearring of type-2, but not of type 1.*

(ii) Let $(G, +)$ be a group (not necessarily Abelian). Define multiplicative operation on G by $a \cdot b = 0$ for all $a, b \in G$, where 0 is additive identity. Then $(G, +,)$ is a null nearring. So $(G, +,)$ is a zero-square nearring of type-2, but not of type-1. Now we can conclude that every group can be made into a zero-square nearring of type-2.

(iii). Suppose that N is a non zero Boolean nearring. Then $x^2 = x$ for all $x \in N$. So N is a non-null nearring and for any $x \neq 0$, we have $x^2 \neq 0$. Hence every non-zero Boolean nearring can neither a zero-square nearring of type-1 nor a zero-square nearring of type-2.

(iv). Let S be a non null nearring (that is, $S^2 \neq 0$). Write $N = S \times S \times S$. Define addition on N component wise. Define multiplication on N by $(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (0, 0, x_1y_2 - x_2y_1)$. Now it is clear that $N^2 \neq 0$ (that is, N is not a null nearring) and $a^2 = 0$ for all $a \in N$. Hence N is a zero-square nearring of type-1.

Theorem 2.3 *Suppose N is a zero-square nearring of type-2. Then*
(i) $aN \neq N$ for all $0 \neq a \in N$.
(ii) If N is simple, then $N^2 = 0$.

Proof: (i) Let N be a zero-square nearring, and $0 \neq a \in N$. Suppose $aN = N$. Then $a \in N = aN$ implies $a = ar$ for some $r \in N$. Now $a = ar = (ar)r = ar^2 = a0 = 0$ (since N is zero symmetric nearring), a contradiction.

(ii) Suppose $N^2 \neq 0$. Then there exist $s, a \in N$ such that $as \neq 0$. Now $0 \neq as \in aN$. Since N is simple and $aN \neq 0$, we have that $aN = N$, a contradiction. Hence $N^2 = 0$.

Corollary 2.4 *A primitive nearring cannot be a zero-square nearring of type-2.*

Proof: Since N is primitive by definition 4.2 of Pilz [2], it is faithful and simple. Let $0 \neq r \in N$. Since N is faithful we have $Nr \neq 0$. Now $0 \neq Nr \subseteq NN$. This means $NN \neq 0$, a contradiction by Theorem 2.3 (ii).

Corollary 2.5 *Let N be a zero-square nearring of type-2.*
(i) If I is a non zero left ideal of N, then I can not be a monogenic left ideal; and
(ii) If I is a non zero right ideal of N, then I can not be a monogenic right ideal.

Proof: (i) In a contrary way, suppose that I is a monogenic left ideal. Then there exist $0 \neq a \in I$ such that $Na = I$, a contradiction to the Theorem 2.3(i).

(ii) Similar to (i).

Corollary 2.6 *If N is non-zero zero-square nearring of type-2, then*
(i) $Nr \neq N$ for all $r \in N$; and
(ii) $rN \neq N$ for all $r \in N$.

Proof: The proof follows by taking N instead of I in Corollary 2.5.

3 Zero-square Ideals

In this section we define the substructure zero-square ideal of type-1 (respectively, type-2) of nearring and obtained related results.

Definition 3.1 *A proper ideal I of N is said to be a zero-square ideal of type-1 (respectively, type-2) if the quotient nearring N/I is a zero-square nearring of type-1 (respectively, type-2).*

Remark 3.2 *(i) If N is a zero-square nearring of type-2, then every ideal I of N is a zero-square ideal of type-2. The converse of this statement is not true. For this observe the following example 3.3.*

(ii) If N is a zero-square nearring of type-2, then every ideal of N is also a zero-square nearring of type-2.

Example 3.3 *Consider Z_2, the nearring of integers modulo 2. Z_2 is not a zero-square nearring of type-2. Let G be a non-zero additive group and define $a \cdot b = 0$ for all $a, b \in G$. Now $(G, +,)$ is a zero-square nearring of type-2. Write $N = Z_2 \oplus G$, the direct sum of nearrings Z_2 and G. Now $I = Z_2$ is an ideal of N; for any $x + I \in N/I$, we get that $(x + I)^2 = 0 + I$; and hence I is a zero-square ideal of type-2. Since $1 = 1 + 0 \in Z_2 + G = N$ and $1^2 = 1 \neq 0$, it follows that N is not a zero-square nearring of type-2.*

Example 3.4 *Consider $N = Z_8$, the additive group of integers modulo 8. Let us define the multiplication on N as it is given by the table. Observe that: $I_2 = 0, 4$ is an ideal of N. Now $N^2 = \{0, 4\} \subseteq I_2$; and $N/I_2 = \{0 + I_2, 1 + I_2, 2 + I_2, 3 + I_2\}$.*

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	4	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	4	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	4	0

Also observe that $(N/I_2)^2 = 0$, and so N/I_2 is not a zero-square nearring, where as N is a zero-square nearring. Note that I_2 is not a zero-square ideal.

Remark 3.5 *If I, J be two zero-square ideals of type-2, then $I \cap J$ is also a zero-square ideal of type-2.*

Verification: Let $x \in N/(I \cap J)$. Now, $x + I \in N/I$. Since I is zero-square ideal of type 2, we have $x^2 + I = 0 + I$ implies $x^2 \in I$. Similarly we can show that $x^2 \in J$ and so $x^2 \in I \cap J$. Now $x^2 + (I \cap J) = 0 + (I \cap J)$ implies $(x + (I \cap J))^2 = 0$ in $N/(I \cap J)$. Hence $N/(I \cap J)$ is a zero-square nearring of type-2. Therefore $I \cap J$ is a zero-square ideal of type-2.

Definition 3.6 *A class* B *of nearrings is said to be* homomorphically closed *if every homomorphic image of* N *is in* B *for all* N *in* B.

Theorem 3.7 *The class* B *of all zero-square nearrings of type-2 is homomorphically closed.*

Proof: Let $N \in B$ and I an ideal of N. Take $x + I \in N/I$. Now $(x + I)^2 = x^2 + I = 0 + I$ (since N is a zero-square nearring of type-2). So N/I is a zero-square nearring of type-2 and hence $N/I \in B$.

Remark 3.8 *Suppose* I *is an ideal of* N, I *is a zero-square ideal of type-2 and also a zero-square nearring of type-2, then* $x^4 = 0$ *for all* $x \in N$.

Verification: $x \in N \Rightarrow x + I \in N/I$. Now $(x + I)^2 = 0 + I$ (since I is a zero-square ideal of type-2). This means $x^2 \in I$, which implies that $(x^2)^2 = 0$ (since I is a zero-square nearring of type-2). Therefore $x^4 = 0$.

Theorem 3.9 *Let* N *be a zero-square nearring of type-2 and* I *an ideal of* N. *Then the following two conditions are equivalent:*

(i) $N^2 \not\subseteq I$ *; and (ii)* I *is a zero-square ideal of type-1.*

Proof: $(i) \Rightarrow (ii)$: By Remark 3.2, we get that I is a zero-square ideal of type-2. Since $N^2 \not\subseteq I$, there exist $x, y \in N$ with $xy \notin I$ and so $(x+I)(y+I) \neq 0 + I$ in N/I.

Therefore N/I is a zero-square nearring of type-1 and so I is a zero-square ideal of type-1.

$(ii) \Rightarrow (i)$: Since N/I is a zero-square nearring of type-1, there exist two non-zero elements $c + I$ and $d + I$ in N/I whose product is non-zero in N/I. This gives that $cd \notin I$ and so $N^2 \not\subseteq I$.

Corollary 3.10 *(i) Let* I *and* J *be ideals of a zero-square nearring* N *of type-2 with* $I \subseteq J$. *If* J *is a zero-square ideal of type-1, then* I *is also a zero-square ideal of type-1.*

(ii) Intersection of any collection of zero-square ideals of type-1 is also a zero-square ideal.

Corollary 3.11 *Let* **N** *be the class of all zero-square nearrings* N *of type-1 for which* $N^2 \not\subseteq I$ *for all non-zero ideals* I *of* N. *Then the class* **N** *is homomorphically closed.*

Proof: Let $N \in \mathbf{N}$ and $h : N \to N^1$ be an epimorphism. Then $N/I \cong N^1$, where $I = kerh$, an ideal of N.

Case (i): Suppose h is an isomorphism. Then $I = 0$. Since N is a zero-square nearring of type-1, there exists $x, y \in N$ such that $xy \neq 0$. So $N^2 \neq 0$ and $N^2 \nsubseteq I$.

Case (ii): Suppose h is not an isomorphism. Then $I \neq 0$. From the assumption, $N^2 \nsubseteq I$. Now by Theorem 3.8, I is a zero-square ideal of type-1 and hence $N^1 \cong N/I \in \mathbf{N}$.

Corollary 3.12 *In a zero-square nearring N of type-2,*
(i) every semi-prime ideal S of N is a zero-square ideal of type-1; and
(ii) every prime ideal P of N is a zero-square ideal of type-1.

Proof: (i) Suppose S is not a zero-square ideal of type-1. Then by Theorem 3.8 we get that $N^2 \subseteq S$. Since S is semi prime ideal, we have that $S = N$, a contradiction.

(ii) Follows from (i) since every prime ideal is a semi-prime ideal. This completes the proof.

We denote $ZS1(N)$ = the intersection of all zero-square non-zero ideals of N of type-1; and $ZS2(N)$ = the intersection of all zero-square non-zero ideals of N of type-2. If there are no non-zero zero-square ideals of type-1 (respectively, type-2) in N then we define $ZS1(N) = N$ (respectively, $ZS2(N) = N$).

Remark 3.13 *If N is a zero-square nearring of type-2, then we have the following:*

(i). By Theorem 3.8, we get that if N is a zero-square nearring of type-2, then $ZS1(N) = \cap\{I : I$ is a non-zero ideal of N with $N^2 \nsubseteq I\}$;

(ii). If $ZS2(N) = 0$ (respectively, $ZS1(N) = 0$); then by Theorem 1.2, it follows that N is a subdirect product of the zero-square nearrings N/I, where I runs over all non-zero zero-square ideals of type-2 (respectively, type-1) in N. If $ZS2(N) \neq 0$ (respectively, $ZS1(N) \neq 0$), then $ZS2(N)$ (respectively, $ZS1(N)$) is the smallest non-zero zero-square ideal of type-2 (respectively, type-1), among all non-zero zero-square ideals of type-2 (respectively, type-1).

(iii). In Example 3.3, $N = \mathbb{Z}_2 \oplus G$ is not a zero-square nearring of type-2. In this case $ZS2(N) = \mathbb{Z}_2$. Note that $(0) \neq ZS2(N) \neq N$.

(iv). If N is a zero-square nearring of type-2 and N contains a zero-square ideal of type 1, then by Corollary 3.9, we get that $ZS1(N) \subseteq I$.

(v). If $N^2 = 0$, then N contains no zero-square ideals of type-1 and so $ZS1(N) = N$.

Theorem 3.14 *If there exists a chain $N = I_0 \supset I_1 \supset I_2 \supset ... \supset I_k = (0)$ of ideals of N such that I_{s+1} is a zero-square ideal of type-2 in the nearring I_s, then N is a nil ideal of N. In particular, $x^{(2^k)} = 0$ for all $x \in N$.*

Proof: Let $x \in N = I_0$. Since I_1 is zero-square ideal of type-2 in the nearring I_0 and $x \in I_0$ we have that $(x + I_1)^2 = 0$ in I_0/I_1. So $x^2 \in I_1$. Since $x^2 \in I_1$ and I_2 is a zero-square ideal of type-2 in the nearring I_1, it follows that $(x^2 + I_2)^2 = 0$ in I_1/I_2 and so $x^4 \in I_2$. If we continue this process, eventually, we get that $x^{(2^k)} \in 0$. Thus $x^{(2^k)} = 0$ and this is true for all $x \in N$. Therefore N is a nil ideal.

Corollary 3.15 *Let $I_1, ..., I_k$ be as in the Theorem 3.14. For any ideal I of N, I and N/I are also nil.*

4 Direct Products of Zero-square Nearrings

We observe that the direct product of zero-square nearrings $N_i, 1 \leq i \leq k$ of type-1 is also a zero-square nearring of type-1, but the converse need not be true, in general. We also obtain some important consequences.

If $N_1, N_2, ..., N_k$ are nearrings, then the nearring $N_1 \times N_2 \times ... \times N_k$, the direct product of $N_i, 1 \leq i \leq k$ is denoted by $\prod N_i, 1 \leq i \leq k$. For any nearring N, let us write $N^k = \prod_k N$ for the direct product of k copies of N.

A straightforward verification provides the following.

Theorem 4.1 *(i). If $Ni, 1 \leq i \leq k$ are zero-square nearrings of type-1, then $\prod N_i, 1 \leq i \leq k$ is also a zero-square nearring of type-1;*

(ii). Each $N_i, 1 \leq i \leq k$ are zero-square nearring of type-2 if and only if $\prod N_i, 1 \leq i \leq k$ is a zero-square nearring of type-2.

Remark 4.2 *The converse of the above Lemma 4.1(i) is not true, in general. For this let us observe the following example.*

Example 4.3 *Write $(N, +) = Z_2$ additive group of integers modulo 2. Consider the zero product on N (that is, $xy = 0$ for all $x, y \in N$). Then N is nearring which is not a zero-square nearring of type-1. Let M be a zero-square nearring of type-1. Consider the nearring $N \times M$ which is the direct product of N and M. Now $N \times M$ is a zero-square nearring of type-1, where as N is not a zero-square nearring of type-1.*

Theorem 4.4 *Let $N_i, 1 \leq i \leq k$ be nearrings. The direct product $\prod N_i, 1 \leq i \leq k$ is a zero-square nearrings of type-1 if and only if there exists a non-empty subset $I \subseteq \{1, 2, ..., k\}$ such that N_i is a zero-square nearrings of type-1 for all $i \in I$ and N_j is a zero-square nearring of type-2 but not of type-1 for all $j \in \{1, 2, ..., k\} - I$.*

Proof: Suppose that $\prod N_i, 1 \leq i \leq k$ is a zero-square nearring of type-1. Let $s \in \{1, 2, ..., k\}$ and $x_s \in N_s$. Consider the element $(0, ..., 0, x_s, 0, ..., 0) \in \prod N_i, 1 \leq i \leq k$, the s^{th} co-ordinate is x_s and zero elsewhere. Now $0 = (0, ..., 0, x_s, 0, ..., 0)^2 = (0, ..., 0, x_s^2, 0, ..., 0)$ and $x_s^2 = 0$. Thus $a^2 = 0$ for all $a \in N_s$, and this is true for all $1 \leq s \leq k$. So each N_s is a zero-square nearring of type-2.

Write $I = \{s : 1 \leq s \leq k$ and there exist elements $x, y \in N_s$ such that $xy \neq 0\}$.

Now it is clear that N_i, is a zero-square nearring of type-1 for all $i \in I$. Since $\prod N_i, 1 \leq i \leq k$ is a zero-square nearring of type-1, there exist at least two elements $(x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k)$ in $\prod N_i, 1 \leq i \leq k$ with $(x_1y_1, x_2y_2, ..., x_ky_k) \neq 0$. Thus there exist $t(1 \leq t \leq k)$ such that $x_ty_t \neq 0$. Now $t \in I$ and so $I \neq \phi$. It is clear that for all $j \in J = \{1, 2, ..., k\}\ I$, we have that $xy = 0$ for all $x, y \in N_j$. Hence N_j is not a zero-square nearring of type-1, for all $j \in J$.

Converse: Since I is non-empty, there exists $i \in I$ such that N_i is not a zero-square nearring of type-1. So there exist $x_i, y_i \in N_i$ with $x_iy_i \neq 0$. Now $(0, ..., x_i, ..., 0), (0, ..., y_i, ..., 0) \in \prod N_i, 1 \leq i \leq k$ and the product of these elements is non-zero. By Lemma 4.1, $\prod N_i, 1 \leq i \leq k$ is a zero-square nearring of type-2. Hence it is a zero-square nearring of type-1.

Corollary 4.5 *For any positive integer k, we have that N is a zero-square nearring of type-2 (respectively, type-1) if and only if N^k is a zero-square nearring of type-2 (respectively, type-1).*

5 Zero-square Dimension

In this section, we introduce zero-square dimension of type-1/type-2. We consider a class of nearrings N and obtained some relations between the concepts dimension of N, zero-square dimension of type-1/type-2. Finally, we apply this result for the direct sum of nearrings.

Definition 5.1 *Let N has FDI. We define the* zero-square dimension *of N (denoted by $ZSd(N)$) as follows:*

$ZSd(N) = \{s :$ *there exist uniform ideals $U_i, 1 \leq i \leq s$ in N such that the sum $U_1 + U_2 + ... + U_s$ is direct and each U_i is a zero-square nearring of type-2*$\}$.

Lemma 5.2 *(i) If N has FDI, and N is a zero-square nearring of type-2, then ZSd(N) = dim N.*

(ii). If $N_i, 1 \leq i \leq n$ are nearrings with FDI and each N_i is a zero-square nearring of type-2, then $ZSd(\prod_{i=1}^{n} N_i) = \sum_{i=1}^{n} ZSd(N_i)$.

Proof: (i) Suppose $k = dimN$. Since $k = dimN$, there exist uniform ideals $U_1, U_2, ..., U_k$ in N such that $U_1 \oplus U_2 \oplus ... \oplus U_k \leq_e N$. Since N is a zero-square nearring of type-2, by Remark 3.2 (ii), each U_i is also zero-square nearring of type-2. By definition 5.1, $ZSd(N) = k$. Hence $ZSd(N) = dimN$.

(ii). By Theorem 4.1(ii), $\prod N_i$ is also a zero-square nearring of type-2. Now $= ZSd(\prod N_i) = dim(\prod N_i)$ (by (i)) $= \sum_{i=1}^{n} dim(N_i)$ (by Theorem 1.4) $= \sum_{i=1}^{n} ZSd(N_i)$ (by (i)).

Lemma 5.3 *Suppose N has FDI and satisfies the condition $< xy > = < x >< y >$ for all $x, y \in N$ with $xy \neq 0$. If N is zero-square nearring of type-1, then there exists a uniform ideal U in N such that U itself a zero-square nearring of type-1.*

Proof: Since N has FDI, by Theorem 1.2, $dimN = k$, and there exist uniform ideals $I_1, I_2, ..., I_k$ such that $I_1 \oplus I_2 ... \oplus I_k \leq_e N$. Write $E = I_1 \oplus I_2 \oplus ... \oplus I_k$. Since N is a zero-square nearring of type-1, there exist $x, y \in N$ with $xy \neq 0$. Since $0 \neq xy \in < xy >$, and $E \leq_e N$, we have that $< xy > \cap E \neq 0$. Now $< x >< y > \cap E \neq 0$. This implies there exists $x^1 \in < x >, y^1 \in < y >$ such that $0 \neq x^1 y^1 \in E$. So $E = I_1 \oplus I_2 \oplus ... \oplus I_k$ is a zero-square nearring of type-1. By Theorem 4.4, there exists $t \in \{1, 2, ..., k\}$ such that I_t is a zero-square nearring of type-1.

Definition 5.4 *Let N has FDI and $dimN = k$. If N contains no uniform ideal which is a zero-square nearring of type-1, then we define the* zero-square-1 dimension *of N (denoted by $ZS1d(N)$)is equal to zero. We write $ZS1d(N) = 0$. If N contains a uniform ideal which is a zero-square nearring of type-1, then we define the zero-square-1 dimension of N as follows:*

$ZS1d(N) = max\{t : U_1, U_2, ..., U_t, U_{t+1}, ..., U_k$ are uniform ideals of N, whose sum is direct and essential in N (that is, $U_1 \oplus U_2 \oplus ... \oplus U_k \leq_e N$), $U_1, U_2, ..., U_t$ are zero-square nearrings of type-1, $U_{t+1}, ..., U_k$ are not zero-square nearrings of type-1$\}$.

Note 5.5 *(i). If N has FDI, N is a zero-square nearring of type-1 and satisfies the condition $< xy >=< x >< y >$ for all $x, y \in N$ with $xy \neq 0$. By Lemma 5.3, there exist uniform ideals $U_1, U_2, ..., U_k$ in N whose sum is direct and essential in N. Also at least one of the U_i's is a zero-square nearring of type-1. Thus, in this case, $ZS1d(N) \geq 1$.*

(ii). If N is a zero-square nearring of type-2 but not of type-1, then there exist no uniform ideal in N which is a zero-square nearring of type-1. So in this case $ZS1d(N) = 0$.

Theorem 5.6 *If N_1, N_2 are nearrings with FDI and $N = N_1 \oplus N_2$, the direct sum of nearrings, then $ZS1d(N_1 \oplus N_2) \geq ZS1d(N_1) + ZS1d(N_2)$.*

Proof: Suppose $ZS1d(N_1) = n$ and $ZS1d(N_2) = m$. Then there exists uniform ideals $I_1, I_2, ..., I_k$ of N_1 such that $I_1 \oplus I_2 \oplus ... \oplus I_k \leq_e N_1$, where $I_i, 1 \leq i \leq n$ are zero-square nearrings of type-1. Similarly there exists uniform ideals $J_1, J_2, ..., J_s$ of N_2 such that $J_1 \oplus J_2 \oplus ... \oplus J_s \leq_e N_2$, $J_i, 1 \leq i \leq m$ are zero-square nearrings of type-1. Since $N = N_1 \oplus N_2$, we have that the ideals of N_1 and the ideals of N_2 are also ideals of N. Now $I_1 \oplus I_2 \oplus ... \oplus I_n \oplus J_1 \oplus J_2 \oplus ... \oplus J_m \oplus I_{n+1} \oplus I_{n+2} \oplus ... \oplus I_k \oplus J_{m+1} \oplus ... \oplus J_s \leq_e N$ (by Th. 1.3); $I_1 \oplus I_2 \oplus ... \oplus I_n \oplus J_1 \oplus J_2 \oplus ... \oplus J_m$ is a sum of $(n+m)$ uniform ideals which are zero-square nearrings of type-1. So by the Definition 5.4, it follows that $ZS1d(N_1 \oplus N_2) \geq n + m = ZS1d(N_1) + ZS1d(N_2)$.

Corollary 5.7 *If $N_i, 1 \leq i \leq k$ are nearrings with FDI, then $ZS1d(N_1 \times N_2 \times ... \times N_k) \geq \sum_{i=1}^{k} ZS1d(N_i)$.*

Definition 5.8 *Let N be a nearring with FDI. We define $ZS2d(N)$, the zero-square-2 dimension of N as follows:*

$ZS2d(N) = min\{t : U_1, U_2, ..., U_k$ are uniform ideals of N such that $U_1 \oplus U_2 \oplus ... \oplus U_k \leq_e N, U_1, U_2, ..., U_t$ are zero-square nearrings of type-2 but not of type-1$\}$.

Note 5.9 *Suppose N has FDI, $dimN = k$ and N is a zero-square nearring of type-2 but not of type-1. Then by Note 5.5, $ZS1d(N) = 0$. Since every representation $E = U_1 \oplus U_2 \oplus ... \oplus U_k$ that is equal to a direct sum of uniform ideals with $E \leq_e N$, contains exactly k uniform ideals, we have that $ZS2d(N) = k$. So in this case, $ZS1d(N) = 0$ and $ZS2d(N) = dimN$.*

Theorem 5.10 *(i) If N has FDI and N is a zero-square nearring of type-1, then $dim(N) = ZSd(N) = ZS1d(N) + ZS2d(N)$.*

(ii) If $Ni, 1 \leq i \leq k$ are nearrings with FDI, and also zero-square nearrings of type-1, then $dim(N_1 \times N_2 \times ... \times N_k) = ZSd(N_1 \times N_2 \times ... \times N_k) \geq \sum_{i=1}^{k} ZS1d(N_i) + \sum_{i=1}^{k} ZS2d(N_i)$.

Proof: (i) By Lemma 5.2(i), $dim(N) = ZSd(N)$. Suppose $dim(N) = k$ and $ZS1d(N) = n$. Then there exist uniform ideals $I_1, I_2, ..., I_k$ in N such that $I_1 \oplus I_2 \oplus ... \oplus I_k \leq_e N$ and $I_i, 1 \leq i \leq n$ are zero-square nearrings of type-1, n is maximum among such n. Also $I_{n+1}, ..., I_k$ are uniform ideals of $N(k - n$ in number) which are zero-square nearrings of type-2 (but not of type-1).

So $ZS2d(N) \leq k - n$. Suppose $m = ZS2d(N)$. Then there exist uniform ideals $U_1, U_2, ..., U_k$ in N such that $U_1 \oplus U_2 \oplus ... \oplus U_k \leq_e N, U_i, 1 \leq i \leq m$ are zero-square-nearrings of type-2 (but not type-1) and m is the minimum among these numbers. This means that the remaining $k - m$ uniform ideals $U_{m+1}, ..., U_k$ are zero-square nearrings of type-1 (we get this because of the hypothesis that N is a zero-square nearring of type-2). By the Definition

5.4, we conclude that $k - m \leq n$, which imply that $m \geq k - n$. Hence $ZS2d(N) = m = k - n = dimN - ZS1d(N)$. Finally we get that $dimN = ZSd(N) = ZS1d(N) + ZS2d(N)$.

Proof for (ii) follows by using (i) and mathematical induction.

Corollary 5.11 *(i) If N_1, N_2 are zero-square nearrings of type-2 with FDI, then $ZS2d(N_1 \oplus N_2) \leq ZS2d(N_1) + ZS2d(N_2)$.*

(ii) If $N_i, 1 \leq i \leq k$ are zero-square nearrings with FDI, then $ZS2d(N_1 \oplus N_2 \oplus ... \oplus N_k) \leq \sum_{i=1}^{k} ZS2d(N_i)$.

Proof: (i) $ZS1d(N_1 \oplus N_2) + ZS2d(N_1 \oplus N_2) = ZSd(N_1 \oplus N_2)$ (by Theorem 5.10) $= ZSd(N_1) + ZSd(N_2)$ (by Lemma 5.2(ii)) $= ZS1d(N_1) + ZS2d(N_1) + ZS1d(N_2) + ZS2d(N_2)$ (by Theorem 5.10) $\leq ZS1d(N_1 \oplus N_2) + ZS2d(N_1) + ZS2d(N_2)$ (by Theorem 5.6). Therefore $ZS2d(N_1 \oplus N_2) \leq ZS2d(N_1) + ZS2d(N_2)$.

Proof for (ii) follows by using (i) and mathematical induction.

An Application: We give an application to graph theory by considering a graph of N denoted by $G_N = (V, E)$ defined as $V = N$ and $E = \{ab : a, b \in N, a \neq b, a \cdot b = 0\}$. For instance, consider $S = \{0, 1\}$, the nearring of integers modulo 2. Write $N = S \times S \times S$. Define addition and multiplication defined as in example 2.2. Note that a graph is said to be *complete* if there exists exactly one edge between any two vertices. Take the set of vertices $V = \{A = (0,0,0), B = (0,0,1), C = (0,1,0), D = (0,1,1), E = (1,0,0), F = (1,0,1), G = (1,1,0), H = (1,1,1)\}$. It is clear that there are no edges between F and G; E and D; D and H. Therefore it is not a complete graph. Further, since $F \neq 0$ and $G \neq 0$, there is no edge between F and G. Similarly, there are no edges between E and D; D and H etc. This means that the graph cannot be a complete graph. In general, a zero-square nearring N of type 2 is a zero-square nearring of type 1 if and only if its related graph is not complete.

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