Ideals and Direct Products of Zero-Square Nearrings

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Abstract

We consider a zero symmetric right nearring $N$. The concepts, zero-square nearring of type-1/type-2, zero-square ideal of type-1/type-2, and zero square dimension of a nearring were introduced and obtained several important results. Finally, some relations between the zero-square dimension of the direct sum of finite number of nearrings, and the sum of the zero-square dimension of individual nearrings are obtained. Necessary examples are provided.

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1 Introduction

Throughout the paper $N$ stands for nearring. A nearring $N$ a zero-square if $x^2 = 0$ for all $x \in N$.

A nearring $N$ is said to be monogenic if there exists $0 \neq a \in N$ such that $Na = N$. A left (respectively, right) ideal $I$ of $N$ is said to be monogenic if there exists $0 \neq a \in I$ such that $Na = I$ (respectively, $aN = I$). A monogenic nearring $N$ is primitive if and only if $N$ is faithful and simple (which is same as $0 -$ primitive as in 4.2 of Pilz [2]).

A proper ideal $P$ of $N$ is said to be a prime ideal if it satisfies the condition: $A, B$ are ideals of $N$ such that $AB \subseteq P$ imply $A \subseteq P$ or $B \subseteq P$. A proper ideal $S$ of $N$ is said to be a semiprime ideal if it satisfies the condition: $A$ is an ideal of $N$ with $A^2 \subseteq S$ implies $A \subseteq S$. A nearring $N$ is a subdirect product of family of nearrings $\{S_i : i \in I\}$ if there is a monomorphism $k : N \rightarrow S = \prod_{i \in I} S_i$ such that $\pi_i ok$ is epimorphism for all $i \in I$, where $\pi_i : S \rightarrow S_i$ canonical epimorphism.

An element $x \in N$ is said to be nilpotent if there exists a positive integer $n$ such that $x^n = 0$. If every element of an ideal $I$ of $N$ is nilpotent, then we say that $I$ is a nil ideal. For an ideal $I$ of $N$, the quotient nearring of $N$ with respect to $I$ is denoted by $N/I$. $N$ is said to be nil if every element of $N$ is nilpotent.

Let $I, J$ be two ideals of $N$ such that $I \subseteq J$. We say that $I$ is essential in $J$ (denoted by $I \lesssim J$) if it satisfies the condition: $K$ is an ideal of $N$, $K \subseteq J$, $I \cap K = (0)$ imply $K = (0)$. If $I$ is essential in $J$ and $I \neq J$, then we say that $J$ is a proper essential extension of $I$. A non-zero ideal $I$ of $N$ is said to be uniform if $B$ is a non zero ideal of $N$, and $B \subseteq I$ implies $B \leq_e I$.

We say that $N$ has finite dimension on ideals (denoted by $FDI$) if $N$ do not contain infinite number of non zero ideals whose sum is direct.

**Theorem 1.1** $N$ is a subdirect product of the nearrings $\{S_i : i \in I\}$, if and only if $S_i \cong N/K_i$, $K_i$ an ideal of $N$ and $\bigcap_{i \in I} K_i = 0$.

**Theorem 1.2** (Satyanarayana et.al. [5]): Suppose $N$ is a nearring with $FDI$. Then
(i) (existence) there exist uniform (two sided) ideals $U_1, U_2, ..., U_n$ in $N$ whose sum is direct and essential in $N$;
(ii) (uniqueness) if $V_i, 1 \leq i \leq k$, are uniform ideals of $N$ whose sum is direct and essential in $N$, then $k = n$.

The number $n$ of the above Theorem is independent of the choice of the uniform ideals, and this number $n$ is called the dimension of $N$ (it is denoted by $dimN$).
Theorem 1.3 (Satyanarayana et.al. [5]): Suppose $I_i, 1 \leq i \leq k$ are ideals of the nearrings $N_i, 1 \leq i \leq k$ respectively. Then the following two conditions are equivalent:

(i) $I_i \leq_e N_i, 1 \leq i \leq k$;
(ii) $I_1 \oplus I_2 \oplus ... \oplus I_k \leq_e N_1 \oplus N_2 \oplus ... \oplus N_k$.

From Theorems 1.2 and 1.3, we get the following theorem.

Theorem 1.4 (Satyanarayana et.al. [5]): If $N_i, 1 \leq i \leq k$ are nearrings with FDI, then $\dim(N_1 \oplus N_2 \oplus ... \oplus N_k) = \dim N_1 + \dim N_2 + ... + \dim N_k$.

The ideal generated by an element $x \in N$ is denoted by $\langle x \rangle$. We do not present the proofs of some results in this paper when they are simple or parallel to those results in the literature on nearring theory.

2 Zero-square Nearrings

In this section we define and study the concepts zero-square nearring of type-1/type-2. Zero-square nearring of type-2 is same as the zero-square nearring studied by the earlier authors. We prove that every zero-square nearring of type-1 is a zero-square nearring of type-2, but the converse need not be true, in general.

Definition 2.1 (i) A nearring $N$ is said to be a zero-square nearring of type-1 if $x^2 = 0$ for all $x \in N$, and there exists two elements $a, b \in N$ such that $ab \neq 0$.

(ii) A nearring $N$ is said to be a zero-square nearring of type-2 if $x^2 = 0$ for all $x \in N$.

Zero-square nearrings of type-2 are same as the zero-square nearrings studied by the earlier authors like Stanley. Every zero-square nearring of type-1 is a zero-square nearring of type-2.

Example 2.2 (i) Every null nearring (that is $N^2 = 0$) is a zero-square nearring of type-2, but not of type 1.

(ii) Let $(G, +)$ be a group (not necessarily Abelian). Define multiplicative operation on $G$ by $a \cdot b = 0$ for all $a, b \in G$, where 0 is additive identity. Then $(G, +, \cdot)$ is a null nearring. So $(G, +, \cdot)$ is a zero-square nearring of type-2, but not of type-1. Now we can conclude that every group can be made into a zero-square nearring of type-2.
(iii). Suppose that $N$ is a non zero Boolean nearring. Then $x^2 = x$ for all $x \in N$. So $N$ is a non-null nearring and for any $x \neq 0$, we have $x^2 \neq 0$. Hence every non-zero Boolean nearring can neither a zero-square nearring of type-1 nor a zero-square nearring of type-2.

(iv). Let $S$ be a non null nearring (that is, $S^2 \neq 0$). Write $N = S \times S \times S$. Define addition on $N$ component wise. Define multiplication on $N$ by $(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (0, 0, x_1y_2 - x_2y_1)$. Now it is clear that $N^2 \neq 0$ (that is, $N$ is not a null nearring) and $a^2 = 0$ for all $a \in N$. Hence $N$ is a zero-square nearring of type-1.

**Theorem 2.3** Suppose $N$ is a zero-square nearring of type-2. Then

(i) $aN \neq N$ for all $0 \neq a \in N$.

(ii) If $N$ is simple, then $N^2 = 0$.

**Proof**: (i) Let $N$ be a zero-square nearring, and $0 \neq a \in N$. Suppose $aN = N$. Then $a \in N = aN$ implies $a = ar$ for some $r \in N$. Now $a = ar = (ar)r = ar^2 = a0 = 0$ (since $N$ is zero symmetric nearring), a contradiction.

(ii) Suppose $N^2 \neq 0$. Then there exist $s, a \in N$ such that $as \neq 0$. Now $0 \neq as \in aN$. Since $N$ is simple and $aN \neq 0$, we have that $aN = N$, a contradiction. Hence $N^2 = 0$.

**Corollary 2.4** A primitive nearring cannot be a zero-square nearring of type-2.

**Proof**: Since $N$ is primitive by definition 4.2 of Pilz [2], it is faithful and simple. Let $0 \neq r \in N$. Since $N$ is faithful we have $Nr \neq 0$. Now $0 \neq Nr \subseteq NN$. This means $NN \neq 0$, a contradiction by Theorem 2.3 (ii).

**Corollary 2.5** Let $N$ be a zero-square nearring of type-2.

(i) If $I$ is a non zero left ideal of $N$, then $I$ can not be a monogenic left ideal; and

(ii) If $I$ is a non zero right ideal of $N$, then $I$ can not be a monogenic right ideal.

**Proof**: (i) In a contrary way, suppose that $I$ is a monogenic left ideal. Then there exist $0 \neq a \in I$ such that $Na = I$, a contradiction to the Theorem 2.3(i).

(ii) Similar to (i).

**Corollary 2.6** If $N$ is non-zero zero-square nearring of type-2, then

(i) $Nr \neq N$ for all $r \in N$; and

(ii) $rN \neq N$ for all $r \in N$.

**Proof**: The proof follows by taking $N$ instead of $I$ in Corollary 2.5.
3 Zero-square Ideals

In this section we define the substructure zero-square ideal of type-1 (respectively, type-2) of nearring and obtained related results.

**Definition 3.1** A proper ideal $I$ of $N$ is said to be a zero-square ideal of type-1 (respectively, type-2) if the quotient nearring $N/I$ is a zero-square nearring of type-1 (respectively, type-2).

**Remark 3.2** (i) If $N$ is a zero-square nearring of type-2, then every ideal $I$ of $N$ is a zero-square ideal of type-2. The converse of this statement is not true. For this observe the following example 3.3.

(ii) If $N$ is a zero-square nearring of type-2, then every ideal of $N$ is also a zero-square nearring of type-2.

**Example 3.3** Consider $Z_2$, the nearring of integers modulo 2. $Z_2$ is not a zero-square nearring of type-2. Let $G$ be a non-zero additive group and define $a \cdot b = 0$ for all $a, b \in G$. Now $(G, +)$ is a zero-square nearring of type-2. Write $N = Z_2 \oplus G$, the direct sum of nearrings $Z_2$ and $G$. Now $I = Z_2$ is an ideal of $N$; for any $x + I \in N/I$, we get that $(x + I)^2 = 0 + I$; and hence $I$ is a zero-square ideal of type-2. Since $1 = 1 + 0 \in Z_2 + G = N$ and $1^2 = 1 \neq 0$, it follows that $N$ is not a zero-square nearring of type-2.

**Example 3.4** Consider $N = Z_8$, the additive group of integers modulo 8. Let us define the multiplication on $N$ as it is given by the table. Observe that: $I_2 = 0, 4$ is an ideal of $N$. Now $N^2 = \{0, 4\} \subseteq I_2$; and $N/I_2 = \{0 + I_2, 1 + I_2, 2 + I_2, 3 + I_2\}$.

\[
\begin{array}{cccccccc}
\cdot & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
\end{array}
\]

Also observe that $(N/I_2)^2 = 0$, and so $N/I_2$ is not a zero-square nearring, where as $N$ is a zero-square nearring. Note that $I_2$ is not a zero-square ideal.

**Remark 3.5** If $I, J$ be two zero-square ideals of type-2, then $I \cap J$ is also a zero-square ideal of type-2.
Verification: Let $x \in N/(I \cap J)$. Now, $x + I \in N/I$. Since $I$ is zero-square ideal of type 2, we have $x^2 + I = 0 + I$ implies $x^2 \in I$. Similarly we can show that $x^2 \in J$ and so $x^2 \in I \cap J$. Now $x^2 + (I \cap J) = 0 + (I \cap J)$ implies $(x + (I \cap J))^2 = 0$ in $N/(I \cap J)$. Hence $N/(I \cap J)$ is a zero-square nearring of type-2. Therefore $I \cap J$ is a zero-square ideal of type-2.

**Definition 3.6** A class $B$ of nearrings is said to be homomorphically closed if every homomorphic image of $N$ is in $B$ for all $N$ in $B$.

**Theorem 3.7** The class $B$ of all zero-square nearrings of type-2 is homomorphically closed.

Proof: Let $N \in B$ and $I$ an ideal of $N$. Take $x + I \in N/I$. Now $(x + I)^2 = x^2 + I = 0 + I$ (since $N$ is a zero-square nearring of type-2). So $N/I$ is a zero-square nearring of type-2 and hence $N/I \in B$.

**Remark 3.8** Suppose $I$ is an ideal of $N$, $I$ is a zero-square ideal of type-2 and also a zero-square nearring of type-2, then $x^4 = 0$ for all $x \in N$.

Verification: $x \in N \Rightarrow x + I \in N/I$. Now $(x + I)^2 = 0 + I$ (since $I$ is a zero-square ideal of type-2). This means $x^2 \in I$, which implies that $(x^2)^2 = 0$ (since $I$ is a zero-square nearring of type-2). Therefore $x^4 = 0$.

**Theorem 3.9** Let $N$ be a zero-square nearring of type-2 and $I$ an ideal of $N$. Then the following two conditions are equivalent:

(i) $N^2 \not\subseteq I$; and (ii) $I$ is a zero-square ideal of type-1.

Proof: (i) $\Rightarrow$ (ii): By Remark 3.2, we get that $I$ is a zero-square ideal of type-2. Since $N^2 \not\subseteq I$, there exist $x, y \in N$ with $xy \not\in I$ and so $(x + I)(y + I) \neq 0 + I$ in $N/I$.

Therefore $N/I$ is a zero-square nearring of type-1 and so $I$ is a zero-square ideal of type-1.

(ii) $\Rightarrow$ (i): Since $N/I$ is a zero-square nearring of type-1, there exist two non-zero elements $c + I$ and $d + I$ in $N/I$ whose product is non-zero in $N/I$. This gives that $cd \notin I$ and so $N^2 \not\subseteq I$.

**Corollary 3.10** (i) Let $I$ and $J$ be ideals of a zero-square nearring $N$ of type-2 with $I \subseteq J$. If $J$ is a zero-square ideal of type-1, then $I$ is also a zero-square ideal of type-1.

(ii) Intersection of any collection of zero-square ideals of type-1 is also a zero-square ideal.

**Corollary 3.11** Let $N$ be the class of all zero-square nearrings $N$ of type-1 for which $N^2 \not\subseteq I$ for all non-zero ideals $I$ of $N$. Then the class $N$ is homomorphically closed.
Proof: Let \( N \in \mathbf{N} \) and \( h : N \rightarrow N^1 \) be an epimorphism. Then \( N/I \cong N^1 \), where \( I = \ker h \), an ideal of \( N \).

Case (i): Suppose \( h \) is an isomorphism. Then \( I = 0 \). Since \( N \) is a zero-square nearring of type-1, there exists \( x, y \in N \) such that \( xy \neq 0 \). So \( N^2 \neq 0 \) and \( N^2 \not\subseteq I \).

Case (ii): Suppose \( h \) is not an isomorphism. Then \( I \neq 0 \). From the assumption, \( N^2 \not\subseteq I \). Now by Theorem 3.8, \( I \) is a zero-square ideal of type-1 and hence \( N^1 \cong N/I \in \mathbf{N} \).

**Corollary 3.12** In a zero-square nearring \( N \) of type-2,
(i) every semi-prime ideal \( S \) of \( N \) is a zero-square ideal of type-1; and
(ii) every prime ideal \( P \) of \( N \) is a zero-square ideal of type-1.

Proof: (i) Suppose \( S \) is not a zero-square ideal of type-1. Then by Theorem 3.8 we get that \( N^2 \subseteq S \). Since \( S \) is semi prime ideal, we have that \( S = N \), a contradiction.

(ii) Follows from (i) since every prime ideal is a semi-prime ideal. This completes the proof.

We denote \( ZS1(N) = \) the intersection of all zero-square non-zero ideals of \( N \) of type-1; and \( ZS2(N) = \) the intersection of all zero-square non-zero ideals of \( N \) of type-2. If there are no non-zero zero-square ideals of type-1 (respectively, type-2) in \( N \) then we define \( ZS1(N) = N \) (respectively, \( ZS2(N) = N \)).

**Remark 3.13** If \( N \) is a zero-square nearring of type-2, then we have the following:
(i). By Theorem 3.8, we get that if \( N \) is a zero-square nearring of type-2, then \( ZS1(N) = \cap\{ I : I \text{ is a non-zero ideal of } N \text{ with } N^2 \not\subseteq I \} \);

(ii). If \( ZS2(N) = 0 \) (respectively, \( ZS1(N) = 0 \)); then by Theorem 1.2, it follows that \( N \) is a subdirect product of the zero-square nearrings \( N/I \), where \( I \) runs over all non-zero zero-square ideals of type-2 (respectively, type-1) in \( N \). If \( ZS2(N) \neq 0 \) (respectively, \( ZS1(N) \neq 0 \), then \( ZS2(N) \) (respectively, \( ZS1(N) \)) is the smallest non-zero zero-square ideal of type-2 (respectively, type-1), among all non-zero zero-square ideals of type-2 (respectively, type-1).

(iii). In Example 3.3, \( N = Z_2 \oplus G \) is not a zero-square nearring of type-2. In this case \( ZS2(N) = Z_2 \). Note that \( (0) \neq ZS2(N) \neq N \).

(iv). If \( N \) is a zero-square nearring of type-2 and \( N \) contains a zero-square ideal of type 1, then by Corollary 3.9, we get that \( ZS1(N) \subseteq I \).

(v). If \( N^2 = 0 \), then \( N \) contains no zero-square ideals of type-1 and so \( ZS1(N) = N \).
Theorem 3.14 If there exists a chain $N = I_0 \supset I_1 \supset I_2 \supset ... \supset I_k = (0)$ of ideals of $N$ such that $I_{s+1}$ is a zero-square ideal of type-2 in the nearring $I_s$, then $N$ is a nil ideal of $N$. In particular, $x^{(2^k)} = 0$ for all $x \in N$.

Proof: Let $x \in N = I_0$. Since $I_1$ is zero-square ideal of type-2 in the nearring $I_0$ and $x \in I_0$ we have that $(x + I_1)^2 = 0$ in $I_0/I_1$. So $x^2 \in I_1$. Since $x^2 \in I_1$ and $I_2$ is a zero-square ideal of type-2 in the nearring $I_1$, it follows that $(x^2 + I_2)^2 = 0$ in $I_1/I_2$ and so $x^4 \in I_2$. If we continue this process, eventually, we get that $x^{(2^k)} \in 0$. Thus $x^{(2^k)} = 0$ and this is true for all $x \in N$. Therefore $N$ is a nil ideal.

Corollary 3.15 Let $I_1, ..., I_k$ be as in the Theorem 3.14. For any ideal $I$ of $N$, $I$ and $N/I$ are also nil.

4 Direct Products of Zero-square Nearrings

We observe that the direct product of zero-square nearrings $N_i, 1 \leq i \leq k$ of type-1 is also a zero-square nearring of type-1, but the converse need not be true, in general. We also obtain some important consequences.

If $N_1, N_2, ..., N_k$ are nearrings, then the nearring $N_1 \times N_2 \times ... \times N_k$, the direct product of $N_i, 1 \leq i \leq k$ is denoted by $\prod N_i, 1 \leq i \leq k$. For any nearring $N$, let us write $N^k = \prod_k N$ for the direct product of $k$ copies of $N$.

A straightforward verification provides the following.

Theorem 4.1 (i). If $N_i, 1 \leq i \leq k$ are zero-square nearrings of type-1, then $\prod N_i, 1 \leq i \leq k$ is also a zero-square nearring of type-1;

(ii). Each $N_i, 1 \leq i \leq k$ are zero-square nearring of type-2 if and only if $\prod N_i, 1 \leq i \leq k$ is a zero-square nearring of type-2.

Remark 4.2 The converse of the above Lemma 4.1(i) is not true, in general. For this let us observe the following example.

Example 4.3 Write $(N, +) = Z_2$ additive group of integers modulo 2. Consider the zero product on $N$ (that is, $xy = 0$ for all $x, y \in N$). Then $N$ is nearring which is not a zero-square nearring of type-1. Let $M$ be a zero-square nearring of type-1. Consider the nearring $N \times M$ which is the direct product of $N$ and $M$. Now $N \times M$ is a zero-square nearring of type-1, where as $N$ is not a zero-square nearring of type-1.

Theorem 4.4 Let $N_i, 1 \leq i \leq k$ be nearrings. The direct product $\prod N_i, 1 \leq i \leq k$ is a zero-square nearrings of type-1 if and only if there exists a non-empty subset $I \subseteq \{1, 2, ..., k\}$ such that $N_i$ is a zero-square nearrings of type-1 for all $i \in I$ and $N_j$ is a zero-square nearrings of type-2 but not of type-1 for all $j \in \{1, 2, ..., k\} - I$. 
Now (0 \neq x, y) such that \( xy = 0 \) for all \( x, y \in N_j \). Hence \( N_j \) is not a zero-square nearring of type-1, for all \( j \in J \).

Converse: Since \( I \) is non-empty, there exists \( i \in I \) such that \( N_i \) is not a zero-square nearring of type-1. So there exist \( x_i, y_i \in N_i \) with \( x_i y_i \neq 0 \). Now \( (0, ..., x_i, ..., 0), (0, ..., y_i, ..., 0) \in \prod N_i, 1 \leq i \leq k \) and the product of these elements is non-zero. By Lemma 4.1, \( \prod N_i, 1 \leq i \leq k \) is a zero-square nearring of type-2. Hence it is a zero-square nearring of type-1.

**Corollary 4.5** For any positive integer \( k \), we have that \( N \) is a zero-square nearring of type-2 (respectively, type-1) if and only if \( N^k \) is a zero-square nearring of type-2 (respectively, type-1).

## 5 Zero-square Dimension

In this section, we introduce zero-square dimension of type-1/type-2. We consider a class of nearrings \( N \) and obtained some relations between the concepts dimension of \( N \), zero-square dimension of type-1/type-2. Finally, we apply this result for the direct sum of nearrings.

**Definition 5.1** Let \( N \) has FDI. We define the zero-square dimension of \( N \) (denoted by \( ZSd(N) \)) as follows:

\[
ZSd(N) = \{ s : \text{there exist uniform ideals } U_i, 1 \leq i \leq s \text{ in } N \text{ such that the sum } U_1 + U_2 + \ldots + U_s \text{ is direct and each } U_i \text{ is a zero-square nearring of type-2} \}.
\]

**Lemma 5.2** (i) If \( N \) has FDI, and \( N \) is a zero-square nearring of type-2, then \( ZSd(N) = \dim N \).

(ii). If \( N_i, 1 \leq i \leq n \) are nearrings with FDI and each \( N_i \) is a zero-square nearring of type-2, then \( ZSd(\prod_{i=1}^n N_i) = \sum_{i=1}^n ZSd(N_i) \).
Proof: (i) Suppose \( k = \dim N \). Since \( k = \dim N \), there exist uniform ideals \( U_1, U_2, \ldots, U_k \) in \( N \) such that \( U_1 \oplus U_2 \oplus \ldots \oplus U_k \leq e \). Since \( N \) is a zero-square nearring of type-2, by Remark 3.2 (ii), each \( U_i \) is also zero-square nearring of type-2. By definition 5.1, \( ZSd(N) = k \). Hence \( ZSd(N) = \dim N \).

(ii) By Theorem 4.1(ii), \( \prod N_i \) is also a zero-square nearring of type-2. Now \( ZSd(\prod N_i) = \dim(\prod N_i) \) (by (i)) = \( \sum_{i=1}^{n} \dim(N_i) \) (by Theorem 1.4) = \( \sum_{i=1}^{n} ZSd(N_i) \) (by (i)).

**Lemma 5.3** Suppose \( N \) has FDI and satisfies the condition \( <xy> = <x><y> \) for all \( x, y \in N \) with \( xy \neq 0 \). If \( N \) is zero-square nearring of type-1, then there exists a uniform ideal \( U \) in \( N \) such that \( U \) itself a zero-square nearring of type-1.

Proof: Since \( N \) has FDI, by Theorem 1.2, \( \dim N = k \), and there exist uniform ideals \( I_1, I_2, \ldots, I_k \) such that \( I_1 \oplus I_2 \oplus \ldots \oplus I_k \leq e \). Write \( E = I_1 \oplus I_2 \oplus \ldots \oplus I_k \). Since \( N \) is a zero-square nearring of type-1, there exist \( x, y \in N \) with \( xy \neq 0 \). Since \( 0 \neq xy \in <xy> \), and \( E \leq e \), we have that \( <xy> \cap E \neq 0 \). Now \( <x><y> \cap E \neq 0 \). This implies there exists \( x^1 \in <x>, y^1 \in <y> \) such that \( 0 \neq x^1y^1 \in E \). So \( E = I_1 \oplus I_2 \oplus \ldots \oplus I_k \) is a zero-square nearring of type-1. By Theorem 4.4, there exists \( t \in \{1, 2, \ldots, k\} \) such that \( I_t \) is a zero-square nearring of type-1.

**Definition 5.4** Let \( N \) has FDI and \( \dim N = k \). If \( N \) contains no uniform ideal which is a zero-square nearring of type-1, then we define the zero-square-1 dimension of \( N \) (denoted by \( ZS1d(N) \)) is equal to zero. We write \( ZS1d(N) = 0 \). If \( N \) contains a uniform ideal which is a zero-square nearring of type-1, then we define the zero-square-1 dimension of \( N \) as follows:

\[
ZS1d(N) = \max\{t : U_1, U_2, \ldots, U_t, U_{t+1}, \ldots, U_k \text{ are uniform ideals of } N, \text{ whose sum is direct and essential in } N \text{ (that is, } U_1 \oplus U_2 \oplus \ldots \oplus U_k \leq e \text{ )}, U_1, U_2, \ldots, U_t \text{ are zero-square nearrings of type-1, } U_{t+1}, \ldots, U_k \text{ are not zero-square nearrings of type-1}\}.
\]

**Note 5.5** (i) If \( N \) has FDI, \( N \) is a zero-square nearring of type-1 and satisfies the condition \( <xy> = <x><y> \) for all \( x, y \in N \) with \( xy \neq 0 \). By Lemma 5.3, there exist uniform ideals \( U_1, U_2, \ldots, U_k \) in \( N \) whose sum is direct and essential in \( N \). Also at least one of the \( U_i \)'s is a zero-square nearring of type-1. Thus, in this case, \( ZS1d(N) \geq 1 \).

(ii) If \( N \) is a zero-square nearring of type-2 but not of type-1, then there exist no uniform ideal in \( N \) which is a zero-square nearring of type-1. So in this case \( ZS1d(N) = 0 \).

**Theorem 5.6** If \( N_1, N_2 \) are nearrings with FDI and \( N = N_1 \oplus N_2 \), the direct sum of nearrings, then \( ZS1d(N_1 \oplus N_2) \geq ZS1d(N_1) + ZS1d(N_2) \).
Proof: Suppose $ZS1d(N_1) = n$ and $ZS1d(N_2) = m$. Then there exists uniform ideals $I_1, I_2, ..., I_k$ of $N_1$ such that $I_1 \oplus I_2 \oplus ... \oplus I_k \leq_e N_1$, where $I_i, 1 \leq i \leq n$ are zero-square nearrings of type-1. Similarly there exists uniform ideals $J_1, J_2, ..., J_s$ of $N_2$ such that $J_1 \oplus J_2 \oplus ... \oplus J_s \leq_e N_2$, $1 \leq i \leq m$ are zero-square nearrings of type-1. Since $N = N_1 \oplus N_2$, we have that the ideals of $N_1$ and the ideals of $N_2$ are also ideals of $N$. Now $I_1 \oplus I_2 \oplus ... \oplus I_n \oplus J_1 \oplus J_2 \oplus ... \oplus J_m$ is a sum of $(n + m)$ uniform ideals which are zero-square nearrings of type-1. So by the Definition 5.4, it follows that $ZS1d(N_1 \oplus N_2) = n + m = ZS1d(N_1) + ZS1d(N_2)$.

**Corollary 5.7** If $N_i, 1 \leq i \leq k$ are nearrings with FDI, then $ZS1d(N_1 \times N_2 \times ... \times N_k) \geq \Sigma_{i=1}^k ZS1d(N_i)$.

**Definition 5.8** Let $N$ be a nearring with FDI. We define $ZS2d(N)$, the zero-square-2 dimension of $N$ as follows:

$$ZS2d(N) = \min \{t : U_1, U_2, ..., U_k \text{ are uniform ideals of } N \text{ such that } U_1 \oplus U_2 \oplus ... \oplus U_k \leq_e N, U_1, U_2, ..., U_t \text{ are zero-square nearrings of type-2 but not of type-1}\}.$$ 

**Note 5.9** Suppose $N$ has FDI, $dim N = k$ and $N$ is a zero-square nearring of type-2 but not of type-1. Then by Note 5.5, $ZS1d(N) = 0$. Since every representation $E = U_1 \oplus U_2 \oplus ... \oplus U_k$ that is equal to a direct sum of uniform ideals with $E \leq_e N$, contains exactly $k$ uniform ideals, we have that $ZS2d(N) = k$. So in this case, $ZS1d(N) = 0$ and $ZS2d(N) = dim N$.

**Theorem 5.10** (i) If $N$ has FDI and $N$ is a zero-square nearring of type-1, then $dim(N) = ZSd(N) = ZS1d(N) + ZS2d(N)$.

(ii) If $N_i, 1 \leq i \leq k$ are nearrings with FDI, and also zero-square nearrings of type-1, then $dim(N_1 \times N_2 \times ... \times N_k) = ZSd(N_1 \times N_2 \times ... \times N_k) \geq \Sigma_{i=1}^k ZS1d(N_i) + \Sigma_{i=1}^k ZS2d(N_i)$.

Proof: (i) By Lemma 5.2(i), $dim(N) = ZSd(N)$. Suppose $dim(N) = k$ and $ZS1d(N) = n$. Then there exist uniform ideals $I_1, I_2, ..., I_k$ in $N$ such that $I_1 \oplus I_2 \oplus ... \oplus I_k \leq_e N$ and $I_i, 1 \leq i \leq n$ are zero-square nearrings of type-1, $n$ is maximum among such $n$. Also $I_{n+1}, ..., I_k$ are uniform ideals of $N(k - n$ in number) which are zero-square nearrings of type-2 (but not of type-1).

So $ZS2d(N) \leq k - n$. Suppose $m = ZS2d(N)$. Then there exist uniform ideals $U_1, U_2, ..., U_k$ in $N$ such that $U_1 \oplus U_2 \oplus ... \oplus U_k \leq_e N, U_i, 1 \leq i \leq m$ are zero-square-nearrings of type-2 (but not type-1) and $m$ is the minimum among these numbers. This means that the remaining $k - m$ uniform ideals $U_{m+1}, ..., U_k$ are zero-square nearrings of type-1 (we get this because of the hypothesis that $N$ is a zero-square nearring of type-2). By the Definition
5.4, we conclude that \( k - m \leq n \), which imply that \( m \geq k - n \). Hence\[ ZS2d(N) = m = k - n = \text{dim}N - ZS1d(N). \] Finally we get that \( \text{dim}N = ZSd(N) = ZS1d(N) + ZS2d(N) \).

Proof for (ii) follows by using (i) and mathematical induction.

**Corollary 5.11** (i) If \( N_1, N_2 \) are zero-square nearrings of type-2 with FDI, then \( ZS2d(N_1 \oplus N_2) \leq ZS2d(N_1) + ZS2d(N_2) \).

(ii) If \( N_i, 1 \leq i \leq k \) are zero-square nearrings with FDI, then \( ZS2d(N_1 \oplus N_2 \oplus ... \oplus N_k) \leq \sum_{i=1}^{k} ZS2d(N_i) \).

Proof: (i) \( ZS1d(N_1 \oplus N_2) + ZS2d(N_1 \oplus N_2) = ZSd(N_1 \oplus N_2) \) (by Theorem 5.10) = \( ZSd(N_1) + ZSd(N_2) \) (by Lemma 5.2(ii)) = \( ZS1d(N_1) + ZS2d(N_1) + ZS1d(N_2) + ZS2d(N_2) \) (by Theorem 5.10) \( \leq ZS1d(N_1 \oplus N_2) + ZS2d(N_1) + ZS2d(N_2) \). Therefore \( ZS2d(N_1 \oplus N_2) \leq ZS2d(N_1) + ZS2d(N_2) \).

Proof for (ii) follows by using (i) and mathematical induction.

**An Application**: We give an application to graph theory by considering a graph of \( N \) denoted by \( G_N = (V, E) \) defined as \( V = N \) and \( E = \{ab : a, b \in N, a \neq b, a \cdot b = 0\} \). For instance, consider \( S = \{0, 1\} \), the nearring of integers modulo 2. Write \( N = S \times S \times S \). Define addition and multiplication defined as in example 2.2. Note that a graph is said to be complete if there exists exactly one edge between any two vertices. Take the set of vertices \( V = \{A = (0, 0, 0), B = (0, 0, 1), C = (0, 1, 0), D = (0, 1, 1), E = (1, 0, 0), F = (1, 0, 1), G = (1, 1, 0), H = (1, 1, 1)\} \). It is clear that there are no edges between \( F \) and \( G \); \( E \) and \( D \); \( D \) and \( H \). Therefore it is not a complete graph. Further, since \( F \neq 0 \) and \( G \neq 0 \), there is no edge between \( F \) and \( G \). Similarly, there are no edges between \( E \) and \( D \); \( D \) and \( H \) etc. This means that the graph cannot be a complete graph. In general, a zero-square nearring \( N \) of type 2 is a zero-square nearring of type 1 if and only if its related graph is not complete.

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**References**


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