International Journal of Algebra, Vol. 4, 2010, no. 15, 735 - 741

A Note on the Paper of Wang and Li[1]

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Abstract. The main purpose of this short note is to make (Theorem 2.8, [3]) be a necessary and sufficient condition. Moreover, some applications of the minimal Horseshoe Lemma are also given.

Mathematics Subject classification: Primary 16S37, 16W50; Secondary 16E30, 16E40

Keywords: nice modules, quasi-Koszul modules, minimal Horseshoe Lemma

1. Introduction

In 2008, Wang and Li gave some sufficient conditions for the Horseshoe Lemma to be true in the minimal case in [3]. In particular, (Theorem 2.8, [3]) is the main result of that paper:

- Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of nice modules with $JK = K \cap JM$. Then the minimal Horseshoe Lemma holds with respect to such an exact sequence.

The main purpose of this short note is to make the above result be a necessary and sufficient condition. That is, we prove the following:

[1]This work was supported by Zhejiang Innovation Project (Grant No. T200905).

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Theorem 1.1. *Let* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence of nice modules. Then* $JK = K \cap JM$ *if and only if the minimal Horseshoe Lemma holds with respect to such an exact sequence.*

Further, we also give some applications of the minimal Horseshoe Lemma. More precisely, in 1996, Green and Martínez-Villa introduced the notions of quasi-Koszul algebra and quasi-Koszul module in [1] as a nongraded version of Koszul algebras and Koszul modules. In particular, they gave conditions such that the category of quasi-Koszul modules is closed under extensions (Proposition 5.3 (a), [1]). However, they didn't give any conditions such that the category of quasi-Koszul modules preserves kernels of epimorphisms and preserves cokernels of monomorphisms. Now as an application of minimal Horseshoe Lemma, we will obtain some sufficient conditions. In particular, we obtain

Theorem 1.2. *Let* $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be a short exact sequence of finitely generated* R*-modules, where* R *is a Noetherian semiperfect algebra with Jacobson radical* J*. Then we have the following statements:*

(1) If minimal Horseshoe Lemma is true with respect to ξ*, then* K *is quasi-Koszul provided that* M *and* N *are quasi-Koszul.*

(2) If minimal Horseshoe Lemma is true with respect to ξ *and for all* $i \geq 0$*,* $J^2\Omega^i(K) = \Omega^i(K) \cap J^2\Omega^i(M)$*, then* N *is quasi-Koszul provided that* K *and* M *are quasi-Koszul.*

2. Proofs of the Main Results

In this section, a standard graded algebra means a positively graded $\Bbbk$-algebra $A = \bigoplus_{i \geq 0} A_i$ such that (a) $A_0 = \Bbbk \times \cdots \times \Bbbk$, a finite product of $\Bbbk$; (b) $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$; and (c) $\dim_{\Bbbk} A_i < \infty$ for all $i \geq 0$, where $\Bbbk$ denotes an arbitrary ground field. Clearly, the graded Jacobson radical J of a standard graded algebra A is obvious $\bigoplus_{i \geq 1} A_i$.

Definition 2.1. [3] Let A be a standard graded algebra with Jacobson radical J and M a finitely generated A-module. Let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

a minimal projective resolution of M. Then we call M a *nice module* if and only if for all $n \geq 0$, we have $\ker d_n \subseteq JP_n$ and $J \ker d_n = \ker d_n \cap J^2 P_n$.

In particular, A is called a *nice algebra* if and only if A_0 is a nice module.

It is easy to see that quasi-Koszul modules (see [1] for the details) are nice modules.

Lemma 2.2. *Let* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence of finitely generated graded* A*-modules. Then* $JK = K \cap JM$ *if and only if we have the following commutative diagram with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^1(K) \arrow[r] \arrow[d] & \Omega^1(M) \arrow[r] \arrow[d] & \Omega^1(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & L_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0, \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $P_0 \to K \to 0$, $Q_0 \to M \to 0$ *and* $L_0 \to N \to 0$ *are projective covers.*

Proof. ($\Rightarrow$) Clearly, we obtain the exact sequence $0 \to K/JK \to M/JM \to N/JN \to 0$. Note that for any finitely generated A-module M, $A \otimes_{A_0} M/JM \longrightarrow M \longrightarrow 0$ is a projective cover. Now setting $P_0 := A \otimes_{A_0} K/JK$, $Q_0 := A \otimes_{A_0} M/JM$ and $L_0 := A \otimes_{A_0} N/JN$. We have the following exact sequence $0 \longrightarrow P_0 \longrightarrow Q_0 \longrightarrow L_0 \longrightarrow 0$ since A_0 is semisimple. Therefore, we have the desired diagram.

($\Leftarrow$) Suppose that we have the above commutative diagram. Note that the projective cover of a module is unique up to isomorphisms. We may assume that $P_0 := A \otimes_{A_0} K/JK$, $Q_0 := A \otimes_{A_0} M/JM$ and $L_0 := A \otimes_{A_0} N/JN$. From the middle row of the diagram, we have the following exact sequence

$$0 \longrightarrow A \otimes_{A_0} K/JK \longrightarrow A \otimes_{A_0} M/JM \longrightarrow A \otimes_{A_0} N/JN \longrightarrow 0.$$

Thus, we have the following short exact sequence as A_0-modules

$$0 \longrightarrow K/JK \longrightarrow M/JM \longrightarrow N/JN \longrightarrow 0$$

since A_0 is semisimple, which implies $JK = K \cap JM$. □

Lemma 2.3. *Let* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be a short exact sequence of finitely generated graded* A*-modules. Then* $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ *for all* $i \geq 0$ *if and only if the minimal Horseshoe Lemma holds.*

Proof. By Lemma 2.2, $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$ if and only if, for all $i \geq 0$, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{i+1}(K) & \longrightarrow & \Omega^{i+1}(M) & \longrightarrow & \Omega^{i+1}(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_i & \longrightarrow & Q_i & \longrightarrow & L_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^i(K) & \longrightarrow & \Omega^i(M) & \longrightarrow & \Omega^i(N) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where P_i, Q_i and L_i are projective covers of $\Omega^i(K)$, $\Omega^i(M)$ and $\Omega^i(N)$, respectively. Now putting this commutative diagrams together, we finish the proof. $\square$

Now we can prove Theorem 1.1.

Proof. ($\Rightarrow$) It has been proved in [3].

($\Leftarrow$) By Lemma 2.3, the minimal Horseshoe Lemma is true if and only if $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$. In particular, we have $JK = K \cap JM$. $\square$

Remark 2.4. If we replace the standard graded algebra by Noetherian semiperfect algebra with Jacobson radical J, all the results and proofs above are valid.

Now it is the time to prove Theorem 1.2.

Proof. (1) By assumption, we have the following commutative diagram with exact rows and columns for all $i \geq 0$,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^{i+1}(K) \arrow[r] \arrow[d] & \Omega^{i+1}(M) \arrow[r] \arrow[d] & \Omega^{i+1}(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_i \arrow[r] \arrow[d] & Q_i \arrow[r] \arrow[d] & L_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^{i}(K) \arrow[r] \arrow[d] & \Omega^{i}(M) \arrow[r] \arrow[d] & \Omega^{i}(N) \arrow[r] \arrow[d] & 0, \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that $P_i \to \Omega^i(K) \to 0$, $Q_i \to \Omega^i(M) \to 0$ and $L_i \to \Omega^i(N) \to 0$ are projective covers, respectively. Thus we have the following commutative diagram with exact rows and columns for all $i \geq 0$,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^{i+1}(K) \arrow[r] \arrow[d] & \Omega^{i+1}(M) \arrow[r] \arrow[d] & \Omega^{i+1}(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & JP_i \arrow[r] & JQ_i \arrow[r] & JL_i \arrow[r] & 0.
\end{tikzcd}$$

Applying the functor $R/J \otimes_R -$ to the above diagram, note that M and N are quasi-Koszul modules, then we have the following commutative diagram with exact rows and columns for all $i \geq 0$,

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & R/J \otimes_R \Omega^{i+1}(K) \arrow[r] \arrow[d, "\theta"'] & R/J \otimes_R \Omega^{i+1}(M) \arrow[r] \arrow[d] & R/J \otimes_R \Omega^{i+1}(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R/J \otimes_R JP_i \arrow[r] & R/J \otimes_R JQ_i \arrow[r] & R/J \otimes_R JL_i \arrow[r] & 0.
\end{tikzcd}$$

Therefore, θ is a monomorphism, which implies that $J\Omega^{i+1}(K) = \Omega^{i+1}(K) \cap J^2P_i$ for all $i \geq 0$. Thus K is a quasi-Koszul module, as desired.

(2) Similar to the proof of (2), we have the following commutative diagram with exact rows and columns for all $i \geq 0$,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^{i+1}(K) \arrow[r] \arrow[d] & \Omega^{i+1}(M) \arrow[r] \arrow[d] & \Omega^{i+1}(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & JP_i \arrow[r] \arrow[d] & JQ_i \arrow[r] \arrow[d] & JL_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J\Omega^i(K) \arrow[r] \arrow[d] & J\Omega^i(M) \arrow[r] \arrow[d] & J\Omega^i(N) \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note that

$$\begin{aligned}
J\Omega^i(K) \cap J(J\Omega^i(M)) &= J\Omega^i(K) \cap J^2\Omega^i(M) \\
&= J\Omega^i(K) \cap J^2\Omega^i(M) \cap \Omega^i(K) \\
&= J\Omega^i(K) \cap J^2\Omega^i(K) \\
&= J\Omega^i(K).
\end{aligned}$$

Thus, we have the following commutative diagram with exact rows and columns for all $i \geq 0$,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R/J \otimes_R \Omega^{i+1}(K) \arrow[r] \arrow[d] & R/J \otimes_R \Omega^{i+1}(M) \arrow[r] \arrow[d] & R/J \otimes_R \Omega^{i+1}(N) \arrow[r] \arrow[d, "\zeta"'] & 0 \\
0 \arrow[r] & R/J \otimes_R JP_i \arrow[r] \arrow[d] & R/J \otimes_R JQ_i \arrow[r] \arrow[d] & R/J \otimes_R JL_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R/J \otimes_R J\Omega^i(K) \arrow[r] \arrow[d] & R/J \otimes_R J\Omega^i(M) \arrow[r] \arrow[d] & R/J \otimes_R J\Omega^i(N) \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Thus we have $J\Omega^{i+1}(N) = \Omega^{i+1}(N) \cap J^2L_i$ for all $i \geq 0$ since ζ is a monomorphism. Thus N is a quasi-Koszul module, as desired. $\square$

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Received: March, 2010