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Regular Elements of Some Order-Preserving Transformation Semigroups

[1]W. Mora and [2]Y. Kemprasit

Department of Mathematics, Faculty of Science
Chulalongkorn University, Bangkok 10330, Thailand
[1]winita.m@student.chula.ac.th, [2]yupaporn.k@chula.ac.th

Abstract

Let X be a chain and $OT(X)$ the full order-preserving transformation semigroup on X. In this paper, we give a necessary and sufficient condition for an element of $OT(X)$ to be regular. For $\emptyset \neq Y \subseteq X$, we may count the order-preserving transformation semigroup $OT(X,Y) = \{\alpha \in OT(X) \mid \operatorname{ran}\alpha \subseteq Y\}$ as a generalization of $OT(X)$. In addition, we show that an element $\alpha \in OT(X,Y)$ is regular in $OT(X,Y)$ if and only if $\operatorname{ran}\alpha = Y\alpha$ and α is regular in $OT(X)$.

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1 Introduction and Preliminaries

An element a of a semigroup S is called *regular* if $a = axa$ for some $x \in S$, and S is called a *regular semigroup* if every element of S is regular. Let $\operatorname{Reg}(S)$ be the set of all regular elements of S.

The image of x in the domain of a mapping α under α is written as $x\alpha$ and the range (image) of α is denoted by $\operatorname{ran}\alpha$.

For a nonempty set X, let $T(X)$ be the full transformation semigroup on X, i.e., $T(X)$ is the semigroup under composition of all mappings $\alpha : X \to X$. It is well-known that $T(X)$ is a regular semigroup ([1, p.4] and [2, p.63]) and every semigroup can be embedded in $T(X)$ for some nonempty set X([1, p.3] and [2, p.7]).

A mapping φ from a poset X into a poset Y is said to be *order-preserving* if for all $x, x' \in X$, $x \leq x'$ in X implies $x\varphi \leq x'\varphi$ in Y. The posets X and Y

are said to be *order-isomorphic* if there is an order-preserving bijection φ from X onto Y such that $\varphi^{-1} : Y \to X$ is order-preserving.

For a poset X, let $OT(X)$ be the subsemigroup of $T(X)$ consisting of all order-preserving mappings $\alpha : X \to X$. It is known from [1, p.203] that $OT(X)$ is a regular semigroup if X is a finite chain. In 2000, Kemprasit and Changphas [4] extended this result to any chain order-isomorphic to a subset of $\mathbb{Z}$, the set of integers with their natural order.

Theorem 1.1. [4] *If X is any chain which is order-isomorphic to a subset of $\mathbb{Z}$ with usual order, then $OT(X)$ is a regular semigroup.*

Note that a chain X in Theorem 1.1 is a countable chain. In fact, Kim and Kozhukhov [6] characterized a countable chain X such that $OT(X)$ is a regular semigroup. We have that Theorem 1.1 becomes a consequence of their characterization.

In [4], the authors also considered the regularity of $OT(X)$ where X is an interval X in $\mathbb{R}$, the set of real numbers with their natural order as follows:

Theorem 1.2. [4] *For an interval X in $\mathbb{R}$ with usual order, $OT(X)$ is a regular semigroup if and only if X is closed and bounded.*

Rungrattrakoon and Kemprasit [8] extended Theorem 1.2 by considering intervals in any proper subfield of $\mathbb{R}$ as follows:

Theorem 1.3. [8] *For a nontrivial interval X in a proper subfield F of $\mathbb{R}$, $OT(X)$ is not a regular semigroup.*

In fact, Theorem 1.3 is a consequence of a main theorem in [3]. As a particular case of Theorem 1.3, we have that $OT(\mathbb{Q})$ is not a regular semigroup where $\mathbb{Q}$ is the set of rational numbers with their natural order. This result may be considered as a consequence of a lemma in [6] which states that for a countable chain X having no maximum and minimum, $OT(X)$ is regular if and only if X is order-isomorphic to $\mathbb{Z}$.

In [5], the authors generalized the semigroup $OT(X)$ by using sandwich multiplication and then the regularity was investigated.

The above theorems motivate us to characterize the regular elements of $OT(X)$ when X is a chain. Then these known results become its consequences.

In 1975, Symons [9] considered the subsemigroup $T(X,Y)$ of $T(X)$ where Y is a nonempty subset of a nonempty set X and $T(X,Y) = \{\alpha \in T(X) \mid \operatorname{ran}\alpha \subseteq Y\}$. He studied the automorphisms of $T(X,Y)$ and considered when two $T(X,Y)$ are isomorphic. Since $T(X,X) = T(X)$, we may count $T(X,Y)$ as a generalization of $T(X)$. However, $T(X,Y)$ may not be regular. In [7], the authors characterized the regular elements of $T(X,Y)$ as follows:

Theorem 1.4. [7] *Let Y be a nonempty subset of a set X. For $\alpha \in T(X,Y)$, $\alpha \in \mathrm{Reg}(T(X,Y))$ if and only if* $\mathrm{ran}\,\alpha = Y\alpha$.

We define $OT(X,Y)$ analogously where Y is a nonempty subset of a poset X, i.e., $OT(X,Y) = \{\alpha \in OT(X) \mid \mathrm{ran}\,\alpha \subseteq Y\}$. Then $OT(X,Y)$ may be also considered as a generalization of $OT(X)$. Notice that $OT(X,Y)$ is a subsemigroup of both $T(X,Y)$ and $OT(X)$. We show in this paper that for a chain X and $\emptyset \neq Y \subseteq X, \mathrm{Reg}(OT(X,Y)) = \mathrm{Reg}(T(X,Y)) \cap \mathrm{Reg}(OT(X))$ and determine when $OT(X,Y)$ is a regular semigroup.

For a nonempty subset A of a chain X, we let $\max(A)$ and $\min(A)$ denote the maximum and the minimum of A, respectively if they exist. Also, for nonempty subsets A and B of X, let $A < B$ mean that $a < b$ for all $a \in A$ and $b \in B$. For $x \in X$, let $x < A$ stand for $\{x\} < A$. We define $A > B, A \leq B, A \geq B, x > A, x \leq A$ and $x \geq A$ analogously. Notice that x is an upper bound (u.b.) of A in X if and only if $x \geq A$, and x is a lower bound (l.b.) of A in X if and only if $x \leq A$.

The cardinality of a set S is denoted by $|S|$.

2 Regular Element of $OT(X)$

To characterize the regular elements of the semigroup $OT(X)$ where X is a chain, the following series of lemma is needed. The first lemma is evident.

Lemma 2.1. *Let X be a chain. If $\alpha \in OT(X)$ and $a, b \in \mathrm{ran}\,\alpha$ satisfy $a < b$, then $a\alpha^{-1} < b\alpha^{-1}$.*

Lemma 2.2. *Let X be a chain and $\alpha \in \mathrm{Reg}(OT(X))$.*

(i) *If $\mathrm{ran}\,\alpha$ has an u.b. in X, then $\max(\mathrm{ran}\,\alpha)$ exists.*

(ii) *If $\mathrm{ran}\,\alpha$ has a l.b. in X, then $\min(\mathrm{ran}\,\alpha)$ exists.*

Proof. (i) Let $\beta \in OT(X)$ and $u \in X$ be such that $\alpha = \alpha\beta\alpha$ and $u \geq \mathrm{ran}\,\alpha$. Then $\mathrm{ran}\,\alpha = X\alpha = X\alpha\beta\alpha = (\mathrm{ran}\,\alpha)\beta\alpha \leq u\beta\alpha \in \mathrm{ran}\,\alpha$. It follows that $u\beta\alpha = \max(\mathrm{ran}\,\alpha)$.

(ii) can be proved similarly. □

Lemma 2.3. *Let X be a chain and $\alpha \in \mathrm{Reg}(OT(X))$. If $x \in X \setminus \mathrm{ran}\,\alpha$ is neither an u.b. nor a l.b. of $\mathrm{ran}\,\alpha$, then $\max(\{t \in \mathrm{ran}\,\alpha \mid t < x\})$ or $\min(\{t \in \mathrm{ran}\,\alpha \mid t > x\})$ exists.*

Proof. Let $\beta \in OT(X)$ be such that $\alpha = \alpha\beta\alpha$. We have from the assumption that both $\{t \in \mathrm{ran}\,\alpha \mid t < x\}$ and $\{t \in \mathrm{ran}\,\alpha \mid t > x\}$ are nonempty sets and $\mathrm{ran}\,\alpha$ is a disjoint union of these two sets. Since $x\beta\alpha \in \mathrm{ran}\,\alpha$, it follows that

$x\beta\alpha < x$ or $x\beta\alpha > x$. For $t \in X$, if $t\alpha < x$, then $t\alpha = (t\alpha)\beta\alpha \leq x\beta\alpha$. If $t\alpha > x$, then $t\alpha = (t\alpha)\beta\alpha \geq x\beta\alpha$. This shows that

$$x\beta\alpha = \begin{cases} \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) & \text{if } x\beta\alpha < x, \\ \min(\{t \in \operatorname{ran}\alpha \mid t > x\}) & \text{if } x\beta\alpha > x, \end{cases}$$

so the desired result follows. □

Theorem 2.4. *Let X be a chain and $\alpha \in OT(X)$. Then $\alpha \in \operatorname{Reg}(OT(X))$ if and only if the following three conditions hold.*

(i) *If* $\operatorname{ran}\alpha$ *has an u.b. in X, then* $\max(\operatorname{ran}\alpha)$ *exists.*
(ii) *If* $\operatorname{ran}\alpha$ *has a l.b. in X, then* $\min(\operatorname{ran}\alpha)$ *exists.*
(iii) *If $x \in X \smallsetminus \operatorname{ran}\alpha$ is neither an u.b. nor a l.b. of* $\operatorname{ran}\alpha$*, then* $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ *or* $\min(\{t \in \operatorname{ran}\alpha \mid t > x\})$ *exists.*

Proof. If $\alpha \in \operatorname{Reg}(OT(X))$, then from Lemma 2.2(i), Lemma 2.2(ii) and Lemma 2.3, (i), (ii) and (iii) hold, respectively.

For the converse, assume that (i), (ii) and (iii) hold. If $\operatorname{ran}\alpha$ has an u.b. in X, let $u = \max(\operatorname{ran}\alpha)$. If $\operatorname{ran}\alpha$ has a l.b. in X, let $l = \min(\operatorname{ran}\alpha)$. If $x \in X \smallsetminus \operatorname{ran}\alpha$ is neither an u.b. nor a l.b. of $\operatorname{ran}\alpha$, let

$$m_x = \begin{cases} \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) & \text{if } \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) \text{ exists}, \\ \min(\{t \in \operatorname{ran}\alpha \mid t > x\}) & \text{if } \max(\{t \in \operatorname{ran}\alpha \mid t < x\}) \text{ does not exist} \\ & \quad \text{and} \min(\{t \in \operatorname{ran}\alpha \mid t > x\}) \text{ exists}. \end{cases}$$

For each $x \in \operatorname{ran}\alpha$, choose $x' \in x\alpha^{-1}$. Then $x'\alpha = x$ for all $x \in \operatorname{ran}\alpha$. Thus $(x\alpha)'\alpha = x\alpha$ for all $x \in X$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x' & \text{if } x \in \operatorname{ran}\alpha, \\ u' & \text{if } x \geq \operatorname{ran}\alpha, \\ l' & \text{if } x \leq \operatorname{ran}\alpha, \\ {m_x}' & \text{if } x \in X \smallsetminus \operatorname{ran}\alpha \text{ and } x \text{ is neither an u.b. nor} \\ & \quad \text{a l.b. of } \operatorname{ran}\alpha. \end{cases}$$

Then for every $x \in X$, $x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)'\alpha = x\alpha$. Hence $\alpha = \alpha\beta\alpha$. It remains to show that β is order-preserving. Let $x, y \in X$ be such that $x < y$. We can see from Lemma 2.1 that $u' = \max(\operatorname{ran}\beta)$ if $\operatorname{ran}\alpha$ has an u.b. in X and $l' = \min(\operatorname{ran}\beta)$ if $\operatorname{ran}\alpha$ has a l.b. in X. It follows that if $y \geq \operatorname{ran}\alpha$ or $x \leq \operatorname{ran}\alpha$, then $x\beta \leq y\beta$. Also, by Lemma 2.1, we have that if $x, y \in \operatorname{ran}\alpha$, then $x\beta = x' < y' = y\beta$. Therefore there are three cases to clarify as follows:

Case 1: $x \in \operatorname{ran}\alpha, y \in X \smallsetminus \operatorname{ran}\alpha$ and $y \ngeq \operatorname{ran}\alpha$. Since $x < y$, we have $y \nleq \operatorname{ran}\alpha$. If $m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\})$, then $x \leq m_y$, so by Lemma 2.1, $x\beta = x^{'} \leq {m_y}^{'} = y\beta$. If $m_y = \min(\{t \in \operatorname{ran}\alpha \mid t > y\})$, then $x < y < m_y$, so by Lemma 2.1, $x\beta = x^{'} < {m_y}^{'} = y\beta$.

Case 2 : $x \in X \smallsetminus \operatorname{ran}\alpha$, $x \nleq \operatorname{ran}\alpha$ and $y \in \operatorname{ran}\alpha$. Since $x < y$, we have $x \ngeq \operatorname{ran}\alpha$. If $m_x = \max(\{t \in \operatorname{ran}\alpha \mid t < x\})$, then $m_x < x < y$, so $x\beta = {m_x}^{'} < y^{'} = y\beta$. If $m_x = \min(\{t \in \operatorname{ran}\alpha \mid t > x\})$, then $m_x \leq y$ and hence $x\beta = {m_x}^{'} \leq y^{'} = y\beta$.

Case 3: $x, y \in X \smallsetminus \operatorname{ran}\alpha, x \nleq \operatorname{ran}\alpha$ and $y \ngeq \operatorname{ran}\alpha$. Since $x < y$, it follows that $x \ngeq \operatorname{ran}\alpha$ and $y \nleq \operatorname{ran}\alpha$.

Subcase 3.1 : $m_x = \max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ and $m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\})$. Since $\{t \in \operatorname{ran}\alpha \mid t < x\} \subseteq \{t \in \operatorname{ran}\alpha \mid t < y\}$, we have $m_x \leq m_y$, so $x\beta = {m_x}^{'} \leq {m_y}^{'} = y\beta$.

Subcase 3.2 : $m_x = \max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ and $m_y = \min(\{t \in \operatorname{ran}\alpha \mid t > y\})$. Then $m_x < x < y < m_y$, thus $x\beta < y\beta$.

Subcase 3.3 : $m_x = \min(\{t \in \operatorname{ran}\alpha \mid t > x\})$ and $m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\})$. Then $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ does not exist. It follows that $\{t \in \operatorname{ran}\alpha \mid t < x\} \subsetneq \{t \in \operatorname{ran}\alpha \mid t < y\}$. Hence $x < c < y$ for some $c \in \operatorname{ran}\alpha$. This implies that $m_x \leq c \leq m_y$ and thus $x\beta \leq y\beta$.

Subcase 3.4 : $m_x = \min(\{t \in \operatorname{ran}\alpha \mid t > x\})$ and $m_y = \min(\{t \in \operatorname{ran}\alpha \mid t > y\})$. Since $\{t \in \operatorname{ran}\alpha \mid t > x\} \supseteq \{t \in \operatorname{ran}\alpha \mid t > y\}$, we have $m_x \leq m_y$. Hence $x\beta \leq y\beta$.

The proof is thereby completed. □

By the property of $\mathbb{Z}$ with usual order, Theorem 1.1 is clearly obtained from Theorem 2.4

Corollary 2.5. *If X is any chain which is order-isomorphic to a subset of $\mathbb{Z}$ with usual order, then $OT(X)$ is a regular semigroup.*

Let X be an interval in $\mathbb{R}$ and $\alpha \in OT(X)$. If $x \in X \smallsetminus \operatorname{ran}\alpha$ is neither an u.b nor a l.b. of $\operatorname{ran}\alpha$ in X, then

$$X = \{t \in \operatorname{ran}\alpha \mid t < x\}\alpha^{-1} \,\dot{\cup}\, \{t \in \operatorname{ran}\alpha \mid t > x\}\alpha^{-1}.$$

where $\dot{\cup}$ means a disjoint union and by Lemma 2.1, $\{t \in \operatorname{ran}\alpha \mid t < x\}\alpha^{-1} < \{t \in \operatorname{ran}\alpha \mid t > x\}\alpha^{-1}$. Since X is an interval in $\mathbb{R}$, it follows that

$$\sup(\{t \in \operatorname{ran}\alpha \mid t < x\}\alpha^{-1}) = \inf(\{t \in \operatorname{ran}\alpha \mid t > x\}\alpha^{-1}), \text{ say e.}$$

Then either $e = \max(\{t \in \operatorname{ran}\alpha \mid t < x\}\alpha^{-1})$ or $e = \min(\{t \in \operatorname{ran}\alpha \mid t > x\}\alpha^{-1})$. Since α is order-preserving, it follows that $e\alpha = \max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ for the first case and $e\alpha = \min(\{t \in \operatorname{ran}\alpha \mid t > x\})$ for the second case.

Hence we have

Corollary 2.6. *Let X be an interval in $\mathbb{R}$ and $\alpha \in OT(X)$. Then $\alpha \in \operatorname{Reg}(OT(X))$ if and only if the following two conditions hold.*

(i) *If $\operatorname{ran}\alpha$ has an u.b. in X, then $\max(\operatorname{ran}\alpha)$ exists.*
(ii) *If $\operatorname{ran}\alpha$ has a l.b. in X, then $\min(\operatorname{ran}\alpha)$ exists.*

We can give a simple proof of Theorem 1.2 by making use of Corllary 2.6.

Corollary 2.7. *Let X be an interval in $\mathbb{R}$. Then $OT(X)$ is a regular semigroup if and only if X is closed and bounded.*

Proof. Let X be closed and bounded. Then $X = [a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$. Since α is order-preserving, $a\alpha = \min(\operatorname{ran}\alpha)$ and $b\alpha = \max(\operatorname{ran}\alpha)$. By Corollary 2.6, $OT(X)$ is a regular semigroup.

For the converse, assume that X is not closed or X is unbounded. Then X is one of the forms:

$$\begin{array}{ll} \mathbb{R}, [a,\infty), (a,\infty), (-\infty,a], (-\infty,a) & \text{where } a \in \mathbb{R}, \\ [a,b), (a,b], (a,b) & \text{where } a, b \in \mathbb{R} \text{ with } a < b. \end{array}$$

Case 1: $X = \mathbb{R}$, $[a, \infty)$ or (a, ∞). Let $c \in X$ and define $\alpha : X \to \mathbb{R}$ by

$$x\alpha = \begin{cases} c + \dfrac{x-c}{x-c+1} & \text{if } x \geq c, \\ c & \text{if } x < c. \end{cases}$$

Since the derivative of α at $x > c$ is $\dfrac{1}{(x-c+1)^2} > 0$ and $\operatorname{ran}\alpha = [c, c+1) \subseteq X$, it follows that $\alpha \in OT(X)$. By Corollary 2.6(i), $\alpha \notin \operatorname{Reg}(OT(X))$.

Case 2: $X = (-\infty, a]$ or $(-\infty, a)$. Let $d \in X$ and define $\beta : X \to \mathbb{R}$ by

$$x\beta = \begin{cases} d - \dfrac{x-d}{x-d-1} & \text{if } x \leq d, \\ d & \text{if } x > d. \end{cases}$$

Then the derivative of β at $x < d$ is $\dfrac{1}{(x-d-1)^2} > 0$ and $\operatorname{ran}\beta = (d-1, d] \subseteq X$. It follows that $\beta \in OT(X)$ and by Corollary 2.6(ii), $\beta \notin \operatorname{Reg}(OT(X))$.

Case 3: $X = [a,b)$, $(a,b]$ or (a,b). Define $\gamma : X \to \mathbb{R}$ by

$$x\gamma = \frac{1}{4}(x-a) + \frac{a+b}{2} \quad \text{for all } x \in X.$$

Then the derivative of γ at $x \in X$ is $\frac{1}{4}$, $a < \frac{a+b}{2} < \frac{a+3b}{4} < b$ and

$$\operatorname{ran}\gamma = \begin{cases} [\frac{a+b}{2}, \frac{a+3b}{4}) & \text{if } X = [a,b), \\ (\frac{a+b}{2}, \frac{a+3b}{4}] & \text{if } X = (a,b], \\ (\frac{a+b}{2}, \frac{a+3b}{4}) & \text{if } X = (a,b). \end{cases}$$

Then $\gamma \in OT(X)$ and by Corollary 2.6, $\gamma \notin \operatorname{Reg}(OT(X))$.

The proof is thereby completed. □

Next, we shall prove Theorem 1.3 as a consequence of Theorem 2.4.

Corollary 2.8. *If X is a nontrivial interval of a proper subfield F of $\mathbb{R}$, then $OT(X)$ is not a regular semigroup.*

Proof. Let $a, b \in X$ be such that $a < b$. Since $\mathbb{Q} \subseteq F \subsetneq \mathbb{R}$, there is an irrational number $c \in \mathbb{R} \smallsetminus F$. Thus $a - c < d < b - c$ for some $d \in \mathbb{Q}$. Thus $a < c + d < b, c + d \in \mathbb{R} \smallsetminus F$ and $c + d$ is an irrational number. Let $e = c + d$. Consequently, $X = \big((-\infty, a) \cap X\big) \cup \big([a, e) \cap X\big) \cup \big((e, \infty) \cap X\big)$. Define $\mu : \mathbb{R} \to F$ by

$$x\mu = \begin{cases} x & \text{if } x \in (-\infty, a), \\ \frac{a+x}{2} & \text{if } x \in [a, e), \\ x & \text{if } x \in (e, \infty), \end{cases}$$

and let $\alpha = \mu|_X$. Clearly α is order-preserving. We claim that $([a, e) \cap X)\,\alpha = [a, \frac{a+e}{2}) \cap X$. If $x \in [a, e) \cap X$, then $\frac{a+x}{2} \in F$ and $a \leq \frac{a+x}{2} = x\alpha < \frac{a+e}{2} < \frac{a+b}{2} < b$, so $x\alpha \in [a, \frac{a+e}{2}) \cap X$ since X is an interval in F. For the reverse inclusion, let $y \in [a, \frac{a+e}{2}) \cap X$. Then $a \leq 2y - a < e < b$ and $2y - a \in F$. It follows that $2y - a \in X$ and $(2y - a)\alpha = y$. Hence the claim holds. Consequently,

$$\begin{aligned} \operatorname{ran}\alpha &= \big((-\infty, a) \cap X\big) \cup \big([a, \tfrac{a+e}{2}) \cap X\big) \cup \big((e, \infty) \cap X\big) \\ &= \big((-\infty, \tfrac{a+e}{2}) \cap X\big) \cup \big((e, \infty) \cap X\big) \subseteq X. \end{aligned}$$

Therefore we have $\alpha \in OT(X)$. Let $p \in \mathbb{Q}$ be such that $\frac{a+e}{2} < p < e$. Then $p \notin \operatorname{ran}\alpha$. Since $\mathbb{Q} \subseteq F$ and $a < \frac{a+e}{2} < p < e < b$, it follows that $p \in X$. Hence $p \in X \smallsetminus \operatorname{ran}\alpha$, $\{t \in \operatorname{ran}\alpha \mid t < p\} = (-\infty, \frac{a+e}{2}) \cap X$ and $\{t \in \operatorname{ran}\alpha \mid x > p\} = (e, \infty) \cap X$. If $\max\big((-\infty, \frac{a+e}{2}) \cap X\big)$ exists, say m, then $m \in X$ and $a \leq m < \frac{a+e}{2} < b$. Let $q \in \mathbb{Q}$ be such that $m < q < \frac{a+e}{2}$. Then $q \in F$ and $a < q < b$ which imply that $m < q \in (-\infty, \frac{a+e}{2}) \cap X$. This is a contradiction. Then $\max\big((-\infty, \frac{a+e}{2}) \cap X\big)$ does not exist. We can show similarly that $\min\big((e, \infty) \cap X\big)$ does not exist. By Theorem 2.4, $\alpha \notin \operatorname{Reg}(OT(X))$. This proves that $OT(X)$ is not a regular semigroup, as desired. $\square$

3 Regular Elements of $OT(X, Y)$

In this section, we characterize the regular elements of the semigroup $OT(X, Y)$ where Y is a nonempty subset of a chain X. Then we determine when $OT(X, Y)$ is a regular semigroup.

Theorem 3.1. *Let X be a chain and $\emptyset \neq Y \subseteq X$. Then for $\alpha \in OT(X, Y)$, $\alpha \in \operatorname{Reg}(OT(X, Y))$ if and only if $\alpha \in \operatorname{Reg}(T(X, Y))$ and $\alpha \in \operatorname{Reg}(OT(X))$.*

Proof. Assume that $\alpha \in \operatorname{Reg}(OT(X, Y))$. Since $OT(X, Y)$ is a subsemigroup of $T(X, Y)$ and $OT(X)$, it follows that α is regular in $T(X, Y)$ and $OT(X)$, i.e., $\alpha \in \operatorname{Reg}(T(X, Y))$ and $\alpha \in \operatorname{Reg}(OT(X))$.

For the converse, assume that $\alpha \in \operatorname{Reg}(T(X, Y))$ and $\alpha \in \operatorname{Reg}(OT(X))$. By Theorem 1.4, $\operatorname{ran}\alpha = Y\alpha$ or equivalently, $x\alpha^{-1} \cap Y \neq \emptyset$ for all $x \in \operatorname{ran}\alpha$. For each $x \in \operatorname{ran}\alpha$, choose $y_x \in x\alpha^{-1} \cap Y$. Then $y_x\alpha = x$ for all $x \in \operatorname{ran}\alpha$. Let $\beta \in OT(X)$ be such that $\alpha = \alpha\beta\alpha$. Then $X\alpha = X\alpha\beta\alpha \subseteq X\beta\alpha \subseteq X\alpha = \operatorname{ran}\alpha$. It follows that $\operatorname{ran}\alpha = \operatorname{ran}(\beta\alpha)$. Thus $X = \dot{\bigcup}_{x \in \operatorname{ran}(\beta\alpha)} x(\beta\alpha)^{-1} = \dot{\bigcup}_{x \in \operatorname{ran}\alpha} x(\beta\alpha)^{-1}$. Define $\beta' : X \to Y$ by a bracket notation as follows:

$$\beta' = \begin{pmatrix} x(\beta\alpha)^{-1} \\ y_x \end{pmatrix}_{x \in ran\alpha}.$$

If $x \in X$, then $x\alpha = (x\alpha)\beta\alpha$, so $x\alpha \in (x\alpha)(\beta\alpha)^{-1}$ which implies that $x\alpha\beta'\alpha = y_{x\alpha}\alpha = x\alpha$. Hence $\alpha = \alpha\beta'\alpha$. To show that β' is order-preserving, let $x_1, x_2 \in X$ be such that $x_1 < x_2$. Then $x_1\beta\alpha \leq x_2\beta\alpha$. If $x_1\beta\alpha = x_2\beta\alpha$, then $x_1, x_2 \in (x_1\beta\alpha)(\beta\alpha)^{-1}$, so $x_1\beta' = y_{x_1\beta\alpha} = x_2\beta'$. If $x_1\beta\alpha < x_2\beta\alpha$, then by Lemma 2.1, $(x_1\beta\alpha)\alpha^{-1} < (x_2\beta\alpha)\alpha^{-1}$. It follows that $y_{x_1\beta\alpha} < y_{x_2\beta\alpha}$. Since $((x_1\beta\alpha)(\beta\alpha)^{-1})\beta' = \{y_{x_1\beta\alpha}\}$ and $((x_2\beta\alpha)(\beta\alpha)^{-1})\beta' = \{y_{x_2\beta\alpha}\}$, we have that $x_1\beta' = y_{x_1\beta\alpha} < y_{x_2\beta\alpha} = x_2\beta'$.

Hence the proof is completed. □

The following theorem is a direct consequence of Theorem 1.4, Theorem 2.4 and Theorem 3.1.

Theorem 3.2. *Let X be a chain and $\emptyset \neq Y \subseteq X$. Then for $\alpha \in OT(X,Y)$, $\alpha \in \mathrm{Reg}(OT(X,Y))$ if and only if the following four conditions hold.*

(i) $\operatorname{ran}\alpha = Y\alpha$.

(i) *If* $\operatorname{ran}\alpha$ *has an u.b. in* X*, then* $\max(\operatorname{ran}\alpha)$ *exists.*

(ii) *If* $\operatorname{ran}\alpha$ *has a l.b. in* X*, then* $\min(\operatorname{ran}\alpha)$ *exists.*

(iii) *If* $x \in X \smallsetminus \operatorname{ran}\alpha$ *is neither an u.b. nor a l.b. of* $\operatorname{ran}\alpha$*, then* $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ *or* $\min(\{t \in \operatorname{ran}\alpha \mid t > x\})$ *exists.*

Finally, the regularity of the semigroup $OT(X,Y)$ is determined. The following series of lemmas is needed.

Lemma 3.3. *Let X be a chain and $Y \subseteq X$ and $|Y| \geq 2$. If there is an element $a \in X$ such that $a > Y$ or $a < Y$, then the semigroup $OT(X,Y)$ is not regular.*

Proof. Let $e, f \in Y$ be such that $e < f$. Define $\alpha : X \to Y$ by

$$\alpha = \begin{pmatrix} u & v \\ e & f \end{pmatrix}_{\substack{u<a \\ v \geq a}} \text{ if } a > Y \quad \text{and} \quad \alpha = \begin{pmatrix} u & v \\ e & f \end{pmatrix}_{\substack{u \leq a \\ v > a}} \text{ if } a < Y.$$

Then $\alpha \in OT(X,Y), \operatorname{ran}\alpha = \{e,f\}, Y\alpha = \{e\}$ for $a > Y$ and $Y\alpha = \{f\}$ for $a < Y$. By Theorem 3.2, $\alpha \notin \mathrm{Reg}(OT(X,Y))$. Hence $OT(X,Y)$ is not regular. □

Lemma 3.4. *Let X be a chain. If $Y \subsetneq X$ and $|Y| \geq 3$, then the semigroup $OT(X,Y)$ is not regular.*

Proof. Let $e, f, g \in Y$ be such that $e < f < g$ and let $a \in X \smallsetminus Y$. If $a > Y$ or $a < Y$, then by Lemma 3.3, $OT(X,Y)$ is not regular. Assume that $a \not> Y$ and $a \not< Y$. Then $\{t \in Y \mid t < a\}$ and $\{t \in Y \mid t > a\}$ are nonempty. Define $\alpha : X \to Y$ by

$$\alpha = \begin{pmatrix} u & a & v \\ e & f & g \end{pmatrix}_{\substack{u<a \\ v>a}}.$$

Then $\alpha \in OT(X,Y)$ and $\operatorname{ran}\alpha = \{e,f,g\} \neq \{e,g\} = Y\alpha$. By Lemma 3.2, $\alpha \notin \mathrm{Reg}(OT(X,Y))$ and so $OT(X,Y)$ is not regular. □

Lemma 3.5. *Let X be a chain, $Y \subseteq X$ and $|Y| = 2$. Then $OT(X,Y)$ is a regular semigroup if and only if* $\min(X)$ *and* $\max(X)$ *exist and* $Y = \{\min(X), \max(X)\}$.

Proof. Let $Y = \{e, f\}$ be such that $e < f$. Assume that $OT(X,Y)$ is regular. Then by Lemma 3.3, for every $a \in X$, $a \ngtr Y$ and $a \nless Y$. Thus $e \le a \le f$ for all $a \in X$. This implies that $e = \min(X)$ and $f = \max(X)$.

For the converse, assume that $\min(X)$ and $\max(X)$ exist, $e = \min(X)$ and $f = \max(X)$. Let $\alpha \in OT(X,Y)$. If $|\operatorname{ran}\alpha| = 1$, then $\alpha^2 = \alpha$, so $\alpha \in \operatorname{Reg}(OT(X,Y))$. If $\operatorname{ran}\alpha = \{e, f\}$, then $e\alpha = e$ and $f\alpha = f$ since α is order-preserving. Thus $\operatorname{ran}\alpha = Y\alpha$, so α satisfies (i) of Theorem 3.2. It is evident that α satisfies (ii) - (iv) of Theorem 3.2. It follows that $\alpha \in \operatorname{Reg}(OT(X,Y))$. □

Theorem 3.6. *Let X be a chain and $\emptyset \neq Y \subseteq X$. Then $OT(X,Y)$ is a regular semigroup if and only if one of the following statements holds.*

(i) *$Y = X$ and $OT(X)$ is a regular semigroup.*
(ii) $|Y| = 1$.
(iii) $|Y| = 2$, $\min(X)$ *and* $\max(X)$ *exist and* $Y = \{\min(X), \max(X)\}$.

Proof. Assume that $OT(X,Y)$ is regular and suppose that (i) and (ii) are false. Then ($Y \subsetneq X$ or $OT(X)$ is not regular) and $|Y| \ge 2$.

Case 1: $Y \subsetneq X$ and $|Y| \ge 2$. Then the regularity of $OT(X,Y)$ and Lemma 3.4 yield $|Y| = 2$. Hence (iii) holds by Lemma 3.5.

Case 2: $OT(X)$ is not regular and $|Y| \ge 2$. Since $OT(X,Y)$ is regular, it follows that $Y \subsetneq X$, so by Lemma 3.4, $|Y| = 2$. Thus (iii) holds by Lemma 3.5.

Conversely, $OT(X,Y)$ is obviously regular if (i) or (ii) holds. We have by Lemma 3.5 that $OT(X,Y)$ is regular if (iii) holds.

Therefore the theorem is proved. □

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