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An Intersection Theorem in Free Group Rings

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Abstract

Let $\mathbb{Z}F$ denote the free group ring of a free group F and $\Delta(F)$ its augmentation ideal. For normal subgroups R and S of F, we determine the intersection $\Delta^3(R) \cap \Delta^3(S)$.

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1 Introduction

Let $\mathbb{Z}G$ denote the integral group ring of a group G and $\Delta(G)$ its augmentation ideal. Let $\{\gamma_n(G)\}_{n\geq 1}$ be the lower central series of G. We also write G' for $\gamma_2(G) = [G, G]$, the derived group of G. When G is free, then integral group ring is known as free group ring. Intersections of ideals contained in the augmentation ideal have been used by mathematicians for certain specific investigations in integral group rings. Hartley [3] used his intersection to study residual nilpotence of augmentation ideal. An intersection of Hartley was used by Passi and Sucheta [8] in showing that divisible groups have dimension property. That intersections can be beneficial in other areas also was shown by Levin [7]. He used an intersection to obtain a necessary condition for the generalized free product of two finitely generated free groups to be again free.

Intersections of ideals are very helpful in the identification of subgroups determined by the ideals in integral group rings. Let R be a subgroup of a free group F. Then $F \cap (1 + \Delta(R)\Delta^3(F)) \subset F \cap (1 + \Delta(R)\Delta(F)) = R'$ and $R' - 1 \subset \Delta^2(R)$. Therefore if $w \in F \cap (1 + \Delta(R)\Delta^3(F))$, then $w - 1 \in \Delta(R)\Delta^3(F) \cap \Delta^2(R)$ and thus computation of $\Delta(R)\Delta^3(F) \cap \Delta^2(R)$ can lead to the identification of the subgroup $F \cap (1 + \Delta(R)\Delta^3(F))$. Such type of intersections were obtained by Karan and Vermani [6,10] and Vermani and Razdan [11]. Karan and Vermani [6,10] also computed some intersections for the study of augmentation quotients.

For an Abelian group G and a subgroup H of G, Karan and Vermani [6] proved that $\Delta^2(G)\cap\Delta(H) = \Delta^2(H)$. For any group G, and a subgroup H of G, Vermani and Karan [10] proved that $\Delta^3(G)\cap\Delta(G') = \Delta^2(G')+\Delta(\gamma_3(G))$, while Vermani and Razdan [11] proved that $\Delta^2(G)\cap\Delta(H) = \Delta^2(H) + \Delta(H \cap G')$. For a free group F and a subgroup R of F, Karan, Kumar and Vermani [5] proved that $\Delta^3(F)\cap\Delta^2(R) = \Delta^3(R) + \Delta(R)\Delta(R\cap F') + \Delta(R'\cap\gamma_3(F))$ and $\Delta^3(F)\cap\Delta(R) = \Delta^3(R) + \Delta(R)\Delta(R\cap F') + \Delta(R\cap\gamma_3(F))$. For R a normal subgroup of F, Karan and Kumar [4] computed the intersection $\Delta^{n+1}(F)\cap\Delta^n(R)$ for all $n \geq 1$.

Henceforth in this note, unless or otherwise stated, R and S are normal subgroups of a free group F and A denotes $R\cap S$. In section 2 of this note, we identify two subgroups $A\cap(1+\Delta^3(A)+\Delta(A)\Delta(R'\cap S)+\Delta(A)\Delta(R\cap S'))$ and $A\cap(1+\Delta^3(R)+\Delta(A)\Delta(R\cap S'))$. Using these identifications, we compute the intersection $\Delta^3(R)\cap\Delta^3(S)$ in section 3.

2 Identifications

The following is a simple observation from [9] which we will use without citing it.

Lemma 2.1 *Let G be a group, K a subgroup of G, and J an ideal of $\mathbb{Z}G$ containing $\Delta^2(K)$. Then*

$$G\cap(1+J+\Delta(K)) = (G\cap(1+J))K.$$

Karan, Kumar and Vermani [5] proved the following:

Theorem 2.2 *If F is a free group and R and T are subgroups of F with $T\subset R$ and $R/R'T$ torsion-free, then*

$$(\Delta^2(F)\Delta(R)+\Delta(F)\Delta(T))\cap\Delta^2(R) = \Delta^3(R)+\Delta(R\cap F')\Delta(R)+\Delta(R)\Delta(T).$$

The following generalization of this can be proved in a similar manner.

Theorem 2.3 *If F is a free group and R, S and T are subgroups of F with $T\subset R\subset S$ and $R/R'T$ torsion-free, then*

$$(\Delta^2(F)\Delta(R)+\Delta(F)\Delta(T))\cap\Delta(S)\Delta(R) = \Delta^2(S)\Delta(R)+\Delta(S\cap F')\Delta(R)+\Delta(S)\Delta(T).$$

As a consequence of this, we have the following:

Corollary 2.4 *With F, R and S as above*

$$\Delta^3(F)\cap\Delta(S)\Delta(R) = \Delta^2(S)\Delta(R)+\Delta(S\cap F')\Delta(R)+\Delta(S)\Delta(R\cap F').$$

Proof. The proof follows by taking intersection via $\Delta(F)\Delta(R)$ and using the fact that $\Delta(F)$ is a free right $\mathbb{Z}F$-module. □

Gupta [1] identified the subgroup $F\cap(1+\Delta(R)\Delta(F)\Delta(S))$. She proved that $F\cap(1+\Delta(R)\Delta(F)\Delta(S)) = (R'\cap S)'(R\cap S')'[A, R'\cap S']$. This subgroup, which is contained in A', equals $A\cap(1+\Delta^3(A)+\Delta(R'\cap S)\Delta(A)+\Delta(A)\Delta(R\cap S'))$ in view of the following

Lemma 2.5 *The intersection* $\Delta(R)\Delta(F)\Delta(S)\cap\Delta^2(A)$ *equals*

$$\Delta^3(A)+\Delta(R'\cap S)\Delta(A)+\Delta(A)\Delta(R\cap S').$$

Proof. Since $\mathbb{Z}F\Delta(R)$ is a free right $\mathbb{Z}F$-module [2, Proposition I.1.2], it follows that

$$\begin{aligned}
& \Delta(R)\Delta(F)\Delta(S)\cap\Delta^2(A)\\
= \; & \Delta(R)\Delta(F)\Delta(S)\cap\Delta(R)\Delta(A)\cap\Delta^2(A)\\
= \; & \Delta(R)(\Delta(F)\Delta(S)\cap ZR\Delta(A))\cap\Delta^2(A)\\
= \; & \Delta(R)(\Delta(R)\Delta(A)+\Delta(F)\Delta(S)\cap\Delta(A))\cap\Delta^2(A)\\
= \; & \Delta(R)(\Delta(R)\Delta(A)+\Delta^2(S)\cap\Delta(A))\cap\Delta^2(A)\\
= \; & \Delta(R)(\Delta(R)\Delta(A)+\Delta^2(A)+\Delta(A\cap S'))\cap\Delta^2(A)\\
= \; & (\Delta^2(R)\Delta(A)+\Delta(R)\Delta(R\cap S'))\cap\Delta^2(A)\\
= \; & \Delta^3(A)+\Delta(A)\Delta(R\cap S')+\Delta(R'\cap S)\Delta(A),
\end{aligned}$$

where last equality follows from Theorem 2.2. □

The subgroup $A\cap(1+\Delta^3(A)+\Delta(A)\Delta(R'\cap S)+\Delta(A)\Delta(R\cap S'))$ is equal to $A\cap(1+\Delta^3(A)+\Delta(A)\Delta((R'\cap S)(R\cap S')))$ and therefore [9, Theorem 2.2]

$$A\cap(1+\Delta^3(A)+\Delta(A)\Delta(R'\cap S)+\Delta(A)\Delta(R\cap S')) = \gamma_3(A)U,$$

where $U=<[x^m,y]\mid x^m,y^m\in(R'\cap S)(R\cap S')\, for\ some\ m\geq 1, x,y\in A>$. We here give a different identification of this subgroup.

Proposition 2.6 $A\cap(1+\Delta^3(A)+\Delta(A)\Delta(R'\cap S)+\Delta(A)\Delta(R\cap S'))$ *equals*

$$\gamma_3(A)(R'\cap S)'(R\cap S')'[R'\cap S, R\cap S'].$$

Proof. Since $\gamma_3(A)-1\subset\Delta^3(A)$, $(R'\cap S)'-1\subset\Delta(A)\Delta(R'\cap S)$, $(R\cap S')'-1\subset\Delta(A)\Delta(R\cap S')$ and $[R'\cap S, R\cap S']-1\subset\Delta(A)\Delta(R'\cap S)+\Delta(A)\Delta(R\cap S')$, it follows that $\gamma_3(A)(R'\cap S)'(R\cap S')'[R'\cap S, R\cap S']$ is contained in $A\cap(1+\Delta^3(A)+\Delta(A)\Delta(R'\cap S)+\Delta(A)\Delta(R\cap S'))$. For the reverse inequality, we let $w\in A$ such that $w-1\in\Delta^3(A)+\Delta(A)\Delta(R'\cap S)+\Delta(A)\Delta(R\cap S')$ and proceed to show that $w \equiv 1 \pmod{\gamma_3(A)(R' \cap S)'(R \cap S')'[R' \cap S, R \cap S']}$. Now

$$\begin{aligned} & A \cap (1 + \Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta(A)\Delta(R \cap S')) \\ \subset \ & A \cap (1 + \Delta^3(A) + \Delta(R' \cap S)\Delta(A) + \Delta(A)\Delta(R \cap S') + \Delta([A, R' \cap S])) \\ = \ & [A, R' \cap S](R \cap S')' \subset \gamma_3(R)(R \cap S')'. \end{aligned}$$

Thus

$$\begin{aligned} w - 1 \ \in \ & (\Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta(A)\Delta(R \cap S')) \\ & \cap (\Delta^3(R) + \Delta((R \cap S')')) \\ = \ & \Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta((R \cap S')') \\ & + \Delta^3(R) \cap \Delta(A)\Delta(R \cap S') \\ = \ & \Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta((R \cap S')') \\ & + \Delta(R' \cap S)\Delta(R \cap S'), \end{aligned}$$

where last equality follows from Corollary 2.4. Therefore

$$\begin{aligned} w \ \in \ & A \cap (1 + \Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta((R \cap S')') \\ & + \Delta(R' \cap S)\Delta(R \cap S')) \\ \subset \ & A \cap (1 + \Delta^3(A) + \Delta(A)\Delta(R' \cap S))(R \cap S')'[R' \cap S, R \cap S'] \\ = \ & \gamma_3(A)(R' \cap S)'(R \cap S')'[R' \cap S, R \cap S'], \end{aligned}$$

where last equality follows from [12, Theorem 4.4]. □

Proposition 2.7 $A \cap (1 + \Delta^3(R) + \Delta(A)\Delta(R \cap S')) = (R \cap S')'(S \cap \gamma_3(R))$.

Proof: Observe that $A \cap (1 + \Delta^3(R) + \Delta(A)\Delta(R \cap S')) \subset A \cap (1 + \Delta^3(R) + \Delta^2(A)) = A'(S \cap \gamma_3(R))$ [5, Lemma 3.4]. Therefore if $w \in A \cap (1 + \Delta^3(R) + \Delta(A)\Delta(R \cap S'))$, then

$$\begin{aligned} w - 1 \ \in \ & (\Delta^3(R) + \Delta(A)\Delta(R \cap S')) \cap (\Delta(S \cap \gamma_3(R)) + \Delta^2(A)) \\ = \ & \Delta(S \cap \gamma_3(R)) + \Delta(A)\Delta(R \cap S') + \Delta^3(R) \cap \Delta^2(A) \\ = \ & \Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta(A)\Delta(R \cap S') + \Delta(S \cap \gamma_3(R)) \end{aligned}$$

and thus

$$\begin{aligned} w \ \in \ & A \cap (1 + \Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta(A)\Delta(R \cap S'))(S \cap \gamma_3(R)) \\ = \ & (R \cap S')'(S \cap \gamma_3(R)). \end{aligned}$$

□

3 The Intersection $\Delta^3(R) \cap \Delta^3(S)$

We begin with the following easy observation

Lemma 3.1 $\Delta(R) \cap \Delta(S) = \Delta(A)$.

Lemma 3.2 $\Delta^2(R) \cap \Delta^2(S) = \Delta^2(A) + \Delta(R' \cap S')$.

Proof: Taking intersection via $\Delta(R) \cap \Delta(S)$, we have

$$\begin{aligned} \Delta^2(R) \cap \Delta^2(S) &= \Delta^2(R) \cap \Delta(R) \cap \Delta(S) \cap \Delta^2(S) \\ &= \Delta^2(R) \cap \Delta(A) \cap \Delta^2(S) \\ &= \Delta^2(R) \cap (\Delta^2(A) + \Delta(R \cap S')) \\ &= \Delta^2(A) + \Delta^2(R) \cap \Delta(R \cap S') \\ &= \Delta^2(A) + \Delta(R' \cap S'). \end{aligned}$$

□

Lemma 3.3 $\Delta^3(R) \cap \Delta(S) = \Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta(\gamma_3(R) \cap S)$.

Proof: Observe that

$$\begin{aligned} \Delta^3(R) \cap \Delta(S) &= \Delta^3(R) \cap \Delta(R) \cap \Delta(S) = \Delta^3(R) \cap \Delta(A) \\ &= \Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta(\gamma_3(R) \cap S). \end{aligned}$$

□

Lemma 3.4 $\Delta^3(R) \cap \Delta^2(S) = \Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta(\gamma_3(R) \cap S')$.

Proof: Taking intersection via $\Delta^2(R)$, we have

$$\Delta^3(R) \cap \Delta^2(S) = \Delta^3(R) \cap (\Delta^2(A) + \Delta(R' \cap S')).$$

Let $x \in \Delta^3(R) \cap (\Delta^2(A) + \Delta(R' \cap S'))$, then $x = y + z$, where $y \in \Delta^2(A)$, $z \in \Delta(R' \cap S')$. Write $z = \sum_{i=1}^n m_i(a_i - 1)$, where $a_i \in R' \cap S'$. Let $a = \prod_{i=1}^n a_i^{m_i}$, then $a - 1 \equiv z \pmod{\Delta^2(R' \cap S')}$. This gives $a - 1 \in \Delta^3(R) + \Delta^2(A)$ and $a \in A'(\gamma_3(R) \cap S)$. Thus $z \in \Delta^2(R' \cap S') + \Delta(A'(\gamma_3(R) \cap S))$, and therefore

$$\begin{aligned} &\Delta^3(R) \cap \Delta^2(S) \\ =\ &\Delta^3(R) \cap (\Delta^2(A) + \Delta(\gamma_3(R) \cap S')) \\ =\ &\Delta(\gamma_3(R) \cap S') + \Delta^3(R) \cap \Delta^2(A) \\ =\ &\Delta(\gamma_3(R) \cap S') + \Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta(A' \cap \gamma_3(R)) \\ =\ &\Delta^3(A) + \Delta(A)\Delta(R' \cap S) + \Delta(\gamma_3(R) \cap S'). \end{aligned}$$

□

Theorem 3.5 *The intersection* $\Delta^3(R) \cap \Delta^3(S)$ *equals*

$$\Delta^3(A) + \Delta(A)\Delta(R' \cap S') + \Delta(R' \cap S)\Delta(R \cap S') + \Delta(\gamma_3(R) \cap \gamma_3(S))$$

Proof : Taking intersection via $\Delta^2(R)$, we have

$$\begin{aligned} & \Delta^3(R) \cap \Delta^3(S) \\ = \; & \Delta^3(R) \cap \Delta^2(R) \cap \Delta^3(S) \\ = \; & \Delta^3(R) \cap (\Delta^3(A) + \Delta(A)\Delta(R \cap S') + \Delta(\gamma_3(S) \cap R')) \\ = \; & \Delta^3(A) + \Delta^3(R) \cap (\Delta(A)\Delta(R \cap S') + \Delta(\gamma_3(S) \cap R')) \qquad (1) \end{aligned}$$

Let $z \in \Delta^3(R) \cap (\Delta(A)\Delta(R \cap S') + \Delta(\gamma_3(S) \cap R'))$. Then $z = a + b$, where $a \in \Delta(A)\Delta(R \cap S')$ and $b \in \Delta(\gamma_3(S) \cap R')$. As in Lemma 3.4 and using Proposition 2.7, we can show that $b \in \Delta^2(\gamma_3(S) \cap R') + \Delta((R \cap S')'(\gamma_3(R) \cap \gamma_3(S)))$, and therefore by (1),

$$\begin{aligned} & \Delta^3(R) \cap \Delta^3(S) \\ = \; & \Delta^3(A) + \Delta^3(R) \cap (\Delta(A)\Delta(R \cap S') + \Delta(\gamma_3(R) \cap \gamma_3(S))) \\ = \; & \Delta^3(A) + \Delta(\gamma_3(R) \cap \gamma_3(S)) + \Delta^3(R) \cap \Delta(A)\Delta(R \cap S') \\ = \; & \Delta^3(A) + \Delta(A)\Delta(R' \cap S') + \Delta(R' \cap S)\Delta(R \cap S') + \Delta(\gamma_3(R) \cap \gamma_3(S)), \end{aligned}$$

by Corollary 2.4. □

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