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Lift of Spanning Trees for Diagram Group in Graphs of Semigroup Presentation

$$S = \langle x, y, z \mid x = y, y = z, x = z \rangle$$

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Abstract

The aim of this paper is to determine a spanning tree for the complex of diagram groups obtained from Semigroup presentation $P = \langle x, y, z \mid x = y, y = z, x = z \rangle$. The tree will be systematically selected using lifting method of graph according to the length words. The general polynomials of the numbers all lifts and edges then will be computed.

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1 Introduction

Let $P = \langle X \mid R \rangle$ be a semigroup presentation, where $X$is a set of generator where elements of relations in R is of the form $R_{\varepsilon} = R_{-\varepsilon}(R_{\pm\varepsilon}$ are reduced positive words on $X)$. We may construct the diagram group $D(P, w)$ where w

is a positive word on X as described for example in Guba and Sapir [3] and Kilibarda [4] and Pride [2], and Abd Ghafur Ahmad [1]. For any semigroup presentation P, we may obtain a Squier complex $K(P)$. Vertices of $K(P)$ are all positive words on Xwhile edges are atomic picture labeled by $A = (u, l \to r, v)$ where u, v are words on X and $(l = r) \in R$. The 2-cell of $K(P)$ are 5-tuples of the form $(u, l_1 \to r_1, v, l_2 \to r_2, w)$ where $u, v, w \in X^*$and $(l_i = r_i) \in R$. Such a 2-cell has the following defining specific path: $(ul_1v, l_2 \to r_2, w)(u, l_1 \to r_1, vr_2w)(ur_1v, l_2 \to r_2, w)^{-1}(u, l_1 \to r_1, vl_2w)^{-1}$.
It is easy to see that 2-cell correspond to independent applications of the relations from R. See Guba and Sapir [3] for details. As a complex, we may obtain the fundamental group $\pi_1(K(P), w)$ which is isomorphic to $D(P, w)$.First note that $\pi_1(K(P), u) \cong \pi_1(K(P), v)$ if and only if length(u)= length(v)(see lemma 3.1). In our previous research we obtained the general polynomial for component of graphs from diagram group of semigroup presentation
$P = \langle x, y, z \mid x = y, y = z, x = z\rangle$. See Gheisari and Abd Ghafur Ahmad [5]. Now we want to expand that study.
In this paper we will consider the fundamental group $\pi_1(K(P), w)$ constructed from semigroup presentation $P = \langle x, y, z \mid x = y, y = z, x = z\rangle$. Guba and Sapir [3] have shown that $\pi_1(K(P), x)$ is infinite cyclic.
As a group, it is sufficient to determine its generators. The generators of this group can be determined from the complex $K(P)$ by identifying the spanning tree. Then the edges which are not in the spanning tree will be the generators. The objective of this paper is to determine and to select a spanning tree in $K(P)$ systematically for presentation $P = \langle x, y, z \mid x = y, y = z, x = z\rangle$. This selection is obtained from a collection of lift (refer to section 2).
In section 2 we will give some definitions about graphs, paths, lifting of paths and spanning trees. Then a suitable selection of lift will be shown in section 3. This section includes determining the general polynomial of all lifts and the number of those lifts. Similarly the general polynomial of all edges and its number will be computed in the spanning tree.

2 Basic definition

Definition 2.1 *Let $\Gamma = (V, E, \iota, \tau, -1)$ be a graph, V, E are disjoint set (V set of vertices and E set of edges) with maps*

$$\iota : E \to V$$

$$\tau : E \to V$$

$$-1 : E \to V = E$$

Let $P = \langle X \mid R\rangle$ be a semigroup presentation. Then Pcan be viewed as a graph as follows:the vertices are all words in X^* and the positive edges are triple $(u, l \to r, v)$ and the negative edges are triple $(u, l \to r, v)$ where $u, v \in X^*$and$(l = r) \in R$. If $e = (u, s \to t, v)$ is an edges of $K(P)$ then $e^{-1} = (u, t \to s, v)$, $\iota(e) = usv$, $\tau(e) = utv$.

Example 2.1 Let $P = \langle x, y, z \mid x = y, y = z, x = z\rangle$. Note that the graph obtained from P is collections of several sub graphs, and the graph Γ obtained from P is just a union of Γ_n connecting all vertices of length n and respective edges. For example in figure1, is shown as:

$$(1, x \to y, 1)(1, x \to y, 1)(1, x \to z, 1)(1, x \to z, 1)(1, y \to z, 1)(1, y \to z, 1)$$

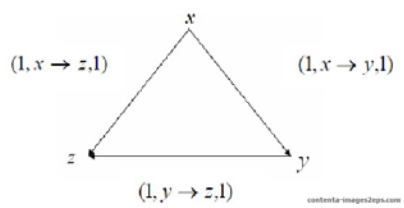

Figure 1: Graph Γ_1

While Γ_2 is shown in figure2.

$$(y, y \to z, , 1)(y, y \to z, , 1)(1, x \to y, y)(1, x \to y, y)(x, x \to y, 1)(x, x \to y, 1)$$

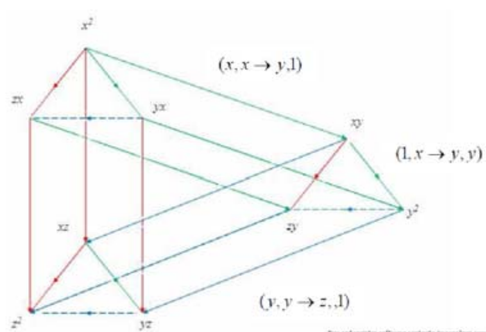

Figure 2: Graph Γ_2

Note that Γ_2 is three copies of and each vertex in each copy are joined together, respectively. Similarly, with three copies of Γ_2, we may obtain Γ_3. Repeat similar procedures for Γ_4 and so on.

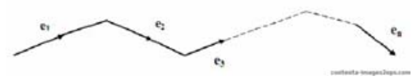

Figure 3: Path of α

Definition 2.2 *A path* α *in the graph* Γ *is a sequence of edges* $e_1, e_2, e_3, ..., e_n$ *where* $\tau(e_i) = \iota(e_{i+1})i = 1, 2, 3, ..., n$ $(e_i \in E)$*. Then* $\iota(\alpha) = \iota(e_1)$ *and* $\tau(\alpha) = \tau(e_n)$*(see figure 3).*

We define α *to be closed path if* $\tau(\alpha) = \iota(\alpha)$*.*

Example 2.2 In Figure 2, $(1, x \to z, 1)^{-1}(1, x \to y, 1)$ is a path and $(1, x \to z, 1)^{-1}(1, x \to y, 1)(1, y \to z, 1)$ is a closed path.

Definition 2.3 *Lifting of path. Let* $\phi : \Gamma^* \to \Gamma$ *be a mapping of graphs. If* v*and* v^* *are the vertices of* Γ *and* Γ^* *respectively such that* $\phi(v^*) = v$ *then* v^* *is said to lie over* v*. Let* α *be a path in* Γ *with* $\iota(\alpha) = v$ *and suppose* v^* *lies over* v*. A path* α^**in* Γ^**is said to be a lift of* α *at* v^* *if* $\phi(\alpha^*) = \alpha$ *(figure 4).*

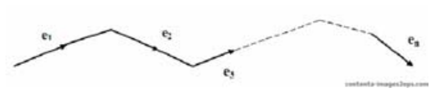

Figure 4: Lifting of path

As in our example $(1, x \to z, 1)^{-1}(1, x \to y, 1)$ *is a path and* $\iota(\alpha) = z, \tau(\alpha) = y$ *and suppose* zy *lies over* z*. Lift of* α *at* zy *is* $\alpha^* = (1, x \to z, y)^{-1}(1, x \to y, y)$*.*

Definition 2.4 *Let* Γ *be a connected graph. A subgraph* T*of* Γ *is called a spanning tree or maximal subtree if* T*is a tree and contains all vertices of* Γ*.*

For example, in figure 5, a spanning tree in is
T_1 $=(1, x \to z, 1)^{-1}(1, x \to y, 1)$
And in figure 6, the spanning tree in Γ_2 *is*

3 Choosing a systematically spanning tree and theorems

Lemma 3.1 *Let* $P = \langle x, y, z \mid x = y, y = z, x = z\rangle$ *be the presentation. If* u *and* v *are two positive words on* $\{x, y, z\}$*, then* $\pi_1(K(P), u)$= $\pi_1(K(P), v)$ *if and only if length*(u)= *length*(v)*.*

Figure 5: Spanning tree in Γ_1

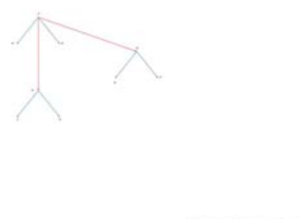

Figure 6: Spanning tree in Γ_2

Proof: Suppose that$D(P, u) \cong D(P, v)$, then $\pi_1(K(P), u)\cong\pi_1(K(P), v)$ and hence u and v are connected. Thus there is a finite sequence of edges $u = u_1, u_2, u_3, ..., u_n = v$. where u_{i+1} is obtained u_{i+1} by an elementary operation. Since any edge of $k(P)$ is an atomic picture or its inverse for some U and then the sequence is actually $u = u_1 \to u_2 \to v$ thus length (u_i) = length (u_{i+1}) and hence we conclude that length (u) = length(v).
Now suppose that length(u)= length(v). It is sufficient for us to show that u and $v$are connected. Since u and$v$ positive word on $\{x, y, z\}$, then
$u = w_1u_1w_2u_2...w_nu_nw_{n+1}$ and

$$v = w_1v_1w_2v_2...w_nv_nw_{n+1}$$

such that w_i words on $\{x, y, z\}$ (possibly empty) while if
u_i and $v_i$are alphabet on $\{x, y, z\}$ such that $u_i \neq v_i$ Then

$$u = w_1u_1w_2u_2...w_nu_nw_{n+1}$$

$$\to w_1v_1w_2u_2...w_nu_nw_{n+1}$$

$$... \rightarrow w_1 v_1 w_2 v_2 ... w_n u_n w_{n+1}$$

$$\rightarrow w_1 v_1 w_2 v_2 ... w_n u_n w_{n+1}$$

$$\rightarrow w_1 v_1 w_2 v_2 ... w_n v_n w_{n+1} = v$$

is a sequence of edges connecting u and v. Hence $\pi_1(K(P), u) \cong \pi_1(K(P), v)$.

Now will compute and determine a spanning tree in Γ_n.
Let T_1 be a spanning tree in that $T_1 = (1, x \rightarrow z, 1)^{-1}(1, x \rightarrow y, 1)$ at $v_1 = z$. then
the collections all lift of T_1in at $v_2 = za \ \forall a \in X$ are as follows:
i) Lift of T_1 at zx are $(x, x \rightarrow z, 1)^{-1}(x, x \rightarrow y, 1)$ and
$(1, x \rightarrow z, x)^{-1}(1, x \rightarrow y, x)$.
ii) Lift of T_1at zy is $(1, x \rightarrow z, y)^{-1}(1, x \rightarrow y, y)$.
iii) Lift of T_1 at z^2 is $(1, x \rightarrow z, z)^{-1}(1, x \rightarrow y, z)$.
These are all lift of T_1 at $v_1 = z$ is exactly spanning tree in Γ_2.

Theorem 3.2 *Let T_n be a collection of all lift of T_1 at zv_{n-1} in Γ_n , where v_{n-1} is a word of length$(n-1)$. Then T_n is a spanning tree in Γ_n.*

Proof: By induction consider T_2 in $\Gamma_2$in figure 6. Since T_2 is a collection of lifts and the number of vertices of T_2 equal to number of vertices in Γ_2, then T_2 is a spanning tree.
Now suppose T_k is a collection of lift of T_1 at zv_{k-1} in Γ_k, thus the number of vertices of T_k equal to number of vertices in Γ_k, then T_k is a spanning tree(figure 7).
Now we will prove that T_{k+1} is a collection of all lift of T_1 at zv_k in Γ_{k+1}. The vertex z^k in the first copy is connected to yv_{k-1} and xv_{k-1}. This is an extra lift of T_1 at xv_{k-1} in Γ_k
By definition T_{k+1} is three copies of T_k. Similarly Γ_{k+1} is three copies of Γ_k, Hence it is a collection of lift of zv_k in Γ_{k+1} and the number of vertices of T_{k+1} equal to number of vertices in Γ_{k+1}, then T_{k+1} is a spanning tree(figure 8).
Next results show how to compute the number of all lifts in Γ_n and the number of edges in spanning tree in Γ_n.

Theorem 3.3 *The general polynomial of degree of every vertex in Γ_n is $a_n = 2n$, where a_i is a degree of every vertex in $\Gamma_i (i = 1, 2, 3, \ldots)$.*

Figure 7: Spanning tree in Γ_k

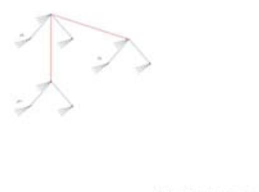

Figure 8: Spanning tree in Γ_{k+1}

Proof: By induction, for $k = 1$ the degree of every vertex in Γ_1 is 2, then $a_1 = 2$ (Figure 1). Now let $a_k = 2k$ be the general polynomial of degree of every vertices in Γ_k. We will prove that the general polynomial of degree of every vertex in Γ_{k+1} is $a_{k+1} = 2(k + 1)$. By theorem 3.2, Figure (4) and figure (5), we can see the degree of every vertices in Γ_{k+1} is $2k + 2$. Then $a_{k+1} = 2(k + 1)$.

Theorem 3.4 *The general polynomial of all lifts of* T_{n-1} *is* $a_n = 3a_{n-1} + 1$ *where* $a_i(i = 1, 2, 3, ..., n)$*is the number of all lift of* T_i *in* Γ_i*.*

Proof: By induction.There is only one lift of T_1=$(1, x \to z, 1)^{-1}(1, x \to y, 1)$ at $v_1 = z$ and we denote this number bya_1. The number of all lift of T_2 is four, and we denote by a_2. Now suppose a_k is the number of all lift of T_{k-1} in Γ_k such that $a_k = 3a_{k-1} + 1$. We will prove that a_{k+1} is the number of all lift of T_k in Γ_{k+1} is $a_{k+1'} = 3a_k + 1$.
By induction T_{k+1} is three copies of T_k plus one (as in proof theorem 3.2). Then $a_{k+1'} = 3a_k + 1$.

Theorem 3.5 *The number of all lift of* T_n *in* Γ_n *is* $a_n = \frac{1}{2}(3^n - 1)$.

Proof: By induction for $k = 1$we have $a_1 = \frac{1}{2}(3 - 1) = 1$. Then $a_1 = 1$. Now suppose $a_k = \frac{1}{2}(3^k - 1)$. We will prove that $a_{k+1} = \frac{1}{2}(3^{k+1} - 1)$.

By corrollary1 we have $a_{k+1} = 3a_k + 1 = 3(\frac{1}{2}(3^k - 1) + 1 = (\frac{1}{2}3^{k+1} - \frac{3}{2}) + 1 = \frac{1}{2}3^{k+1} - \frac{1}{2} = \frac{1}{2}(3^{k+1} - 1)$.

Theorem 3.6 *The general polynomial all edges of spanning tree in* Γ_n *is* $t_n = 3t_{n-1} + 2$ *where* t_n *is the number of all edges of spanning tree in* Γ_n.

Proof: By induction, The number all edges of spanning tree in Γ_2 is 2 (refer to figure(6)).
Now suppose t_k is the number of all edges of spanning tree in Γ_k, that is $t_k = 3t_{k-1} + 2$. We will prove that $t_{k+1} = 3t_k + 2$. By induction t_{k+1} is three copies of t_k plus two (as in proof theorem 3.2). Thus $t_{k+1} = 3t_k + 2$.

Theorem 3.7 *The number of all edges of spanning tree* T_n *in is* $t_n = (3^n - 1)$.

Proof: By induction for $k = 1$ we have $t_1 = (3 - 1) = 2$. Then $t_1 = 2$ (refer to figure (5)).
Now suppose$t_k = (3^k - 1)$.We will prove that $t_{k+1} = (3^{k+1} - 1)$. By corollary 3, we have $t_{k+1} = 3t_k + 2 = 3(3^k - 1) + 2 = 3^{k+1} - 3 + 2 = 3^{k+1} - 1$.

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