**International Journal of Algebra, Vol. 4, 2010, no. 15, 709 - 719**

# Exact, Proper Exact Sequences and Projective Semimodules

**S. K. Bhambri**

Department of Mathematics
Kirorimal College
University of Delhi
Delhi 110 007, India
skbhambri@rediffmail.com

**Manish Kant Dubey**

SAG, Metcalfe House
DRDO Complex
Near civil lines
New Delhi 110054, India
kantmanish@yahoo.com

**Abstract.** In this paper, exact and proper exact sequences of semimodules are introduced for generalizing some theorems and definitions in module theory to be applied and defined in semimodule theory and used to prove some results on projective semimodules analogous to those in module theory.

**Mathematics Subject Classification:** 16Y60

## 1. Introduction and preliminaries

Throughout this paper $S$ will denote a semiring. A semiring is a commutative monoid $(S, +, 0_S)$ having additive identity zero $0_S$ and $0_S.x = x.0_S = 0_S$ for all $x \in S$ and a semigroup $(S, \cdot)$ which are connected by ring like distributivity. A left $S$-semimodule $M$ is a commutative monoid $(M, +)$ which has a zero element $0_M$, together with an operation $S \times M \longrightarrow M$; denoted by $(a, x) \longrightarrow ax$ such that for all $a, b \in S$ and $x, y \in M$, $(i)$ $a(x + y) = ax + ay$,$(ii)(a + b)x = ax + bx$, $(iii)(ab)x = a(bx)$,$(iv)0_Sx = 0_M = a0_M$ .

A right $S$-semimodule is defined in an analogous manner. A non empty subset $A$ of $M$ is said to be subsemimodule of $M$, if $A$ is closed under addition and scalar multiplication. Let $M$ and $N$ be left $S$-semimodule. A homomorphism from $M$ to $N$ is a map $f : M \longrightarrow N$ such that, $(i)$ $f(m_1+m_2) = f(m_1)+f(m_2)$ $(ii)$ $f(am) = af(m)$, for all $m, m_1, m_2 \in M$ and for all $a \in S$.

Recall [5] the following:

(i) Let $A$ and $B$ be $S$-semimodules and $f : A \longrightarrow B$ be homomorphism. Define

$$\begin{aligned} K_f &= \{(a,b) \in A \times A | f(a) + x = f(b) + x \text{ for some } x \in B\} \\ I_f &= \{(c,d) \in B \times B | c + f(a) = d + f(b) \text{ for some } a, b \in A\} \\ \bar{I}_f &= \{(c,d) \in B \times B | c + f(a) + x = d + f(b) + x \text{ for some } a, b \in A, \\ &\qquad \text{some } x \in B\} \end{aligned}$$

Then $f$ is said to be a monic if $K_f = \bar{\Delta}_A$, where $\Delta_A = \{(a,a)|a \in A\}$ and $\bar{\Delta}_A = \{(a,b)|A \times A|a + x = b + x \text{ for some } x \in A\}$ and $f$ is said to be an epic if for any $b \in B$ there exist some $a_i \in A$, $i = 1, 2$, and $x \in B$ such that $b + f(a_1) + x = f(a_2) + x$.

If $f$ is both monic and epic then $f$ is called an equivalence.

(ii) Let $A$ and $B$ be $S$-semimodules. Then an $S$-semimodule homomorphism $f : A \longrightarrow B$ is said to be a $Z$-homomorphism if for each $a \in A$ there exists some $x \in B$ such that $f(a) + x = x$.

(iii) Let $M$ be a left $S$-semimodules. An equivalence relation $\rho$ on $M$ is said to be a congruence relation if $(a,b) \in \rho$ implies $(a + c, b + c) \in \rho$ for all $c \in M$ and $(ra, rb) \in \rho$ for all $r \in S$.

Let $M$ be a left $S$-semimodule and $I$ be a subsemimodule of $M$. The Bourne relation (Latorre 1965) $\rho$ on $M$ defined by

$$\rho = \{(x,y) \in M \times M \,|x + i = y + j \text{ for some } i, j \in I\}.$$

Then $\rho$ is a congruence relation on $M$. Hence $M/\rho$ can be made into a left $S$-semimodule under $\oplus$ and $\odot$ defined by

$$x\rho \oplus y\rho = (x+y)\rho \quad \text{and} \quad a \odot x\rho = (ax)\rho\,.$$

This left $S$-semimodule is called the quotient semimodule of $M$ modulo I and is denoted by $M/I$.

**Result 1.1** ([5], Lemma 3.5)**.** Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be $S$-semimodule homomorphisms. Then $gf : A \longrightarrow C$ is a $Z$-homomorphism iff $\bar{I}_f \subseteq K_g$.

**Definition 1.2.** For any two $S$-semimodules $A$ and $B$,

$$\text{Hom}(A,B) = \{f : A \longrightarrow B \,|\, f \text{ is an } S\text{-homomorphism}\}$$

is a semigroup under addition. Let $P$ be a $S$-semimodule.

Define $\bar{f} : \text{Hom}(P, A) \longrightarrow \text{Hom}(P, B)$ such that $\bar{f}(\phi) = f\phi$, where $f \in \text{Hom}(A, B)$. Then $\bar{f}$ is a homomorphism of semigroups, i.e. $\bar{f}(\phi_1 + \phi_2) = \bar{f}(\phi_1) + \bar{f}(\phi_2)$ under addition.

## 2. Exact and Proper exact sequences of semimodules

In this section we study some results on exact and proper exact sequences of semimodules.

**Definition 2.1.** A sequence of $S$-semimodules and $S$-semimodule homomorphism is a diagram of the form,

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \cdots .$$

Such a sequence is said to be exact if $\bar{I}_{f_{i-1}} = K_{f_i}$ for all $i$.

Similarly, an exact sequence of semigroups and their homomorphism is defined.

**Result 2.2.** The short sequence $A \xrightarrow{f} B \xrightarrow{\phi} B/A \longrightarrow 0$ of $S$-semimodules and their homomorphisms is exact, where $f$ is an inclusion map.

*Proof.* Result is immediate from example 3.3 [5]. □

**Theorem 2.3.** *The short sequence of semimodules* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *where* $0$ *denotes the zero semimodule, is exact iff* $f$ *is a monic and* $g$ *is an epic and* $\bar{I}_f = K_g$.

*Proof.* Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be exact, that is, $K_f = \bar{I}_0$, $K_g = \bar{I}_f$ and $K_0 = \bar{I}_g$. We first show that $f$ is a monic.

Let $(a_1, a_2) \in K_f = \bar{I}_0$. Then $a_1 + x = a_2 + x$ for some $x \in A$, this implies $(a_1, a_2) \in \bar{\Delta}_A$ . Therefore $K_f \subseteq \bar{\Delta}_A$. But converse is obvious. Thus $K_f = \bar{\Delta}_A$, which shows that $f$ is a monic.

Now, we show $g$ is an epic. Let $c \in C$. Then $(c, 0) \in K_0 = \bar{I}_g$. Therefore there exist some $b_1, b_2 \in B$ and $x \in C$ such that $c + g(b_1) + x = 0 + g(b_2) + x$ which shows that $g$ is an epic.

Conversely, let $f$ be a monic, $g$ be an epic and $K_g = \bar{I}_f$. We show $K_f = \bar{I}_0$. Let $(a_1, a_2) \in K_f = \bar{\Delta}_A$. Then $a_1 + x = a_2 + x$ for some $x \in A$, which implies $(a_1, a_2) \in \bar{I}_0$. Therefore $K_f \subseteq \bar{I}_0$.

Again, let $(a_1, a_2) \in \bar{I}_0$. Then $a_1 + x = a_2 + x$ for some $x \in A$, implies $(a_1, a_2) \in \bar{\Delta}_A \subseteq K_f$ . Therefore $\bar{I}_0 \subseteq K_f$. Hence $K_f = \bar{I}_0$.

Finally, we show $K_0 = \bar{I}_g = C \times C$. Let $(c_1, c_2) \in K_0$. Since $g$ is an epic then $c_1 + g(b_1) + x = g(b_2) + x$ for some $b_1, b_2 \in B$, $x \in C$. and $c_2 + g(b_3) + y = g(b_4) + y$ for some $b_3, b_4 \in B$, $y \in C$ . Adding these two equations, we get $c_1 + g(b_1 + b_4) + x + y = c_2 + g(b_2 + b_3) + x + y$, implies $(c_1, c_2) \in \bar{I}_g$. So $K_0 \subseteq \bar{I}_g$. But $\bar{I}_g \subseteq K_0$ is obvious. Therefore $K_0 = \bar{I}_g$. □

**Theorem 2.4.** *In the following commutative diagram of* $S$*-semimodules, assume the upper row is exact*

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & M_2 & \longrightarrow & 0 \\ & & & & & \nearrow g_1 & \downarrow h & & \\ 0 & \longrightarrow & M_0 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_2} & M_3 & & \end{array}$$

*Then the lower row is exact iff right column is exact.*

*Proof.* Suppose the lower row is exact. To show $h$ is a monic. Let $(a, b) \in K_h$. Then

$$(1) \qquad h(a) + p = h(b) + p \quad \text{for some} \quad p \in M_3$$

Since $g_1$ is an epic therefore for each $a \in M_2$, we have $a + g_1(m_1) + m = g_1(m_2) + m$ for some $m_1, m_2 \in M_1$ and $m \in M_2$. Similarly, $b + g_1(m_3) + m' = g_1(m_4) + m'$
for some $m_3, m_4 \in M_1$ and $m' \in M_2$.

Adding these two equations, we get

$$(2) \qquad a + g_1(m_5) + m'' = b + g_1(m_6) + m''$$

where $m_1 + m_4 = m_5$, $m_2 + m_3 = m_6$ and $m + m' = m''$, which implies $h(a) + g_2(m_5) + p + h(m'') = h(b) + g_2(m_6) + p + h(m'')$ (as $hg_1 = g_2$ and adding $p$ on both sides). Using (1), we have $g_2(m_5) + h(m'') + h(b) + p = g_2(m_6) + h(m'') + h(b) + p$ implies $(m_5, m_6) \in K_{g_2} = \bar{I}_{f_1} = K_{g_1}$ (by exactness of both rows), this implies

$$(3) \qquad g_1(m_5) + p' = g_1(m_6) + p' \quad \text{for some} \quad p' \in M_2$$

Adding $p'$ on both sides in (2), we get $a + g_1(m_5) + p' + m'' = b + g_1(m_6) + p' + m''$, or $a + g_1(m_6) + p' + m'' = b + g_1(m_6) + p' + m''$ (using (3)), implies $(a, b) \in \bar{\Delta}_{M_2}$. Therefore $K_h \subseteq \bar{\Delta}_{M_2}$. Hence $h$ is a monic.

Similarly, the converse can also be proved. □

**Definition 2.5.** Let $f : M \longrightarrow N$ be $S$-semimodule homomorphism. Then $f$ is said to be $k$-regular if for $a, b \in M$, $f(a) + x = f(b) + x$ for some $x \in N$ implies $f(a) = f(b)$ and $f$ is said to be $i$-regular if for each $b \in N$, there exist $a_1, a_2 \in M$ such that $b + f(a_1) = f(a_2)$ .

Clearly, $f$ is $i$-regular implies $f$ is an epic.

**Definition 2.6.** A sequence of $S$-semimodules $M \xrightarrow{\alpha} N \xrightarrow{\beta} M'$ is said to be proper exact if $K_\beta = I_\alpha$.

**Result 2.7.** Every proper exact sequence is exact sequence.

*Proof.* Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} M'$ be a proper exact sequence. So $K_\beta = I_\alpha \subseteq \bar{I}_\alpha$.

To show $\bar{I}_\alpha \subseteq K_\beta$. Let $(b_1, b_2) \in \bar{I}_\alpha$. Then $b_1 + \alpha(a_1) + x = b_2 + \alpha(a_2) + x$ implies $(b_1 + x, b_2 + x) \in I_\alpha = K_\beta$(by proper exactness). So$(b_1, b_2) \in$

$K_\beta$ (as $K_\beta$ is AC congruence on $N$, Lemma 2.3 [5]). Therefore $\bar{I}_\alpha \subseteq K_\beta$. Hence $K_\beta = \bar{I}_\alpha$. Which shows that the sequence is an exact sequence. □

**Result 2.8.** Suppose a sequence$M \xrightarrow{\alpha} N \xrightarrow{\beta} M'$ is exact and $\alpha$ is $i$-regular. Then this sequence is proper exact.

*Proof.* Since given sequence is exact i.e. $K_\beta = \bar{I}_\alpha$ and $\alpha$ is $i$-regular, then $I_\alpha \subseteq \bar{I}_\alpha = K_\beta$ (By exactness of sequence). So $I_\alpha \subseteq K_\beta$. Let $(b_1, b_2) \in K_\beta = \bar{I}_\alpha$. Then $b_1 + \alpha(a_1) + x = b_2 + \alpha(a_2) + x$ for some $a_1, a_2 \in M$ and $x \in N$. Since $\alpha$ is $i$-regular, so for $x \in N$ there exist $a_3, a_4 \in M$ such that $x + \alpha(a_3) = \alpha(a_4)$, therefore $b_1 + \alpha(a_1) + x + \alpha(a_3) = b_2 + \alpha(a_2) + x + \alpha(a_3)$, or $b_1 + \alpha(a_1 + a_4) = b_2 + \alpha(a_2 + a_4)$, which implies$(b_1, b_2) \in I_\alpha$. So $\bar{I}_\alpha \subseteq I_\alpha$, which shows that the sequence is exact. □

**Definition 2.9.** Let $\beta : M \longrightarrow M'$ be a homomorphism of $S$-semimodules. Define $Z_\beta = \{a \in M \mid \beta(a) + x = x \quad \text{some } x \in M'\}$. Then $Z_\beta$ is a subsemimodule of $M$.

Let $\alpha : M' \longrightarrow M$ be a homomorphism of $S$-semimodules. Define
$J_\alpha = \{m \in M \mid m + \alpha(m'_1) = \alpha(m'_2) \quad \text{for some } m'_1, m'_2 \in M'\}$ and
$\bar{J}_\alpha = \{m \in M \mid m + \alpha(m'_1) + m_1 = \alpha(m'_2) + m_1 \quad \text{for some } m'_1, m'_2 \in M' \text{ and } m_1 \in M\}$. Then $J_\alpha$ and $\bar{J}_\alpha$ are subsemimodules of $M$.

**Result 2.10.** If a sequence $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is proper exact. Then $Z_\beta = J_\alpha$.

*Proof.* Let $m \in Z_\beta$. Then there exists $m'' \in M''$ such that $\beta(m) + m'' = m''$, or $\beta(m) + m'' = \beta(0) + m''$, implies $(m, 0) \in K_\beta = I_\alpha$ (by proper exactness), therefore $m + \alpha(m'_1) = \alpha(m'_2)$ for some $m'_1, m'_2 \in M'$, this implies $m \in J_\alpha$. So $Z_\beta \subseteq J_\alpha$. Let $m \in J_\alpha$. Then$m + \alpha(m'_1) = \alpha(m'_2)$, or $(m, 0) \in I_\alpha = K_\beta$, implies $\beta(m) + m'' = \beta(0) + m''$ for some $m'' \in M'$, or $\beta(m) + m'' = m''$, implies $m \in Z_\beta$. So$J_\alpha \subseteq Z_\beta$. Hence $Z_\beta = J_\alpha$. □

**Theorem 2.11.** *Let $S$ be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of $S$-semimodules. Then this sequence is exact if there exists a commutative diagram*

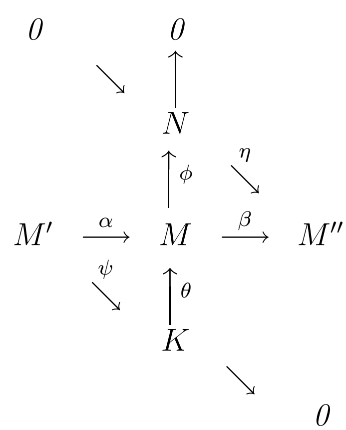


*figure (*)*

*of $S$-semimodules in which the non-horizontal sequences are all exact.*

*Proof.* To show that $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is exact i.e., $K_\beta = \bar{I}_\alpha$. Let $(x_1, x_2) \in K_\beta$. Then for some $x \in M''$, we have $\eta(\phi(x_1)) + x = \eta(\phi(x_2)) + x$ (as $\eta\phi = \beta$) which implies $(\phi(x_1), \phi(x_2)) \in K_\eta$. Since $\eta$ is a monic so $K_\eta = \bar{\Delta}_N$, this gives $\phi(x_1) + n = \phi(x_2) + n$ for some $n \in N$, this implies $(x_1, x_2) \in K_\phi = \bar{I}_\theta$ (by exactness). Therefore

(4) $x_1 + \theta(k_1) + b = x_2 + \theta(k_2) + b$ for some $k_1, k_2 \in K$ and $b \in M$.

Since $\psi$ is an epic therefore for $k_1, k_2$(similary as in equation (2)), we have $k_1 + \psi(m'_1) + a = k_2 + \psi(m'_2) + a$ for some $m'_1, m'_2 \in M'$ and $a \in K$, implies $\theta(k_1) + \theta\psi(m'_1) + \theta(a) + b + x_1 = \theta(k_2) + \theta\psi(m'_2) + \theta(a) + b + x_1$. Using (4) and $\theta\psi = \alpha$, we get $\theta(k_2) + \alpha(m'_1) + \theta(a) + b + x_2 = \theta(k_2) + \alpha(m'_2) + \theta(a) + b + x_1$ implies $(x_1, x_2) \in \bar{I}_\alpha$. So $K_\beta \subseteq \bar{I}_\alpha$. Again, let $(x_1, x_2) \in \bar{I}_\alpha$. Then $x_1 + \alpha(m'_1) + m = x_2 + \alpha(m'_2) + m$ for some $m'_1, m'_2 \in M'$ and some $m \in M$, or $x_1 + \theta(\psi(m'_1)) + m = x_2 + \theta(\psi(m'_2)) + m$, which implies $(x_1, x_2) \in \bar{I}_\theta = K_\phi$ (by exactness). Therefore $\phi(x_1) + x = \phi(x_2) + x$ for some $x \in N$, implies $\beta(x_1) + \eta(x) = \beta(x_2) + \eta(x)$ where$\eta\phi = \beta$, which shows that $(x_1, x_2) \in K_\beta$. Therefore $\bar{I}_\alpha \subseteq K_\beta$. Hence $K_\beta = \bar{I}_\alpha$. □

**Theorem 2.12.** *Let $S$ be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of $S$-semimodules. Then this sequence is proper exact, if there exists a commutative diagram (see figure (*)) of $S$-semimodules in which the non-horizontal sequences are all proper exact.*

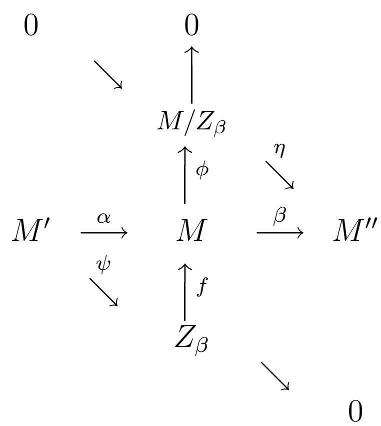
*Proof.* To show $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is proper exact i.e., $K_\beta = I_\alpha$. Since all non horizontal sequences are proper exact therefore they are also exact sequences. By previous theorem, the horizontal sequence is also exact. Therefore$K_\beta = \bar{I}_\alpha$. Now $I_\alpha \subseteq \bar{I}_\alpha = K_\beta$ (by exactness). So $I_\alpha \subseteq K_\beta$. Let $(b_1, b_2) \in K_\beta$. Then for some $x \in M''$, we have $\eta\phi(b_1) + x = \eta\phi(b_2) + x$ (as $\eta\phi = \beta$), which implies $(\phi(b_1), \phi(b_2)) \in K_\eta = \bar{\Delta}_N$ (since $\eta$ is a monic). So $\phi(b_1) + n = \phi(b_2) + n$ for some $n \in N$, which implies $(b_1, b_2) \in K_\phi = \bar{I}_\theta = I_\theta$ (by exactness). Therefore

$$b_1 + \theta(k_1) = b_2 + \theta(k_2) \quad \text{some } k_1, k_2 \in K. \tag{5}$$

Since $\psi$ is an epic (as proper exact is exact) so for $k_1 \in K$, there exist $m'_1, m'_2 \in M'$ such that $k_1 + \psi(m'_1) + k = \psi(m'_2) + k$ for some $k \in K$ for some $m'_1, m'_2 \in M'$ implies $(k_1, 0) \in \bar{I}_\psi = I_\psi$, therefore $k_1 + \psi(m'_1) = \psi(m'_2)$ for some $m'_1, m'_2 \in M'$. So $\psi$ is $i$-regular. Similarly for $k_2 \in K$, we have $k_2 + \psi(m'_3) = \psi(m'_4)$ for some $m'_3, m'_4 \in M'$. Therefore $\theta(k_1) + \theta\psi(m'_1) = \theta\psi(m'_2)$ and $\theta(k_2) + \theta\psi(m'_3) = \theta\psi(m'_4)$. From (5), we have $b_1 + \theta(k_1) + \theta\psi(m'_1) + \theta\psi(m'_3) = b_2 + \theta(k_2) + \theta\psi(m'_1) + \theta\psi(m'_3)$, or $b_1 + \alpha(m'_3 + m'_2) = b_2 + \alpha(m'_4 + m'_1)$ (as $\theta\psi = \alpha$ and using above), this implies$(b_1, b_2) \in I_\alpha$, this shows that $K_\beta \subseteq I_\alpha$. Therefore $K_\beta = I_\alpha$. Hence $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is proper exact. □

**Corollary 2.13.** *Let $S$ be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of $S$-semimodules with $\beta$-being $k$-regular. Then the sequence is proper exact iff there exists a commutative diagram(see figure (*)) of $S$-semimodules in which the non-horizontal sequences are all proper exact.*

*Proof.* Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a proper exact sequence with $\beta$ being $k$-regular. Consider the following diagram:



We define the following mappings

(1) $\phi : M \longrightarrow M/Z_\beta$ such that $\phi(m) = mZ_\beta$, $m \in M$.
It is clearly well defined.

(2) $\eta : M/Z_\beta \longrightarrow M''$ such that $\eta(mZ_\beta) = \beta(m)$, $m \in M$.
It is well defined, as $m_1 Z_\beta = m_2 Z_\beta$ implies $m_1 + x_1 = m_2 + x_2$ for some $x_1, x_2 \in Z_\beta$ (as $Z_\beta$ is congruence on $M$), this implies $\beta(m_1) + \beta(x_1) = \beta(m_2) + \beta(x_2)$. Since $x_1, x_2 \in Z_\beta$ then $\beta(x_1) + x = x$ and $\beta(x_2) + y = y$ for some $x, y \in M''$. Therefore we have $\beta(m_1) + \beta(x_1) + x + y = \beta(m_2) + \beta(x_2) + x + y$ implies $\beta(m_1) + x + y = \beta(m_2) + x + y$ implies $\beta(m_1) = \beta(m_2)$ (as $\beta$ is $k$-regular) which shows that $\eta(m_1 Z_\beta) = \eta(m_2 Z_\beta)$.

(3) $\psi : M' \longrightarrow Z_\beta$ such that $\psi(m') = \alpha(m')$, $m' \in M'$. Clearly, $\alpha(m') \in Z_\beta$ because $\beta\alpha : M' \longrightarrow M''$ is a $Z$-homomorphism, therefore there exists $x \in M''$ such that $\beta(\alpha(m')) + x = x$ implies $\alpha(m') \in Z_\beta$.

(4) $f : Z_\beta \longrightarrow M$ such that $f(x) = x$ for $x \in Z_\beta$.

Diagrams are commutative.

As $\phi(m) = mZ_\beta$ implies $\eta(\phi(m)) = \eta(mZ_\beta) = \beta(m)$ or $\eta\phi(m) = \beta(m)$ for all $m \in M$. Therefore $\eta\phi = \beta$. Again $\psi(m') = \alpha(m')$ implies $f\psi(m') = f(\alpha(m'))$ or $f\psi(m') = \alpha(m')$ for all $m' \in M'$ ($f$ is an inclusion map). Therefore $f\psi = \alpha$. We now show that all non-horizontal sequences are proper exact.

The sequence $Z_\beta \xrightarrow{f} M \xrightarrow{\phi} M/Z_\beta$ is proper exact because it is easily shown that $I_f \subseteq K_\phi$. To show $K_\phi \subseteq I_f$, let $(a, b) \in K_\phi$. Then $\phi(a) + mZ_\beta = \phi(b) + mZ_\beta$ for some $m \in M$, or $aZ_\beta + mZ_\beta = bZ_\beta + mZ_\beta$, or $(a + m)Z_\beta = (b + m)Z_\beta$, implies $a + m + x_1 = b + m + x_2$ for some $x_1, x_2 \in Z_\beta$, implies $\beta(a) + \beta(m) + \beta(x_1) = \beta(b) + \beta(m) + \beta(x_2)$. Since $x_1, x_2 \in Z_\beta$, therefore there exist $x, y \in M''$ such that $\beta(x_1) + x = x$ and $\beta(x_2) + y = y$. Thus we have $\beta(a) + \beta(m) + \beta(x_1) + x + y = \beta(b) + \beta(m) + \beta(x_2) + x + y$, implies $\beta(a) + \beta(m) + x + y = \beta(b) + \beta(m) + x + y$, implies$(a, b) \in K_\beta = I_\alpha$(by proper exactness) which implies $a + \alpha(m_1') = b + \alpha(m_2')$ for some $m_1', m_2' \in M'$, or $a + f\psi(m_1') = b + f\psi(m_2')$ (as $\alpha = f\psi$) implies $(a, b) \in I_f$ which shows that $K_\phi \subseteq I_f$. Thus $K_\phi = I_f$ and hence $Z_\beta \xrightarrow{f} M \xrightarrow{\phi} M/Z_\beta$ is proper exact.

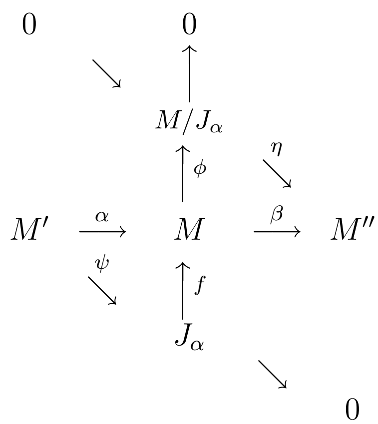
Similarly, it can easily be shown that all other non horizontal sequences are proper exact.

Conversely, see Theorem 2.12. □

**Corollary 2.14.** *Let $S$ be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of $S$-semimodules with $\beta$ being $k$-regular. Then the sequence is exact if and only if there exists a commutative diagram (see figure (*)) of $S$-semimodules in which the non-horizontal sequences are all exact.*

*Proof.* Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be an exact sequence with $\beta$-being $k$-regular.

Consider the following diagram:




We define the following mappings:

(i) $\phi : M \longrightarrow M/J_\alpha$ such that $\phi(m) = mJ_\alpha$, $m \in M$.
It is clearly well defined.

(ii) $\eta : M/J_\alpha \longrightarrow M''$ such that $\eta(mJ_\alpha) = \beta(m)$, $m \in M$.
It is well defined, as $m_1 J_\alpha = m_2 J_\alpha$ implies

$$m_1 + x_1 = m_2 + x_2 \quad \text{for some } x_1, x_2 \in J_\alpha. \tag{6}$$

Since $x_1, x_2 \in J_\alpha$, then $x_1 + \alpha(a_1') = \alpha(a_2')$ for some $a_1', a_2' \in M'$ and $x_2 + \alpha(a_3') = \alpha(a_4')$ for some $a_3', a_4' \in M'$.

Using above two equations in (6), we get $m_1 + x_1 + \alpha(a_1') + \alpha(a_3') = m_2 + x_2 + \alpha(a_1') + \alpha(a_3')$, or $m_1 + \alpha(a_2') + \alpha(a_3') = m_2 + \alpha(a_1') + \alpha(a_4')$, which implies $(m_1, m_2) \in I_\alpha \subseteq \bar{I}_\alpha = K_\beta$. So $\beta(m_1) + x = \beta(m_2) + x$ for some $x \in M''$ implies $\beta(m_1) = \beta(m_2)$ (as $\beta$ is $k$-regular). Therefore $\eta(m_1 J_\alpha) = \eta(m_2 J_\alpha)$

(iii) $\psi : M' \longrightarrow J_\alpha$ such that $\psi(m') = \alpha(m')$. Clearly, $\alpha(m') \in J_\alpha$.

(iv) $f : J_\alpha \longrightarrow M$ such that $f(m) = m$, $m \in J_\alpha$.

As in Corollary 2.13, it can be easily shown that the diagrams are commutative.

Now, the sequence $J_\alpha \xrightarrow{f} M \xrightarrow{\phi} M/J_\alpha$ is exact (by Result 2.2) and it can easily shown that all other non horizontal sequences are exact.

Conversely, see Theorem 2.11. □

## 3. Projective semimodules

In this section, we study some results on projective semimodules. We use Hom functors to prove some results analogous to those in module theory.

**Definition 3.1** (Projective semimodule)**.** A $S$-semimodules $P$ is called projective if it satisfies the following two properties:

(1) If a $S$-homomorphism $f : A \longrightarrow B$ be an epic and $g : P \longrightarrow B$ be $S$-homomorphsim then there exists a $S$-homomorphism $\phi : P \longrightarrow A$ such that

$$\begin{array}{ccc} & & \text{P} \\ & \overset{\phi}{\swarrow} & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

$f \circ \phi = g$.

(2) To every $k$-regular $S$-homomorphism $f : A \longrightarrow B$ and to every $S$-homomorphims $\psi_1, \psi_2 : P \longrightarrow A$ with $f \circ \psi_1 = f \circ \psi_2$, there exist $S$-homomorphisms $k_1, k_2 : P \longrightarrow A$ such that $f \circ k_1$ and $f \circ k_2$ are $Z$-homomorphisms and $\psi_1 + k_1 = \psi_2 + k_2$.

**Lemma 3.2.** *Let $P$ be projective semimodule. If $A \xrightarrow{f} B \xrightarrow{g} C$ is exact. Then for every $S$-homomorphism $\phi : P \longrightarrow B$ with $g \circ \phi$ is a $Z$-homomorphism, there exists a $S$-homomorphism $\phi' : P \longrightarrow A$ with $f \circ \phi' = \phi$.*

$$\begin{array}{ccccc} & & \text{P} & & \\ & \overset{\phi'}{\swarrow} & \downarrow \phi & & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

*Proof.* Let $b \in \bar{J}_\phi$. Then there exist $p_1, p_2 \in P$ and some $b_1 \in B$ such that $b + \phi(p_1) + b_1 = \phi(p_2) + b_1$implies

$$g(b) + g\phi(p_1) + g(b_1) = g\phi(p_2) + g(b_1)\,. \tag{7}$$

Since $g \circ \phi$ is a $Z$-homomorphism, then there exist $x, y \in C$ such that $g\phi(p_1) + x = x$ and $g\phi(p_2) + y = y$. Using this in (7), we have $g(b) + x + y + g(b_1) = x + y + g(b_1)$, implies $(b, 0) \in K_g = \bar{I}_f$ (by exactness). So $b + f(a_1) + b_1 = f(a_2) + b_1$ some $a_1, a_2 \in A$ and $b_1 \in B$ which implies $b \in \bar{J}_f$. Hence $\bar{J}_\phi \subseteq \bar{J}_f$. Consider the diagram:

$$\begin{array}{ccccc} & & P & & \\ & \overset{\phi'}{\swarrow} & \downarrow \phi & & \\ A & \xrightarrow{f} & \bar{J}_f & \longrightarrow & 0 \end{array}$$

Clearly $f$ is an epic. By projectivity of $P$, there exists $S$-homomorphism $\phi' : P \longrightarrow A$ such that $f \circ \phi' = \phi$. □

**Proposition 3.3.** *Let $P$ be projective $S$-semimodule. Then the following holds:*

(i) *If $A \xrightarrow{f} B \xrightarrow{g} C$ is exact and $g$ is $k$-regular, then the sequence* $\mathrm{Hom}(P, A) \xrightarrow{\bar{f}} \mathrm{Hom}(P, B) \xrightarrow{\bar{g}} \mathrm{Hom}(P, C)$ *is exact.*

(ii) *If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact and $f$ and $g$ are $k$-regular, then* $0 \longrightarrow \mathrm{Hom}(P, A) \xrightarrow{\bar{f}} \mathrm{Hom}(P, B) \xrightarrow{\bar{g}} \mathrm{Hom}(P, C) \longrightarrow 0$ *is exact.*

*Proof.* (i) To show $\mathrm{Hom}(P, A) \xrightarrow{\bar{f}} \mathrm{Hom}(P, B) \xrightarrow{\bar{g}} \mathrm{Hom}(P, C)$ is exact i.e. $K_{\bar{g}} = \bar{I}_{\bar{f}}$. Now since $g \circ f$ is a $Z$-homomorphism and $g$ is $k$-regular, $g \circ f = 0 = \bar{g} \circ \bar{f}$. So $\bar{g} \circ \bar{f}$ is a $Z$-homomorphism. Therefore by Result 1.1, $\bar{I}_{\bar{f}} \subseteq K_{\bar{g}}$. Again, let $(\theta_1, \theta_2) \in K_{\bar{g}}$ where $\theta_1, \theta_2 \in \mathrm{Hom}(P, B)$. Then $\bar{g}(\theta_1) + \theta = \bar{g}(\theta_2) + \theta$ for some $\theta \in \mathrm{Hom}(P, C)$, or $g(\theta_1) + \theta = g(\theta_2) + \theta$, implies $g(\theta_1) = g(\theta_2)$ (as $g$ is $k$-regular). By projectivity of $P$, there exist $S$-homomorphisms $k_1, k_2 : P \longrightarrow B$ such that $g \circ k_1$ and $g \circ k_2$ are $Z$-homomorphisms and $\theta_1 + k_1 = \theta_2 + k_2$. By Lemma 3.2,

$$\begin{array}{ccccc} & & P & & \\ & & {\scriptstyle \theta_1,\theta_2}\big\downarrow{\scriptstyle k_1,k_2} & & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

there exist $S$-homomorphisms $\phi_1, \phi_2 \in \mathrm{Hom}(P, A)$ such that $f \circ \phi_1 = k_1$ and $f \circ \phi_2 = k_2$. Adding these two, we get $k_1 + f \circ \phi_2 = k_2 + f \circ \phi_1$, or $\theta_1 + k_1 + \theta_2 + \bar{f}(\phi_2) = \theta_2 + k_2 + \theta_1 + \bar{f}(\phi_1)$ implies $(\theta_1, \theta_2) \in \bar{I}_{\bar{f}}$. So $K_{\bar{g}} \subseteq \bar{I}_{\bar{f}}$. Hence $K_{\bar{g}} = \bar{I}_{\bar{f}}$.

(ii) Since $0 \rightarrow A \xrightarrow{f} B$ is exact and $f$ is $k$-regular. By (i), $\mathrm{Hom}(P, 0) \longrightarrow \mathrm{Hom}(P, A) \xrightarrow{\bar{f}} \mathrm{Hom}(P, B)$ is exact. Therefore $0 \longrightarrow \mathrm{Hom}(P, A) \xrightarrow{\bar{f}} \mathrm{Hom}(P, B)$ is exact. Again $B \xrightarrow{g} C \longrightarrow 0$ is exact and $0$ is $k$-regular, therefore by (i), $\mathrm{Hom}(P, B) \xrightarrow{\bar{g}} \mathrm{Hom}(P, C) \longrightarrow \mathrm{Hom}(P, 0)$ is exact. So $\mathrm{Hom}(P, B) \xrightarrow{\bar{g}} \mathrm{Hom}(P, C) \longrightarrow 0$ is exact. □

**Corollary 3.4.** *If left $S$-semimodule $P$ be projective, then for every proper exact sequence of left $S$-semimodules $M \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ with $\beta$ being $k$-regular and $\alpha$ being $i$-regular, the sequence*

$$\mathrm{Hom}(P, M') \xrightarrow{\bar{\alpha}} \mathrm{Hom}(P, M) \xrightarrow{\bar{\beta}} \mathrm{Hom}(P, M'')$$

*is proper exact.*

*Proof.* The proof is immediate from Proposition 3.3 and Result 2.8. □

The authors wish to express their deep gratitude to the Director, SAG, DRDO, Delhi, India for his support.

## References

[1] D. R. Latorre (1965): On h -ideals and k -ideals in Hemirings, Publ. Math. Debrecen, 12, 219

[2] H.M.J. Al-Thani (1995): A note on projective semimodule, Kobe J. Math., 12, No.2, 89–94.

[3] H.M.J. Al-Thani (2002): Characterizations of projective and $k$-projective semimodule, Internat. J. Math. and Math. Sciences, 32, 7, 439–448.

[4] J.S. Golan (1992): The theory of semirings with application in mathematics and theoretical computer sciences, Pitman Monographs and Surveys in Pure and Applied Mathematics, 54, Longman Sci. Tech. Horlow.

[5] M.R. Adhikari and P. Mukhopadhyay (2002): Exact sequences of semimodules, Bull. Cal. Math. Soc., 94 (1), 23–32.

[6] M. Takahasi (1981): On the Bordism Catagories II, Elementary properties of semomodules, Math. Sem. Notes, 9, 495–530.

[7] M. Takahasi (1983): Extensions of semimodules II, Math. Sem. Notes, 11, 83–118.

[8] S.K. Bhambri and Manish Kant Dubey(2010) : Extensions of Semimodules and Injective Semimodules, South East Asian Bulletine of Mathematics, 34(1), 25–41.

**Received: February, 2010**