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On Generalized Derivations of Semiprime Rings

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Abstract. Let F be a commuting generalized derivation, with associated derivation d, on a semiprime ring R. We show that $d(x)[y,z] = 0$ for all $x, y, z \in R$ and d is central. We define and characterize dependent elements of F and investigate a decomposition of R relative to F.

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1. Introduction and preliminaries

Some researchers have studied the notion of free action on operator algebras (see [11,15] and references therein). Kallman [9] studied the notion of free action of automorphisms on von Neumann algebras by using implicity the dependent elements of an automorphism. Dependent elements of automorphisms were later studied by Choda et al. [5] in the context of C^*-algebras. Several other authors have studied dependent elements of automorphisms in the context of operator algebras (see [6,12,14] and references therein).

It is known that all C^* algebras are semiprime rings and a von Neumann algebra is prime if and only if its centre consists of the scalar multiples of identity. Thus a natural extension of the notion of a dependent element of a mapping on a C^*-algebra or a von Neumann algebra is the study of this notion in the context of semiprime rings.

Laradji and Thaheem [10] initiated the study of dependent elements of endomorphisms of semiprime rings and generalized a number of results of [5] for semiprime rings. Recently, Ali and Chaudhry [2], Vukman and Kosi-Ulbl [16] and Vukman [17,18] have made further study of dependent elements of some mappings on prime and semiprime rings.

Throughout, R will represent an associative ring with centre $Z(R)$. The commutator $xy-yx$ will be denoted by $[x, y]$. We will use the basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = y[x, z] + [x, y]z$. Recall that a ring R is semiprime if $aRa = 0$ implies $a = 0$ and is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \to R$ is called a derivation on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $f : R \to R$ is called commuting if $[f(x), x] = 0$ for all $x \in R$. It is called central if $f(x) \in Z(R)$ for all $x \in R$. Let $a \in R$, then the mapping $d : R \to R$ given by $d(x) = [a, x]$ is a derivation on R. It is called inner derivation on R.

During the past few decades derivations as well as commuting derivations have been extensively studied by researchers in the context of operator algebras, prime rings and semiprime rings (see [3] and references therein). Moreover, a lot of work has been done in operator algebras regarding mappings of the type $F(x) = ax + xb$, a, b are fixed elements of the algebra. Such mappings are called elementary operators. Obviously $F(xy) = axy + xyb = axy+xby+xyb-xby = F(x)y+x[y, b]$. Motivated by this observation, Brešar [4] and Hvala [8] introduced the notion of a generalized derivation. They have studied some properties of such derivations. An additive mapping F of R into itself is called a generalized derivation of R, with associated derivation d, if there is a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Obviously this notion covers the notion of a derivation (in case $F = d$) and a left centralizer (in case $d = 0$). An additive mapping $F : R \to R$ is called a left centralizer if $F(xy) = F(x)y$ for all $x, y \in R$. Thus it is interesting to investigate properties of generalized derivations in the context of semiprime rings. For some properties of generalized derivations we refer to [1,8] and references therein.

Motivated by the work of Laradji and Thaheem [10], Vukman and Kosi-Ulbl [16], and Vukman [17,18] on dependent elements of mappings of semiprime rings and by the work done by various researchers on commuting derivations and commuting generalized derivations on prime and semiprime rings, we investigate some properties of commuting generalized derivations on semiprime rings and their dependent elements.

Motivated by the definition of elementary operator $F(x) = ax + xb = ax - xa + xa + xb = [a, x] + x(a + b)$, which implies $F(x)a = [a, x]a + x(a + b)a$, we call elements $a, b \in R$ dependent elements of a generalized derivation $F : R \to R$ if $F(x)a = [a, x]a + x(a + b)a$ holds for all $x \in R$. Following [5], we say that a generalized derivation $F : R \to R$ acts freely on R (or is a free action) if $a = 0$, whenever a, b are its dependent elements.

We show that the associated derivation d of a commuting generalized derivation F on a semiprime ring R is central, that is, $d(x) \in Z(R)$ for all $x \in R$. We also characterize the dependent elements of F and show that if a, b are dependent elements of F, then there exist ideals U and V of R such that $U \oplus V$ is an essential ideal of R, $F(u) = (a + b)u$ for all $u \in U$, $F(V) \subseteq V$ and F acts freely on V.

It is known that a semiprime ring R has no central nilpotent elements. Further, the left and right annihilators of an ideal U of a semiprime ring coincide. We will denote it by $Ann(U)$. It is known that $U \oplus Ann(U)$ is an essential ideal. If R is semiprime, then $aRb = 0$ implies $ab = 0$ and $ba = 0$. We will use these facts without further mention.

2. Results

We now prove our results. First we state the following lemma which will be used in the sequel.

Lemma 2.1 [7, Lemma 1.1.4 (page 6)]. Suppose R is a semiprime ring and $a \in R$ is such that $a[a, x] = 0$ for all $x \in R$. Then $a \in Z(R)$.

Theorem 2.2. Let R be a semiprime ring and F a generalized derivation of R, with associated derivation d. If F is commuting on R, then $d(x)[y, z] = 0$ for all $x, y, z \in R$ and $d(x) \in Z(R)$ for all $x \in R$.

Proof. By hypothesis

(1) $$[F(x), x] = 0 \;\; for \;\; all \;\; x \in R.$$

Linearizing (1) in x, we get

(2) $$[F(x), y] + [F(y), x] = 0 \;\; for \;\; all \;\; x, y \in R.$$

Replacing y by yx in (2), we get $[F(x), yx]+[F(yx), x] = 0$. That is, $[F(x), y]x+ y[F(x), x] + [F(y)x + yd(x), x] = 0$. The last equation along with (1) implies $[F(x), y]x + [F(y), x]x + y[d(x), x] + [y, x]d(x) = 0$, which along with (2) gives

(3) $$y[d(x), x] + [y, x]d(x) = 0 \;\; for \;\; all \;\; x, y \in R.$$

Replacing y by zy in (3), we get $zy[d(x), x]+[zy, x]d(x) = 0$. which along with (3) implies

(4) $$[z, x]yd(x) = 0 \;\; for \;\; all \;\; x, y, z \in R.$$

Since R is semiprime, therefore (4) implies

(5) $$d(x)[z, x] = 0 \;\; for \;\; all \;\; x, z \in R.$$

Linearizing (5) in x and then using (5), we get $d(y)[z, x] + d(x)[z, y] = 0$, which implies

(6) $$d(x)[z, y] = -d(y)[z, x] \;\; for \;\; all \;\; x, y, z \in R.$$

Replacing z by uz in (5) and then using (5), we get

(7) $$d(x)u[z, x] = 0 \;\; for \;\; all \;\; x, u, z \in R.$$

Replacing u by $[z, y]u(-d(y))$ in (7), we get $d(x)[z, y]u(-d(y))[z, x] = 0$, which along with (7) and semiprimeness of R implies

(8) $$d(x)[z, y] = 0 \;\; for \;\; all \;\; x, y, z \in R.$$

From (8) and Lemma 2.1, we get $d(x) \in Z(R)$ for all $x \in R$. □

Theorem 2.3. Let F be a commuting generalized derivation, with associated derivation d, of R. Then a, b are dependent elements of F if and only if (i) $a \in Z(R)$, (ii) $ab \in Z(R)$ and (iii) $(F(x) - (a + b)x)a = 0 \;\; for \;\; all \;\; x \in R$.

Proof. Since a, b are dependent elements of F, therefore

(9) $$F(x)a = [a, x]a + x(a + b)a \;\; for \;\; all \;\; x \in R.$$

Replacing x by xy in (9), we get $F(xy)a = [a, xy]a+xy(a+b)a$, which implies $F(x)ya + xd(y)a = x[a, y]a + [a, x]ya + xy(a + b)a$. The last equation along with (9) implies $F(x)ya + xd(y)a = xF(y)a + [a, x]ya$, which gives

(10) $$xd(y)a = [a, x]ya + xF(y)a - F(x)ya \;\; for \;\; all \;\; x, y \in R.$$

Replacing y by xz in (10), we get $xd(xz)a = [a, x]xza + xF(xz)a - F(x)xza$, which implies $x^2d(z)a + xd(x)za = [a, x]xza + xF(x)za + x^2d(z)a - F(x)xza$.

The last equation along with $[F(x), x] = 0$ for all $x \in R$, gives

(11) $$xd(x)za = [a, x]xza \ \ for \ \ all \ \ x, z \in R.$$

Replacing y by x in (10) and then using $[F(x), x] = 0$, we get $xd(x)a = [a, x]xa$, which implies

(12) $$xd(x)az = [a, x]xaz \ \ for \ \ all \ \ x, z \in R.$$

Subtrecting (11) from (12), we get

(13) $$xd(x)[a, z] = [a, x]x[a, z] \ \ for \ \ all \ \ x, z \in R.$$

Using (8), from (13) we get

(14) $$[a, x]x[a, z] = 0 \ \ for \ \ all \ \ x, z \in R.$$

Replacing z by uz in (14) and then using (14) we get

(15) $$[a, x]xu[a, z] = 0 \ \ for \ \ all \ \ x, u, z \in R.$$

Replacing z by x in (15) and then multiplying by x on the right, we get $[a, x]xu[a, x]x = 0$, which along with semiprimeness of R implies

(16) $$[a, x]x = 0 \ \ for \ \ all \ \ x \in R.$$

Replacing x by $a+x$ in (16) and then using (16), we get $[a, x]a = 0$. Replacing x by xz in last equation and then using it we get $[a, x]za = 0$ for all $x, z \in R$, which by semiprimeness of R implies $a[a, x] = 0$ for all $x \in R$. Using Lemma 2.1, we get $a \in Z(R)$. This proves (i).

Since $a \in Z(R)$, therefore (9) implies

(17) $$F(x)a = x(a + b)a \ \ for \ \ all \ \ x \in R.$$

Replacing x by xy in (17), we get $F(xy)a = xy(a+b)a$, which implies $F(x)ya+xd(y)a = xy(a + b)a$. That is,

(18) $$F(x)ay + xd(y)a = xy(a + b)a \ \ for \ \ all \ \ x, y \in R.$$

Using (17), from (18) we get $x(a + b)ay + xd(y)a = xy(a + b)a$. That is, $xd(y)a = x[y,(a + b)a] = x[y, a + b]a + x(a + b)[y, a]$ for all $x, y \in R$. Since $a \in Z(R)$, therefore the last equation along with semiprimeness of R implies $d(y)a = [y, a + b]a = [y, a]a + [y, b]a = [y, b]a$. That is,

(19) $$d(y)a = [y, b]a \ \ for \ \ all \ \ y \in R.$$

From (19), we get $d(y)ad(y)a = d(y)a[y, b]a = ad(y)[y, b]a$, which along with (8) implies $d(y)ad(y)a = (d(y)a)^2 = 0$. Since $a \in Z(R)$ and $d(y) \in Z(R)$ by Theorem 2.2, therefore $d(y)a \in Z(R)$ and R being semiprime has no nonzero central nilpotents, therefore

(20) $$d(y)a = 0 \ \ for \ \ all \ \ y \in R.$$

From (19) and (20) we get $[y,b]a = 0$, which implies $[y,ba] = 0$. Thus $ba \in Z(R)$. Since $a \in Z(R)$, therefore $ba = ab \in Z(R)$. This proves (ii).

Replacing y by $d(y)$ in (20), we get

$$d^2(y)a = 0 \quad for \ all \ \ y \in R. \tag{21}$$

Further (20) implies $d(d(y)a) = d(0) = 0$, which along with (21) gives $d(y)d(a) = 0$, in particular $(d(a))^2 = 0$. Since $d(a) \in Z(R)$ by Theorem 2.2 and R is semiprime, therefore

$$d(a) = 0. \tag{22}$$

Now $d(ab) = d(a)b + ad(b)$, which along with (20) and (22) implies

$$d(ab) = 0. \tag{23}$$

Now using (20) and (22), we get $F(xa) = F(x)a + xd(a) = F(x)a$ and $F(ax) = F(a)x + ad(x) = F(a)x$. Since a and $ab \in Z(R)$, therefore $a^2 + ab = a(a+b) \in Z(R)$. Moreover the last two relations along with (17) give $F(a)x = F(x)a = x(a+b)a = a(a+b)x = (a+b)xa$. Thus $(F(x) - (a+b)x)a = F(x)a - (a+b)xa$, which implies

$$(F(x) - (a+b)x)a = 0 \quad for \ all \ \ x \in R. \tag{24}$$

This proves (iii).

Conversely let F be a generalized derivation of R. Let $a, b \in R$ satisfy (i), (ii) and (iii). Then $(F(x) - (a+b)x)a = 0$, implies $F(x)a = (a+b)xa = a(a+b)x = xa(a+b) = x(a+b)a = [a,x]a + x(a+b)a$. Thus a, b are dependent elements of F. □

Remark 2.4. It is known that if A is an ideal of a semiprime ring R then A is a semiprime subring of R and $Z(A) \subseteq Z(R)$.

We now investigate a decomposition of R using dependent elements a and b of a generalized derivation F.

Theorem 2.5. Let F be a commuting generalized derivation, with associated derivation d. Let a, b be dependent elements of F, then there exist ideals U and V of R such that

(a) $U \oplus V$ is an essential ideal of R

(b) $F(u) = (a+b)u$ for all $u \in U$, and

(c) $F(V) \subseteq V$ and F acts freely on V.

Proof. (a) Since $a \in Z(R)$, therefore aR is a two sided ideal of R. Let $U = aR$ and $V = Ann(U)$. Then $U \oplus V$ is an essential ideal of R.

(b) We consider $F(ax) = F(xa) = F(x)a + xd(a)$, which along with (22) implies $F(ax) = F(x)a$. Since a, b are dependent elements of F, therefore the last equation implies $F(ax) = [a, x]a + x(a + b)a = a(a + b)x = (a + b)ax$. Thus $F(u) = (a + b)u$ for all $u \in U$.

(c) Let $y \in V$. Then $yax = 0$ for all $x \in R$. Let $F \setminus V$ and $d \setminus V$ be restrictions of F and d on V, respectively. Now from (20)we conclude that $d \setminus V(y)a = 0$ for all $y \in V$, which implies $d \setminus V(y)ax = 0$ for all $x \in R$. Thus $d \setminus V(y) \in V$ for all $y \in V$. Further semiprimeness of R and $yax = 0$ for all $x \in R$ implies $ya = 0$. Thus

(25) $$ya = ay = 0 \;\; for \;\; all \;\; y \in V.$$

Hence $F(ya) = 0$, which implies $F(y)a + yd(a) = 0$. The last relation along with (22) implies $F(y)a = 0$. That is, $F(y)ax = 0$ for all $x \in R,\; y \in V$. Thus $F(y) \in V$ for all $y \in V$. So $F \setminus V(V) \subseteq V$ and $d \setminus V(V) \subseteq V$. By Remark 2.4, V is a semiprime ring. Thus $F \setminus V$ is a generalized derivation on V with associated derivation $d \setminus V$.

Let c, d be dependent elements of $F \setminus V$ in V. Then by Theorem 2.3 $c, cd \in Z(V) \subseteq Z(R)$, $cd = dc$ and $d \setminus V(y)c = 0$. Now from (17), we get $F \setminus V(y)c = y(c+d)c$ for all $y \in V$. Let $x \in R$, then $xy \in V$. Thus $F \setminus V(xy)c = xy(c+d)c$, which gives $F(xy)c = xy(c+d)c$. That is, $F(x)yc+xd(y)c = xy(c+d)c$, which implies $F(x)yc+xd\setminus V(y)c = xy(c+d)c$. Since $d\setminus V(y)c = 0$, therefore the last relation implies $F(x)yc = xy(c+d)c$. That is, $F(x)cy = x(c+d)cy = xc(c+d)y$ for all $y \in V$. Since $F(x)c$ and $xc(c+d) \in V$ and V is a semiprime ring so last relation implies $F(x)c = xc(c + d) = x(c + d)c = [c, x]c + x(c + d)c$ because $c \in Z(R)$. Hence c, d are dependent elements of F in R. Thus from (25) we get $cy = 0$ for all $y \in V$. Since $c \in V$ and V is semiprime, therefore $c = 0$. Hence $F \setminus V$ is a free action on V. □

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