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On PIF-Rings

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Abstract

In this paper we introduce and investigate a class of those rings in which every projective ideal is free. We establish the transfer of this notion to the trivial ring extension and direct product and then generate new and original families of rings satisfying this property.

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1 Introduction

Throughout this paper, all rings are commutative with identity elements, and all modules are unitary.

A ring R is called a PIF-ring if every projective ideal is free. A local rings and Bézout domains are examples of PIF-rings. Also, every von Neumann regular ring which is not a field is example of non PIF-rings.

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $p \propto E$, where p is a prime (resp., maximal) ideal of A [6, Theorem 25.1(3)]. Suitable background on commutative trivial ring extensions is [4, 5, 6].

The purpose of this paper is to study the transfer of this notion to the trivial ring extension and direct product. In this line, we provide a new family of examples of non-local PIF-rings and quite far from being Bézout domains.

2 Main Results

In this section, we explores trivial ring extensions of the form $R := D \propto E$, where D is an integral domain and E is a $K(:= qf(D))$-vector space. Notice in this context that $(a, b) \in R$ is regular if and only if $a \neq 0$. The main result (Theorem 2.1) examines the transfer of PIF-property to R and hence generates new examples of non-local PIF-rings with zerodivisors.

Theorem 2.1 *Let D be a domain, $K := qf(D)$ and E be a K-vector space. Let $R := D \propto E$ be the trivial ring extension of D by E. Then R is a P-ring if and only if so is D.*

We need the following Lemma before proving Theorem 2.1.

Lemma 2.2 *Let $T := K \propto E$ be the trivial ring extension of a field K by a K-vector space E. Then there exists no proper flat ideal of T.*

Proof. Let $J := 0 \propto E'$ be a proper ideal of T, where $E'(\subseteq E)$ is a K-vector space. We claim that J is not flat. Deny. Let $\{f_i\}_{i\in I}$ be a basis of the K-vector space E' and consider the T-map $T^{(I)} \xrightarrow{u} J$ defined by $u((a_i, e_i)_{i\in I}) = (0, \sum_{i\in I} a_i f_i)$. Clearly, $Ker(u) = 0 \propto E^{(I)} = (0 \propto E)^{(I)}$. Hence, by [8, Theorem 3.55], we obtain

$$(0 \propto E)^{(I)} = (0 \propto E^{(I)}) \cap (0 \propto E)T^{(I)} = (0 \propto E)^{(I)})(0 \propto E) = 0,$$

a contradiction. Hence, J is not flat and this completes the proof.

Proof of Theorem 2.1.
Assume that R is a PIF-ring and let I be a nonzero projective ideal of D. Then $J := I \otimes_D R = I \propto E$ is a nonzero projective ideal of R. Hence, J is a principal ideal of R generated by a regular element of R since R is a P-ring. Therefore, $J = R(a, e) = Da \propto E$ for some $a(\neq 0) \in D$ and $e \in E$, and so $I = Da \cong D$ as D-modules. Hence, D is a PIF-ring.

Conversely, assume that D is a PIF-ring and let J be a nonzero projective ideal of R. Set $T := K \propto E$ which is a flat R-module since $T = S^{-1}R$, where $S = D - \{0\}$. Hence, $JT(= J \otimes_R T)$ is a nonzero projective ideal of T and so $JT = T = K \propto E$ by Lemma 2.2. Therefore, there exists $(a, e) \in J$ such that $a \neq 0$ which implies that $J = I \propto E$ for some nonzero ideal I of D. We claim that I is a projective ideal of D.
Indeed, for any D-module N, we have by [2, p.118],

$$Ext_D(I, N \otimes_D R) \cong Ext_R(I \otimes_D R, N \otimes_D R) = 0$$

On the other hand, N is a direct summand of $N \otimes_D R$ since D is a direct summand of R. Therefore, $Ext_D(I, N) = 0$ for all D-module N. This means that I is a projective ideal of D.
Therefore, $I = Da$ for some $a(\neq 0) \in I$ since D is a P-ring and so $J = Ra$ and this completes the proof of Theorem 2.1.

Recall that a ring R is called coherent if every finitely generated ideal of R is finitely presented. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, valuation rings, and Bézout domains. See for instance [4].

Theorem 2.1 enriches the literature with new examples of non-local PIF-rings with zerodivisors as shown below.

Example 2.3 *Let Z be the ring of integers, $Q = qf(Z)$ and let $R := Z \propto Q$. Then:*
1) *R is a PIF-ring by Theorem 2.1.*
2) *R is not local by [6, Theorem 25.1(3)] since Z is not local.*
3) *R is not coherent by [5, Theorem 2.8(1)]. In particular R is not Noetherian.*

Now, we gives a class of non PIF-rings.

Proposition 2.4 *Let A be a ring containing an idempotent element $e(\neq 0, 1)$, E an A-module, and let $R := A \propto E$ be the trivial ring extension of A by E. Then R is not a PIF-ring.*

Proof. Let e be an idempotent element of A such that $e \neq 0, 1$. It is easy to show that $R(e, 0) \oplus R(1 - e, 0) = R$ and so $R(e, 0)$ is a projective proper ideal of R. On the other hand, $R(e, 0)$ is not a free ideal of R since $(e, 0)(1 - e, 0) = (0, 0)$. Hence, R is not a PIF-ring, as desired.

Corollary 2.5 *Let A be a von Neumann regular ring which is not a field, E an A-module, and let $R := A \propto E$ be the trivial ring extension of A by E. Then R is not a PIF-ring.*

Now, we show that the PIF-property is not a local property.

Example 2.6 *Let R be a von Neumann regular ring which is not a field. Then:*
1) *R is not a PIF-ring.*
2) *R_M is a PIF-ring for each maximal ideal M of R (since R_M is a field).*

Now, we establish that the finite direct product of rings is never a PIF-ring.

Proposition 2.7 *Let $(R_i)_{i=1,\dots,m}$ be a family of rings. Then $\prod_{i=1}^{m} R_i$ is never a PIF-ring.*

Proof. It suffices to prove the assertion for $m = 2$. Let R_1 and R_2 be two rings. Hence, $R_1 \times 0$ is a projective ideal of $R_1 \times R_2$ which is not free since $(R_1 \times 0)(0, 1) = (0, 0)$ and so $R_1 \times R_2$ is not a PIF-ring, as desired.

We close this paper by this remark.

Remark 2.8 *Let A be a ring and I be an ideal of A. Even if A/I is an PIF-ring and I is a projective principal ideal, then ring A is not, in general, a PIF-ring.*
Indeed, let $A = K \times K$, where K is a field, and let $I := K \times 0$. Then A is not a PIF-ring by the above Proposition, $A/I(= K)$ is a PIF-ring, and $I(= R(0, 1))$ is a principal projective ideal of A.

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