Remarks on Derivations of $\sigma$-Prime Rings

M. Rais Khan, Deepa Arora and M. Ali Khan

Department of Mathematics
Jamia Millia Islamia, Jamia Nagar
New Delhi-110025, India
mohdrais_khan@yahoo.co.in, musk.deepa@gmail.com

Centre for Interdisciplinary Research in Basic Sciences
Jamia Millia Islamia, Jamia Nagar, New Delhi-110025, India
mkhan91@gmail.com

Abstract

Let $R$ be a 2-torsion free $\sigma$-prime ring with involution $\sigma$, $I$ a nonzero $\sigma$-ideal of $R$ and $d: R \rightarrow R$ a nonzero derivation commuting with $\sigma$. In this paper, we first establish that $R$ is commutative if the following conditions: (i) $d(x) \circ x = 0 \ \forall \ x \in R$, (ii) $d(x) \circ d(y) = 0 \ \forall \ x, y \in I$, and (iii) $d(x \circ y) = 0 \ \forall \ x, y \in I$ are satisfied. Moreover, also we prove that $r$ is in $Z(R)$ if $r$ in $Sa_\sigma(R)$ satisfies $d(x) \circ r = 0 \ \forall \ x$ in $I$.

Mathematics Subject Classification: 16W10, 16W25, 16U80

Keywords: Rings with involution, $\sigma$-prime ring, derivations, commutativity

1. INTRODUCTION

Throughout, $R$ will represent an associative ring with center $Z(R)$. Recall that a ring $R$ is a prime if $aRb = 0$ implies $a = 0$ or $b = 0$. If $R$ has an involution $\sigma$, then $R$ is said to be $\sigma$-prime if $aRb = aR\sigma(b) = 0$ implies $a = 0$ or $b = 0$. Every prime ring equipped with an involution is $\sigma$-prime but the converse need not be true in general (for example see [4]). An ideal $I$ of $R$ is said to be a $\sigma$-ideal if $I$ is invariant under $\sigma$(i.e., $\sigma(I) = I$). Let us define the set of symmetric and skew symmetric elements of $R$ as $Sa_\sigma(R) = \{x \in R|\sigma(x) = \pm x\}$. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes the anticommutator $xy + yx$. We will
make use of the following basic commutator identities, for any $x, y, z \in R$,

$$
[x, yz] = y[x, z] + [x, y]z, [xy, z] = x[y, z] + [x, y]z
$$

$$
x o (yz) = (x o y)z - y[x, z] = y(x o z) + [x, y]z
$$

$$
(xy) o z = x(y o z) - [x, z]y = (x o z)y + x[y, z].
$$

Several authors [1, 2, 6] have studied the relationship between the commutativity of a ring and the behavior of a special mapping on that ring. Recently, a major breakthrough has been achieved by Oukhtite et al. [5], where the important results by Posner, Herstein and Bell have been proved for $\sigma$-prime rings. More precisely, Posner’s ([Theorem 2, 6]) of existence of a nonzero centralizing derivation on prime ring which makes the ring commutative if $R$ is a prime ring of characteristic $\neq 2$ with a nonzero derivation $d$ such that $[d(x), d(y)] = 0 \ \forall \ x, y \in R$. Oukhtite et al. ([Theorem 1.2, 5]) proved that same result holds for $\sigma$-prime rings. Motivated by a well known result of Herstein [2], Bell [1] studied derivation $d$ satisfying $d(xy) = d(yx)$ for all $x, y \in R$. This result has been extended for $\sigma$-prime rings ([Theorem 1.3, 5]). The objective of this paper is to extend the results of [3, 5]. Now we define the properties as given below:

$(P_1)$ For any $x \in R$ such that $d(x) o x = 0$

$(P_2)$ For every $x, y \in I$ such that $d(x) o d(y) = 0$

$(P_3)$ For every $x, y \in I$ such that $d(x o y) = 0$

$(P_4)$ For any $x$ in $I$ and $r$ in $Sa_\sigma(R)$ satisfies $d(x) o r = 0$.

2. MAIN RESULTS

Theorem 1.1. Let $d$ be a nonzero derivation of $\sigma$-prime ring $R$ commuting with $\sigma$. Let $R$ satisfy the property $(P_1)$. Then $R$ is commutative.

Theorem 1.2. Let $R$ be a 2 torsion free $\sigma$-prime ring and $I$ a nonzero $\sigma$-ideal of $R$. Let $R$ satisfies $(P_2)$ and admit a nonzero derivation $d$ that commutes with $\sigma$. Then $R$ is commutative.

Theorem 1.3. Let $R$ be a 2 torsion free $\sigma$-prime ring and $I$ a nonzero $\sigma$-ideal of $R$. Let $R$ satisfies $(P_3)$ and admit a nonzero derivation $d$ that commutes with $\sigma$. Then $R$ is commutative.
Remarks on derivations of $\sigma$-prime rings

**Theorem 1.4.** Let $0 \neq d$ be a derivation of $R$ and $I$ a nonzero $\sigma$-ideal of $R$ satisfies $(P_4)$, then $r \in Z(R)$.

In order to prove our results we need the following lemmas.

**Lemma 2.1** ([3, Lemma 3.1]) Let $R$ be a $\sigma$-prime ring and let $I$ be a nonzero $\sigma$-ideal of $R$. If $a, b$ in $R$ are such that $aIb = 0 = aI\sigma(b)$, then $a = 0$ or $b = 0$.

**Lemma 2.2** ([5, Lemma 2.2]) Let $I$ be a nonzero $\sigma$-ideal of $R$ and $0 \neq d$ be a derivation on $R$ which commutes with $\sigma$. If $[x, R] Id(x) = 0$ for all $x \in I$, then $R$ is commutative.

**Lemma 2.3** ([5, Lemma 2.3]) Let $I$ be a nonzero $\sigma$-ideal of $R$. If $R$ admits a derivation $d$ such that $d^2(I) = 0$ and $d$ commutes with $\sigma$ on $R$ then $d = 0$.

**Proof of Theorem 1.1.** By our hypothesis $(P_1)$, we have

$$d(x) \circ x = 0 \quad \forall \ x \in R. \tag{1}$$

Linearising (1), we get

$$d(x) \circ y + d(y) \circ x = 0 \quad \forall \ x, y \in R. \tag{2}$$

Replacing $y$ by $xz$ in (2), we have

$$x(d(x) \circ z) + [d(x), x]z + (d(x)z) \circ x + (xd(z)) \circ x = 0,$$

or

$$x(d(x) \circ z) + [d(x), x]z + (d(x) \circ x)z + d(x)[z, x] + x(d(z) \circ x) - [x, x]d(z) = 0$$

Using (1) and (2) in the above obtained relation, we get

$$[d(x), x]z + d(x)[z, x] = 0 \quad \forall \ x, z \in R. \tag{3}$$

Replacing $z$ by $zy$ in (3) and using (3), we get

$$d(x)z[y, x] = 0 \quad \forall \ x, y, z \in R.$$

Hence, $d(x)R[y, x] = 0 \quad \forall \ x, y \in R.$

As an application of the result in ([5, Theorem 1.1]), yields $R$ is commutative.

**Proof of Theorem 1.2.** From the hypothesis of $(P_2)$, we write

$$d(x) \circ d(y) = 0 \quad \forall \ x, y \in I \tag{4}$$
Replacing $y$ by $xy$ in (4), we have

$$d(x)d(x)y + d(x)xd(y) + d(x)yd(x) + xd(y)d(x) = 0 \quad \forall \ x, y \in I \tag{5}$$

Taking $y = x$ in (4), we have $d(x)d(x) = -d(x)d(x)$. Using above relation in (5), we have

$$-d(x)d(x)y + d(x)xd(y) + d(x)yd(x) - xd(y)d(x) = 0$$

or

$$d(x)[y, d(x)] + [d(x), x]d(y) = 0 \quad \forall \ x, y \in I. \tag{6}$$

For any $r \in R$, replacing $y$ by $yr$ in (6) and employing (6), we obtain

$$d(x)y[r, d(x)] + [d(x), x]yd(r) = 0 \quad \forall \ x, y \in I. \tag{7}$$

Replacing $r$ by $rd(x)$ in (7), we get

$$d(x)yrd(x)d^2(x) = 0 \quad \forall \ x, y \in I \text{ and } r \in R.$$

Using (7) in the above obtained relation, we have

$$[d(x), x]yd^2(x) = 0 \quad \forall \ x, y \in I \text{ and } r \in R.$$ 

Hence,

$$[d(x), x]Id^2(x) = 0 \quad \forall \ x \in I.$$

As $d$ commutes with $\sigma$ and $I$ is a $\sigma$-ideal, we have

$$[d(x), x]Id^2(x) = \sigma([d(x), x])Id^2(x) = 0 \quad \forall \ x \in I.$$

Lemma 2.1 gives $[d(x), x] = 0$ or $d^2(x) = 0 \quad \forall \ x \in I$. If $d^2(x) = 0 \quad \forall \ x \in I$, then by Lemma 2.3 we get $d = 0$, a contradiction.

Next, suppose that $[d(x), x] = 0 \quad \forall \ x \in I$. Then, in view of the result ([5, Theorem 1.2]) one gets $R$ is commutative.

**Proof of Theorem 1.3.** By the hypothesis of (P₃), we have

$$d(x o y) = 0 \quad \forall \ x, y \in I. \tag{8}$$

Replacing $y$ by $xy$ in (8), we get

$$0 = d(x o xy) \quad \forall \ x, y \in I$$

$$= d(x(x o y) + [x, x]y)$$

$$= d(x)(x o y) + xd(x o y).$$
Remarks on derivations of $\sigma$-prime rings

This implies that

$$d(x)(x \circ y) = 0 \ \forall \ x, y \in I. \quad (9)$$

For any $r \in R$, replacing $y$ by $yr$ in (9), we obtain

$$d(x)((x \circ y)r - y[x, r]) = 0.$$  

Using (9) in the above relation we get,

$$d(x)y[x, r] = 0 \ \forall \ x, y \in I, r \in R.$$  

Hence,

$$d(x)I[x, r] = 0 \ \forall \ x \in I, r \in R.$$  

Particularly, we can write $0 = d(\sigma(x))I[\sigma(x), \sigma(r)] = \sigma(d(x))I\sigma([x, r]) \ \forall \ x \in I, r \in R$ as $d$ commutes with $\sigma$. 

Applying $\sigma$ to this last equality, we get

$$[x, r]Id(x) = 0 \ \forall \ x \in I, r \in R.$$  

In view of Lemma 2.2, $R$ is commutative.

**Proof of Theorem 1.4.** From the hypothesis of (P₄), we have

$$d(x) \circ r = 0 \ \forall \ x \in I \text{ and } r \in S_{\sigma}(R). \quad (10)$$

Replacing $x$ by $xy$ in (10), we get

$$(d(x)y + xd(y)) \circ r = 0$$

or

$$(d(x)y) \circ r + (xd(y)) \circ r = 0$$

or

$$(d(x) \circ r)y + d(x)[y, r] + x(d(y) \circ r) - [x, r]d(y) = 0$$

Using (10) in the above obtained relation, we have

$$d(x)[y, r] - [x, r]d(y) = 0 \ \forall \ x, y \in I \text{ and } r \in S_{\sigma}(R). \quad (11)$$

Replacing $y$ by $yr$ in (11) and using (11), we obtain

$$[x, r]yd(r) = 0 \ \forall \ x, y \in I \text{ and } r \in S_{\sigma}(R).$$

Hence,

$$[x, r]Id(r) = 0 \ \forall \ x \in I \text{ and } r \in S_{\sigma}(R).$$

The fact that $I$ is a $\sigma$-ideal together with $r$ in $S_{\sigma}(R)$ gives

$$\sigma([x, r])Id(r) = [x, r]Id(r) = 0. \quad (12)$$
From Lemma 2.1 we get, either \( d(r) = 0 \) or \([x, r] = 0 \) \( \forall \ x \) in \( I \) and \( r \) in \( Sa_\sigma(R) \).

Let \( r \in R \) and \( r + \sigma(r) \in Sa_\sigma(R) \). Then by equation (12) we have

\[
d(r + \sigma(r)) = 0 \quad \text{or} \quad [x, r + \sigma(r)] = 0.
\]

If \( d(r + \sigma(r)) = 0 \), then \( d(r) \in Sa_\sigma(R) \). In view of (12) together with Lemma 2.1, yield \( d(r) = 0 \) or \([x, r] = 0 \).

Next, suppose that \([x, r + \sigma(r)] = 0 \)\( \forall \ x \in I \) and \( r \in Sa_\sigma(R) \).

Since \( ch(R) \neq 2 \), we get \([x, r] = 0 \).

If \( d(r - \sigma(r)) = 0 \), then \( d(r) \in Sa_\sigma(R) \). Similarly, (12) gives \( d(r) = 0 \) or \([x, r] = 0 \).

In both cases, we have \( R = G \cup H \), where \( G = \{r \in R | d(r) = 0 \} \) and \( H = \{r \in R | [x, r] = 0, \ \forall \ x \in I \} \). But a group cannot be a union of two of its proper subgroups. Thus, \( R = G \) or \( R = H \). This implies that either \( d(r) = 0 \) \( \forall \ r \in R \) or \([x, r] = 0 \) \( \forall \ x \in I \). Since \( d \) is a nonzero derivation so \( d(r) \neq 0 \), it implies

\[
[x, r] = 0 \quad \forall \ x \in I.
\]  \( (13) \)

Replace \( x \) by \( tx \) in (13) to get \([t, r]x = 0 \) \( \forall \ x \in I \) and \( r, t \in R \).

Hence, \([t, r]I = 0 \) \( \forall \ r, t \in R \). Let \( x_0 \neq 0 \in I \). Then

\[
[t, r]Ix_0 = [t, r]I\sigma(x_0) = 0.
\]

By an application of Lemma 2.1, we conclude \([t, r] = 0 \) \( \forall \ r, t \in R \), i.e. \( r \in Z(R) \).

This completes the proof.

Remark: Using the arguments of the proof of Theorem 1.4, with slight modification one can easily prove the result as follows: If \( d(I) \subset Z(R) \), then \( R \) is commutative.

References


Received: January, 2010