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Remarks on Derivations of σ-Prime Rings

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Abstract

Let R be a 2-torsion free σ-prime ring with involution σ, I a nonzero σ-ideal of R and $d : R \to R$ a nonzero derivation commuting with σ. In this paper, we first establish that R is commutative if the following conditions: (i) $d(x)\ o\ x = 0\ \ \forall\ x \in R$, (ii) $d(x)\ o\ d(y) = 0\ \ \forall\ x, y \in I$, and (iii) $d(x\ o\ y) = 0\ \ \forall\ x, y \in I$ are satisfied. Moreover, also we prove that r is in $Z(R)$ if r in $Sa_\sigma(R)$ satisfies $d(x)\ o\ r = 0\ \ \forall\ x\ in\ I$.

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1. INTRODUCTION

Throughout, R will represent an associative ring with center $Z(R)$. Recall that a ring R is a prime if $aRb = 0$ implies $a = 0$ or $b = 0$. If R has an involution σ, then R is said to be σ-prime if $aRb = aR\sigma(b) = 0$ implies $a = 0$ or $b = 0$. Every prime ring equipped with an involution is σ-prime but the converse need not be true in general (for example see [4]). An ideal I of R is said to be a σ-ideal if I is invariant under σ(i.e., $\sigma(I) = I$). Let us define the set of symmetric and skew symmetric elements of R as $Sa_\sigma(R) = \{x \in R | \sigma(x) = \pm x\}$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x\ o\ y$ denotes the anticommutator $xy + yx$. We will make use of the following basic commutator identities, for any $x, y, z \in R$,

$$[x, yz] = y[x, z] + [x, y]z, [xy, z] = x[y, z] + [x, z]y$$
$$x \ o \ (yz) = (x \ o \ y)z - y[x, z] = y(x \ o \ z) + [x, y]z$$
$$(xy) \ o \ z = x(y \ o \ z) - [x, z]y = (x \ o \ z)y + x[y, z].$$

Several authors [1, 2, 6] have studied the relationship between the commutativity of a ring and the behavior of a special mapping on that ring. Recently, a major breakthrough has been achieved by Oukhtite et al. [5], where the important results by Posner, Herstein and Bell have been proved for σ-prime rings. More precisely, Posner's ([Theorem 2, 6]) of existence of a nonzero centralizing derivation on prime ring which makes the ring commutative if R is a prime ring of characteristic$\neq$2 with a nonzero derivation d such that $[d(x), d(y)] = 0 \ \ \forall \ x, y \in R$. Oukhtite et al. ([Theorem 1.2, 5]) proved that same result holds for σ-prime rings. Motivated by a well known result of Herstein [2], Bell [1] studied derivation d satisfying $d(xy) = d(yx)$ for all $x, y \in R$. This result has been extended for σ-prime rings ([Theorem 1.3, 5]). The objective of this paper is to extend the results of [3, 5]. Now we define the properties as given below:

P_1) For any $x \in R$ such that $d(x) \ o \ x = 0$

(P_2) For every $x, y \in I$ such that $d(x) \ o \ d(y) = 0$

(P_3) For every $x, y \in I$ such that $d(x \ o \ y) = 0$

(P_4) For any x in I and r in $Sa_\sigma(R)$ satisfies $d(x) \ o \ r = 0$.

2. MAIN RESULTS

Theorem 1.1. Let d be a nonzero derivation of σ-prime ring R commuting with σ. Let R satisfy the property (P_1). Then R is commutative.

Theorem 1.2. Let R be a 2 torsion free σ-prime ring and I a nonzero σ-ideal of R. Let R satisfies (P_2) and admit a nonzero derivation d that commutes with σ. Then R is commutative.

Theorem 1.3. Let R be a 2 torsion free σ-prime ring and I a nonzero σ-ideal of R. Let R satisfies (P_3) and admit a nonzero derivation d that commutes with σ. Then R is commutative.

Theorem 1.4. Let $0 \neq d$ be a derivation of R and I a nonzero σ-ideal of R satisfies (P_4), then r in $Z(R)$.

In order to prove our results we need the following lemmas.

Lemma 2.1 ([3, Lemma 3.1]) Let R be a σ-prime ring and let I be a nonzero σ-ideal of R. If a, b in R are such that $aIb = 0 = aI\sigma(b)$, then $a = 0$ or $b = 0$.

Lemma 2.2 ([5, Lemma 2.2]) Let I be a nonzero σ-ideal of R and $0 \neq d$ be a derivation on R which commutes with σ. If $[x, R]Id(x) = 0$ for all $x \in I$, then R is commutative.

Lemma 2.3 ([5, Lemma 2.3]) Let I be a nonzero σ-ideal of R. If R admits a derivation d such that $d^2(I) = 0$ and d commutes with σ on R then $d = 0$.

Proof of Theorem 1.1. By our hypothesis (P_1), we have

$$d(x)\ o\ x = 0\ \ \forall\ \ x \in R. \tag{1}$$

Linearising (1), we get

$$d(x)\ o\ y + d(y)\ o\ x = 0\ \ \forall\ x, y \in R. \tag{2}$$

Replacing y by xz in (2), we have

$$x(d(x)\ o\ z) + [d(x), x]z + (d(x)z)\ o\ x + (xd(z))\ o\ x = 0,$$

or

$$x(d(x)\ o\ z) + [d(x), x]z + (d(x)\ o\ x)z + d(x)[z, x] + x(d(z)\ o\ x) - [x, x]d(z) = 0$$

Using (1) and (2) in the above obtained relation, we get

$$[d(x), x]z + d(x)[z, x] = 0\ \ \forall\ x, z \in R. \tag{3}$$

Replacing z by zy in (3) and using (3), we get

$$d(x)z[y, x] = 0\ \ \forall\ x, y, z \in R.$$

Hence, $d(x)R[y, x] = 0\ \ \forall\ x, y \in R$.

As an application of the result in ([5, Theorem 1.1]), yields R is commutative.

Proof of Theorem 1.2. From the hypothesis of (P_2), we write

$$d(x)\ o\ d(y) = 0\ \ \forall\ \ x, y \in I \tag{4}$$

Replacing y by xy in (4), we have

$$d(x)d(x)y + d(x)xd(y) + d(x)yd(x) + xd(y)d(x) = 0 \ \ \forall \ \ x, y \in I \qquad (5)$$

Taking $y = x$ in (4), we have $d(x)d(x) = -d(x)d(x)$.
Using above relation in (5), we have

$$-d(x)d(x)y + d(x)xd(y) + d(x)yd(x) - xd(x)d(y) = 0,$$

or

$$d(x)[y, d(x)] + [d(x), x]d(y) = 0 \ \ \forall \ \ x, y \in I. \qquad (6)$$

For any $r \in R$, replacing y by yr in (6) and employing (6), we obtain

$$d(x)y[r, d(x)] + [d(x), x]yd(r) = 0 \ \ \forall \ \ x, y \in I. \qquad (7)$$

Replacing r by $rd(x)$ in (7), we get

$$d(x)yr[d(x), d(x)] + d(x)y[r, d(x)]d(x) + [d(x), x]y\{d(r)d(x) + rd^2(x)\} = 0$$

forall x, y$\in I$ *and* $r \in R$.
Using (7) in the above obtained relation, we have

$$[d(x), x]yrd^2(x) = 0 \ \ \forall \ \ x, y \in I \ \ and \ \ r \in R.$$

Hence,

$$[d(x), x]Id^2(x) = 0 \ \ \forall \ \ x \in I.$$

As d commutes with σ and I is a σ-ideal, we have

$$[d(x), x]Id^2(x) = \sigma([d(x), x])Id^2(x) = 0 \ \ \forall \ \ x \in I.$$

Lemma 2.1 gives $[d(x), x] = 0$ or $d^2(x) = 0 \ \ \forall\ x \in I$. If $d^2(x) = 0 \ \ \forall\ x \in I$, then by Lemma 2.3 we get $d = 0$, a contradiction.
Next, suppose that $[d(x), x] = 0 \ \ \forall \ \ x \in I$. Then, in view of the result ([5, Theorem 1.2]) one gets R is commutative.

Proof of Theorem 1.3. By the hypothesis of (P_3), we have

$$d(x \ o \ y) = 0 \ \ \forall \ \ x, y \in I. \qquad (8)$$

Replacing y by xy in (8), we get

$$\begin{aligned} 0 &= d(x \ o \ xy) \ \ \forall \ \ x, y \in I \\ &= d(x(x \ o \ y) + [x, x]y) \\ &= d(x)(x \ o \ y) + xd(x \ o \ y). \end{aligned}$$

This implies that

$$d(x)(x \ o \ y) = 0 \ \ \forall \ \ x, y \in I. \tag{9}$$

For any $r \in R$, replacing y by yr in (9), we obtain

$$d(x)((x \ o \ y)r - y[x, r]) = 0.$$

Using (9) in the above relation we get,

$$d(x)y[x, r] = 0 \ \ \forall \ \ x, y \in I, r \in R.$$

Hence, $d(x)I[x, r] = 0 \ \ \forall \ \ x \in I, r \in R$.
Particularly, we can write $0 = d(\sigma(x))I[\sigma(x), \sigma(r)] = \sigma(d(x))I\sigma([x, r]) \ \ \forall \ \ x \in I, r \in R$ as d commutes with σ.
Applying σ to this last equality , we get

$$[x, r]Id(x) = \ \ \forall \ \ x \in I, r \in R.$$

In view of Lemma 2.2, R is commutative.

Proof of Theorem 1.4. From the hypothesis of (P_4), we have

$$d(x) \ o \ r = 0 \ \ \forall \ \ x \in I \ \ and \ \ r \ \ in \ \ Sa_\sigma(R). \tag{10}$$

Replacing x by xy in (10), we get

$$(d(x)y + xd(y)) \ o \ r = 0$$

or

$$(d(x)y) \ o \ r + (xd(y)) \ o \ r = 0$$

or

$$(d(x) \ o \ r)y + d(x)[y, r] + x(d(y) \ o \ r) - [x, r]d(y) = 0$$

Using (10) in the above obtained relation, we have

$$d(x)[y, r] - [x, r]d(y) = 0 \ \ \forall \ \ x, y \ \ in \ \ I \ \ and \ \ r \ \ in \ \ Sa_\sigma(R). \tag{11}$$

Replacing y by yr in (11) and using (11), we obtain

$$[x, r]yd(r) = 0 \ \ \forall \ \ x, y \in I \ \ and \ \ r \ \ in \ \ Sa_\sigma(R).$$

Hence,

$$[x, r]Id(r) = 0 \ \ \forall \ \ x \in I \ \ and \ \ r \ \ in \ \ Sa_\sigma(R).$$

The fact that I is a σ-ideal together with r in $Sa_\sigma(R)$ gives

$$\sigma([x, r])Id(r) = [x, r]Id(r) = 0. \tag{12}$$

From Lemma 2.1 we get, either $d(r) = 0$ or $[x, r] = 0 \ \forall \ x \ in \ I \ and \ r \ in \ Sa_\sigma(R)$. Let $r \in R$ and $r + \sigma(r) \in Sa_\sigma(R)$. Then by equation (12) we have

$$d(r + \sigma(r)) = 0 \quad or \quad [x, r + \sigma(r)] = 0.$$

If $d(r + \sigma(r)) = 0$, then $d(r) \in Sa_\sigma(R)$. In view of (12) together with Lemma 2.1, yield $d(r) = 0$ or $[x, r] = 0$.
Next, suppose that $[x, r + \sigma(r)] = 0$
If $[x, r - \sigma(r)] = 0$, then $0 = [x, r + \sigma(r)] + [x, r - \sigma(r)] = 2[x, r]$.
Since $ch(R) \neq 2$, we get $[x, r] = 0$.
If $d(r - \sigma(r)) = 0$, then $d(r) \in Sa_\sigma(R)$. Similarly, (12) gives $d(r) = 0$ or $[x, r] = 0$.
In both cases, we have $R = G \cup H$, where $G = \{r \in R | d(r) = 0\}$ and $H = \{r \in R | [x, r] = 0, \ \forall \ x \in I\}$. But a group cannot be a union of two of its proper subgroups. Thus, $R = G$ or $R = H$. This implies that either $d(r) = 0 \ \forall \ r \in R$ or $[x, r] = 0 \ \forall \ x \in I$. Since d is a nonzero derivation so $d(r) \neq 0$, it implies

$$[x, r] = 0 \ \forall \ x \in I. \tag{13}$$

Replace x by tx in (13) to get $[t, r]x = 0 \ \forall \ x \in I \ \ and \ \ r, t \in R$.
Hence, $[t, r]I = 0 \ \forall \ r, t \in R$. Let $x_0 \neq 0 \in I$. Then

$$[t, r]Ix_0 = [t, r]I\sigma(x_0) = 0.$$

By an application of Lemma 2.1, we conclude $[t, r] = 0 \ \forall \ r, t \in R$, i.e. $r \in Z(R)$.
This completes the proof.

Remark: Using the arguments of the proof of Theorem 1.4, with slight modification one can easily prove the result as follows: If $d(I) \subset Z(R)$, then R is commutative.

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