

On the Structure of Some Groups

Containing $L_2(7) \text{ wr } L_2(11)$

Areej A. Al-Muhaimeed

Department of Mathematics, Taibah University
Madinah Munawwarah, P.O.Box 4599, Saudi Arabia
areej-1-1@hotmail.com

Ibrahim R. Al-Amri

Department of Mathematics, Taibah University
Madinah Munawwarah, Saudi Arabia
profibrahim@hotmail.com

Abstract

In this paper, we will generate the wreath product $L_2(7) \text{ wr } L_2(11)$ using only two permutations. Also, we will show the structure of some groups containing the wreath product $L_2(7) \text{ wr } L_2(11)$. The structure of the groups founded is determined in terms of wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$. Some related cases are also included. Also, we will show that S_{77k+1} and A_{77k+1} can be generated using the wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$ and a transposition in S_{77k+1} and an element of order 3 in A_{77k+1} . We will also show that S_{77k+1} and A_{77k+1} can be generated using the wreath product $L_2(7) \text{ wr } L_2(11)$ and an element of order $k+1$.

Keywords: wreath product, Linear group

1. INTRODUCTION

Hammas and Al-Amri [1], have shown that A_{2n+1} of degree $2n+1$ can be generated using a copy of S_n and an element of order 3 in A_{2n+1} . They also gave the symmetric generating set of Groups A_{kn+1} and S_{kn+1} using S_n [5].

Shafee [2] showed that the groups A_{kn+1} and S_{kn+1} can be generated using the wreath product $A_m \text{ wr } S_a$ and an element of order $k+1$. Also she showed how to generate S_{kn+1} and A_{kn+1} symmetrically using n elements each of order $k+1$.

Al-Amri and Al-Shehri [3] have shown that S_{9k+1} and A_{9k+1} can be generated using the wreath product $M_9 \text{ wr } C_k$ and an element of order 4 in S_{9k+1} and element of order 5 in A_{9k+1} .

The Linear groups $L_2(7)$ and $L_2(11)$ are two groups of the well known simple groups. In [6], they are fully described. In a matter of fact, they can be faintly presented in different ways. They have presentations in [6] as follows :

$$L_2(7) = \langle S, T \mid S^7 = (S^4 T)^4 = (ST)^3 = T^2 = 1 \rangle$$

$$L_2(11) = \langle X, Y \mid X^{11} = Y^2 = (X^4 Y X^6 Y)^2 = (XY)^3 = 1 \rangle$$

$L_2(7)$ can be generated using two permutations, the first is of order 7 and an involution as follows : $L_2(7) = \langle (1, 2, 3, 4, 5, 6, 7), (1, 2)(3, 5) \rangle$.

$L_2(11)$ can be generated using two permutations, the first is of order 11 and an involution as follow: $L_2(11) = \langle (1, 2, 3, \dots, 11), (1, 9)(2, 6)(4, 5)(7, 8) \rangle$.

In this paper, we will generate the wreath product $L_2(7) \text{ wr } L_2(11)$ using only two permutations. Also, we show the structure of some groups containing the wreath product $L_2(7) \text{ wr } L_2(11)$. The structure of the groups founded is determined in terms of wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$. Some related cases are also included. Also, we will show that S_{77k+1} and A_{77k+1} can be generated using the wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$ and a transposition in S_{77k+1} and an element of order 3 in A_{77k+1} . We will also show that S_{77k+1} and A_{77k+1} can be generated using the wreath product $L_2(7) \text{ wr } L_2(11)$ and an element of order $k+1$.

2. PRELIMINARY RESULTS

DEFINITION 2.1. Let A and B be groups of permutations on non empty sets Ω_1 and Ω_2 respectively. The wreath product of A and B is denote by $A \text{ wr } B$ and defined as $A \text{ wr } B = A^{\Omega_2} \times_{\theta} B$, i.e., the direct product of $|\Omega_2|$ copies of A and a mapping θ where $\theta: B \rightarrow \text{Aut}(A^{\Omega_2})$ is defined by $\theta_y(x) = x^y$, for all $x \in A^{\Omega_2}$. It follows that $|A \text{ wr } B| = (|A|)^{|\Omega_2|} |B|$.

THEOREM 2.2 [4] Let G be the group generated by the n -cycle $(1, 2, \dots, n)$ and the 2-cycle (n, a) . If $1 < a < n$ is an integer with $n = am$, then $G \cong S_m \text{ wr } C_a$.

THEOREM 2.3 [4] Let $1 \leq a \neq b < n$ be any integers. Let n be an odd integer and let G be the group generated by the n -cycle $(1, 2, \dots, n)$ and the 3-cycle

(n, a, b) . If the $hcf(n, a, b) = 1$, then $G = A_n$. While if n can be an even then $G = S_n$.

THEOREM 2.4 [4] Let $1 \leq a < n$ be any integer. Let $G = \langle (1, 2, \dots, n), (n, a) \rangle$. If $h.c.f.(n, a) = 1$, then $G = S_n$.

THEOREM 2.5 [4] Let $1 \leq a \neq b < n$ be any integers. Let n be an even integer and let G be the group generated by the $(n-1)$ -cycle $(1, 2, \dots, n-1)$ and 3-cycle (n, a, b) . Then $G = A_n$.

3. THE RESULTS

THEOREM 3.1 The wreath product $L_2(7) \text{ wr } L_2(11)$ can be generated using two permutations, the first is of order 77 and the second is of order 4.

Proof : Let $G = \langle X, Y \rangle$, where: $X = (1, 2, 3, 4, \dots, 77)$, which is a cycle of order 77, $Y = (1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27)(29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74)$, which is the product of two cycles each of order 4 and twenty four transpositions. Let $\alpha_1 = ((XY)^6[X, Y]^5)^{18}$. Then

$$\alpha_1 = (11, 22, 33, 44, 55, 66, 77),$$

which is a cycle of order 7. Let $\alpha_2 = \alpha_1^{-1}X$. It is easy to show that

$$\alpha_2 = (1, 2, 3, \dots, 11)(12, 13, 14, \dots, 22) \dots (67, 68, 69, \dots, 77),$$

which is the product of seven cycles each of order 11. Let: $\beta_1 = (Y^2)^{(XY)^{18}} = (9, 20)(12, 23)(31, 53)(34, 56)$, $\beta_2 = \beta_1 Y^{-1} = (1, 9, 12, 20)(2, 6)(4, 5)(7, 8)(13, 17)(15, 16)(18, 19)(23, 31, 45, 53)(24, 28)(26, 27)(29, 30)(34, 42)(35, 39)(37, 38)(40, 41)(46, 50)(48, 49)(51, 52)(56, 64)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74)$, $\beta_3 = (Y^3 \beta_2)^2 = (1, 45)(12, 23)$, $\beta_4 = \beta_3^{(\alpha_2^{-1} \alpha_1^3)} = (11, 44)(55, 66)$ and $\beta_5 = \beta_4^{\beta_3^{\alpha_2^{-1}}} = (11, 66)(44, 55)$. Let $\alpha_3 = \beta_5^{\beta_3^{(\alpha_2^{-1} \alpha_1)}}$. Hence

$$\alpha_3 = (11, 22)(33, 55).$$

Let $\alpha_4 = YX^{-1} \alpha_3^{-1} X$. We can conclude that

$$\alpha_4 = (1, 9)(2, 6)(4, 5)(7, 8)(12, 20)(13, 17)(15, 16)(18, 19)(23, 31)(24, 28)(26, 27)(29, 30)(34, 42)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(56, 64)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74),$$

which is the product of twenty eight transpositions. Let $K = \langle \alpha_2, \alpha_4 \rangle$. Let $\theta: K \rightarrow L_2(11)$ be the mapping defined by

$$\theta(11i+j) = j \quad \forall 0 \leq i \leq 6, \quad \forall 1 \leq j \leq 11$$

Since $\theta(\alpha_2)=(1, 2, \dots, 11)$ and $\theta(\alpha_4)=(1, 9)(2, 6)(4, 5)(7, 8)$, then $K \cong \theta(K) = L_2(11)$. Let $H_0 = \langle \alpha_1, \alpha_3 \rangle$. Then $H_0 \cong L_2(7)$. Moreover, K conjugates H_0 into H_1 , H_1 into H_2 and so it conjugates H_{10} into H_0 , where $H_i = \langle (i, 11+i, 22+i, 33+i, 44+i, 55+i, 66+i), (i, 11+i)(22+i, 44+i) \rangle$, $\forall 1 \leq i \leq 10$. Hence we get $L_2(7) \text{ wr } L_2(11) \subseteq G$. On the other hand, Since $X = \alpha_1 \alpha_2$ and $Y = \alpha_4 \alpha_3^X$, then $G \subseteq L_2(7) \text{ wr } L_2(11)$. Hence $G = L_2(7) \text{ wr } L_2(11)$. \diamond

THEOREM 3.2 The wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$ can be generated using two permutations, the first is of order $77k$ and an involution, for all integers $k \geq 1$.

Proof : Let $\sigma = (1, 2, \dots, 77k)$ and $\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$. If $k=1$, then we get the group $L_2(7) \text{ wr } L_2(11)$ which can be considered as the trivial wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } \langle id \rangle$. Assume that $k > 1$. Let $\alpha = \prod_{i=0}^{11} \tau^{\sigma^i}$, we get an element $\delta = \alpha^{45} = (k, 2k, 3k, \dots, 77k)$. Let $G_i = \langle \delta^{\sigma^i}, \tau^{\sigma^i} \rangle$, be the groups acts on the sets $\Gamma_i = \{i, k+i, 2k+i, \dots, 76k+i\}$, for all $1 \leq i \leq k$. Since $\bigcap_{i=1}^k \Gamma_i = \varnothing$, then we get the direct product $G_1 \times G_2 \times \dots \times G_k$, where, by theorem 3.1 each $G_i \cong L_2(7) \text{ wr } L_2(11)$. Let $\beta = \delta^{-1} \sigma = (1, 2, \dots, k)(k+1, k+2, \dots, 2k) \dots (76k+1, 76k+2, \dots, 77k)$. Let $H = \langle \beta \rangle \cong C_k$. H conjugates G_1 into G_2 , G_2 into G_3, \dots and G_k into G_1 . Hence we get the wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k \subseteq G$. On the other hand, since $\delta \beta = (1, 2, \dots, k, k+1, k+2, \dots, 2k, \dots, 76k+1, 76k+2, \dots, 77k) = \sigma$, then $\sigma \in (L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$. Hence $G = \langle \sigma, \tau \rangle \cong (L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$. \diamond

THEOREM 3.3 The wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } S_k$ can be generated using three permutations, the first is of order $77k$, the second and the third are involutions, for all $k \geq 2$.

Proof : Let $\sigma = (1, 2, \dots, 77k)$, $\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$ and $\mu = (1,$

$2)(k+1, k+2)(2k+1, 2k+2) \dots (76k+1, 76k+2)$. Since by Theorem 3.2, $\langle \sigma, \tau \rangle \cong (L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$ and $(1, 2, \dots, k)(k+1, k+2, \dots, 2k) \dots (76k+1, \dots, 77k) \in (L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$ then $\langle (1, \dots, k)(k+1, \dots, 2k) \dots (76k+1, \dots, 77k), \mu \rangle \cong S_k$. Hence $G = \langle \sigma, \tau, \mu \rangle \cong (L_2(7) \text{ wr } L_2(11)) \text{ wr } S_k \cdot \diamond$

COROLLARY 3.4 The wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } A_k$ can be generated using three permutations, the first is of order $77k$, the second is an involution and the third is of order 3, for all odd integers $k \geq 3$.

THEOREM 3.5 The wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } (S_m \text{ wr } C_a)$ can be generated using three permutations, the first is of order $77k$, the second and the third are involutions, where $k = am$ be any integer with $1 < a < k$.

Proof : Let $\sigma = (1, 2, \dots, 77k)$, $\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$ and $\mu = (k, a)(2k, k+a)(3k, 2k+a) \dots (77k, 76k+a)$. Since by Theorem 3.2, $\langle \sigma, \tau \rangle \cong (L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$ and $(1, \dots, k)(k+1, \dots, 2k) \dots (76k+1, \dots, 77k) \in (L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$ then $\langle (1, \dots, k)(k+1, \dots, 2k) \dots (76k+1, \dots, 77k), \mu \rangle \cong (S_m \text{ wr } C_a)$. Hence $G = \langle \sigma, \tau, \mu \rangle \cong (L_2(7) \text{ wr } L_2(11)) \text{ wr } (S_m \text{ wr } C_a) \cdot \diamond$

THEOREM 3.6 S_{77k+1} and A_{77k+1} can be generated using the wreath product $(L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$ and a transposition in S_{77k+1} for all integers $k > 1$ and an element of order 3 in A_{77k+1} for all odd integers $k > 1$.

Proof: Let $\sigma = (1, 2, \dots, 77k)$, $\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$, $\mu = (77k+1, 1)$ and $\mu' = (1, k, 77k+1)$ be four permutations, of order $77k, 2, 2$ and 3 respectively. Let $H = \langle \sigma, \tau \rangle$. By theorem 3.2 $H \cong (L_2(7) \text{ wr } L_2(11)) \text{ wr } C_k$.

Case 1: Let $G = \langle \sigma, \tau, \mu \rangle$. Let $\alpha = \sigma\mu$, then $\alpha = (1, 2, \dots, 77k, 77k+1)$, which is a cycle of order $77k+1$. By theorem 2.4 $G = \langle \sigma, \tau, \mu \rangle \cong \langle \alpha, \mu \rangle \cong S_{77k+1}$.

Case 2: Let $G = \langle \sigma, \tau, \mu' \rangle$. By theorem 2.5 $\langle \sigma, \mu' \rangle \cong A_{77k+1}$. Since τ is an even permutation, then $G \cong A_{77k+1} \cdot \diamond$

THEOREM 3.7 S_{77k+1} and A_{77k+1} can be generated using the wreath product $L_2(7) \text{ wr } L_2(11)$ and an element of order $k+1$ in S_{77k+1} and A_{77k+1} for all integers $k \geq 1$.

Proof: Let $G = \langle \sigma, \tau, \mu \rangle$, where, $\sigma = (1, 2, 3, \dots, 77)(77(k-(k-1))+1, \dots, 77(k-(k-1))+77) \dots (77(k-1)+1, \dots, 77(k-1)+77)$, $\tau = (1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27)(29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74) \dots (77(k-1)+1, 77(k-1)+9) \dots (77(k-1)+73, 77(k-1)+74)$, and $\mu = (77, 154, \dots, 77k, 77k+1)$, where $k-i > 0$, be three permutations of order 77, 4 and $k+1$ respectively. Let $H = \langle \sigma, \tau \rangle$. Define the mapping θ as follows ;

$$\theta(11(k-i)+j) = j \quad \forall 1 \leq i \leq k, \quad \forall 1 \leq j \leq 11$$

Hence $H = \langle \sigma, \tau \rangle \cong L_2(7) \text{ wr } L_2(11)$. Let $\alpha = \mu\sigma$ it is easy to show that $\alpha = (1, 2, 3, \dots, 77k+1)$, which is a cycle of order $77k+1$. Let $\mu' = \mu^\sigma = (1, 78, \dots, 77(k-1)+1, 77k+1)$ and $\beta = [\mu, \mu'] = (1, 77, 77k+1)$. Since $h.c.f(1, 77, 77k+1) = 1$, then by theorem 2.3 $G = \langle \sigma, \tau, \mu \rangle \cong \langle \alpha, \beta \rangle \cong S_{77k+1}$ or A_{77k+1} depending on whether k is an odd or an even integer respectively. \diamond

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