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Ordinary Weight Conjecture Implies Brauer's $k(B)$-Conjecture

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Abstract

The aim of this work is to show that Ordinary Weight Conjecture (OWC) for p-blocks with defect group which is an extra special p-group of order p^3 and exponent p for an odd prime number p implies Brauer's $k(B)$-Conjecture (BKC), provided that the inertial quotient of a maximal subpair is isomorphic to $C_2 \times C_{p-1}$.

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Introduction

Let p be a prime number and (K, R, F) a p-modular system. Let G be a finite group and B a p-block of RG. The set of all irreducible (complex) ordinary characters of G is denoted by $Irr(G)$. We write $k(B) = |Irr(B)| = |\{\chi \in Irr(G) \ : \ \chi \in B\}|$. Let d be an arbitrary non-negative integer. Then an irreducible ordinary character $\chi \in Irr(B)$ is said to be of defect d if $p^d\chi(1)_p = |G|_p$ where, $\chi(1)_p$ and $|G|_p$ mean the p-part of the degree of χ and of the order of the finite group G respectively. We write $d = d(\chi)$. In the case that $d(\chi) = 0$ we have $Irr(B) = \{\chi\}$ and we say that B is a p-block of defect zero. We use the notation $k_d(B)$ to denote the number of irreducible ordinary characters of B with defect d. The defect number of the p-block B is $d(B)$, where $d(B) = Max\{d(\chi) \ : \ \chi \in Irr(B)\}$. We write C_r to refer to a cyclic group of order r. For Q a p-subgroup of G, we write $C_G(Q)$ for the centralizer of Q in G and $N_G(Q)$ for the normalizer of Q in G. The Brauer map relative to Q is the projection $Br_Q : RG^Q \rightarrow FC_G(Q)$, where RG^Q is the fixed points of RG under the conjugation action by elements from Q. The defect group of the p-block B is a p-subgroup D of G which is maximal subject to $Br_D(1_B) \neq 1$. This map gives a bijection between p-blocks of G with defect group D and p-blocks of $N_G(D)$ with defect group D. For a p-block b_Q of $C_G(Q)$, we can consider the pair (Q, b_Q) which is called (G, B)-sub-pair. The finite group G acts by conjugation on the set of all (G, B)-sub-pairs. There is an ordering relation on this set.The action of G respects the ordering relation. Also maximal (G, B)-sub-pairs are G-conjugate. The stabilizer of (Q, b_Q) in G is $N_G(Q, b_Q)$.

In this paper, we shall extend [1, Theorem E.] by proving that the ordinary weight conjecture implies Brauer $k(B)$-conjecture.

1 Conjectures we are considering

The conjecture we are dealing with is the Ordinary Weight Conjecture (OWC) see [8]. It is a reformulation of Alperin Weight Conjecture [2]. We shall follow G. R. Robinson's approach which uses the chain of p-subgroups and the cancelation methods [5, 7]. We use normal chains of (G, B)-subpairs. For the implementation of normal chains see for instance [7]. So, we write $\mathcal{N}_B(G)$ to be a full set of representatives for the distinct G-conjugacy classes of normal chains σ of (G, B)-sub-pairs $(V_\sigma, b_\sigma) < \cdots < (V^\sigma, b^\sigma)$. Then the stabilizer of such chain σ will be denoted by $N_G(\sigma)$. We consider a p-block of $N_G(\sigma)$ which is denoted by $B(\sigma) =: Br_{V^\sigma}(1_B) \cdot R[N_G(\sigma)]$. By a remarkable result of G. R. Robinson and Knörr, we know that whenever b is a p-block of $N_G(\sigma)$, the induced p-block b^G of G is defined, see [4, Lemma 3.1 and Lemma 3.2].

Now Let Q be an arbitrary p-subgroup of G. Then if η is an irreducible ordinary character of Q, then $I_{N_G(Q,b_Q)}(\eta)$ will refers to the inertial subgroup of η in $N_G(Q, b_Q)$ and $f_0^{(B)}(I_{N_G(Q,b_Q)}(\eta)/Q)$ refers to the number of p-blocks of defect zero in the section $I_{N_G(Q,b_Q)}(\eta)/Q$ which belong to Brauer correspondent of B.

Let us state both the Ordinary Weight Conjecture and the Brauer's $k(B)$-conjecture.

Conjecture 1.1 *The Ordinary Weight Conjecture (OWC): Let B be a p-block of a finite group G. Then for every non-negative integer d, we have*

$$k_d(B) = \sum_{\phi \neq \sigma \in \mathcal{N}_B(G)/G} (-1)^{dim(\sigma)} \sum_{\mu \in Irr_d(V_\sigma)/N_G(\sigma)} f_0^{(B)}(I_{N_G(\sigma)}(\mu)/V_\sigma).$$

Conjecture 1.2 *The Brauer's $k(B)$-Conjecture (BKC): Let B be a p-block of a finite group G with defect group D. Then $k(B) \leq |D|$, where $|D|$ is the order of the defect group D.*

Our main result is to show that Ordinary Weight Conjecture (OWC) implies Brauer's $k(B)$-Conjecture (BKC) in the case that B is a p-block of G with defect group D which is an extra-special p-group of order p^3 and exponent p under the assumption that $N_G(D, b_D)/DC_G(D)$ is isomorphic to $C_2 \times C_{p-1}$. For some proprieties of an extra-special p-group, we refer the reader to [3] as well as [1].

With the same conditions above, in [1] we proved that the Ordinary Weight Conjecture (OWC) implies that $k(B) = k(D) + p$. Let us recast Theorem E in [1].

Theorem 1.3 *Let G be a finite group, p an odd prime number and B be a p-block of G with defect group D which is an extra special p-group of order p^3 and exponent p. Assume that $N_G(D, b_D)/DC_G(D)$ is isomorphic to $C_2 \times C_{p-1}$. Then OWC implies $k(B) = k(D) + p$.*

Now we show that the above result implies that the order of the defect group is an strict upper bound for the number of the ordinary irreducible characters of G which belong to B.

Theorem 1.4 *Let G be a finite group, p an odd prime number and B be a p-block of G with defect group D which is an extra special p-group of order p^3 and exponent p. Assume that $N_G(D, b_D)/DC_G(D)$ is isomorphic to $C_2 \times C_{p-1}$. Then OWC implies Brauer $k(B)$-conjecture for B.*

Proof: By Theorem 1.3, $k(B) = k(D) + p$. Therefore, $k(B) = p^2 + 2p - 1$. It follows that $k(B) < p^2 + 2p$. However, p is an odd prime number and hence $k(B) < p^3 = |D|$. This means that BKC holds for B in this situation.

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