International Journal of Algebra, Vol. 4, 2010, no. 13, 625 - 630

A-Generated Subgroups of A-Solvable Groups

Ulrich Albrecht

Department of Mathematics, Auburn University
Auburn, AL 36849, USA
albreuf@mail.auburn.edu

Abstract. In the discussion of A-solvable groups, the question arises if a torsion-free abelian group A of finite rank is flat as a module over its endomorphism ring if every A-generated subgroup of a torsion-free A-solvable group is A-solvable. This paper gives a negative answer by constructing a torsion-free group of rank 3 for which all A-generated torsion-free groups are A-solvable, although A is not flat as an $E(A)$-module.

Mathematics Subject Classification: 20K15

Keywords: A-Solvable Group, Flat, Endomorphism Ring

Associated with every abelian group A is an adjoint pair (H_A, T_A) of functors between the category of abelian groups and the category of right modules over its endomorphism ring $E(A)$ defined by $H_A(G) = \mathrm{Hom}(A, G)$ for all abelian groups G and $T_A(M) = M \otimes_E A$ for all right E-modules M. The E-module structure of $H_A(G)$ is induced by the composition of maps. These functors induce a natural transformation $\theta_G : T_A H_A(G) \to G$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$. The *$A$-socle, $S_A(G)$, of* G is the image of θ_G. We call G *A-generated* if θ_G is onto, and *A-solvable* if θ_G is an isomorphism. Clearly G is A-generated if and only if it is an epimorphic image of $A^{(I)} = \oplus_I A$ for some index-set I. If I can be chosen to be finite, we say that G is *finitely A-generated*. It is easy to see that a group G is A-solvable if and only if every finitely A-generated subgroup of G is A-solvable.

The classes of A-generated and A-solvable groups are important tools in the investigation of homological properties of A (e.g., see [2], [3], [5], [7], and [8]). In particular, Ulmer's Theorem [10] states that A is flat as a module over its endomorphism ring if and only if every homomorphism $\phi : G \to H$ with G and

H A-solvable has an A-generated kernel. In this case, A-generated subgroups of torsion-free A-solvable groups are A-solvable. This last result raises the question whether or not the converse is true: Is a torsion-free abelian group A flat as a module over its endomorphism ring if every A-generated subgroup of a torsion-free A-solvable group is A-solvable.

It is the goal of this paper to provide a negative answer to this question by constructing a torsion-free group of rank 3 such that all torsion-free A-generated groups are A-solvable, although A is not flat as an $E(A)$-module. Furthermore, A provides counter-examples to several other open questions concerning A-solvable groups.

Consider a torsion-free group G of finite rank with finite type-set. Denote the distinct minimal types in the type-set of G by $\tau_1, \dots, \tau_n$. For each $i = 1, \dots, n$, choose a subgroup A_i of $\mathbb{Q}$ with $type(A_i) = \tau_i$, and consider $A = A_1 \oplus \dots A_n$. The group A is determined by G up to isomorphism and is called the *cd-generator* of G. The ring $E(A)$ is commutative and hereditary ring since $\mathrm{Hom}(A_i, A_j) = 0$ for $i \neq j$. Although G is A-generated, it need not be A-solvable as the next result shows.

We want to remind the reader that a sequence $X \to Y \to 0$ is A-balanced if the induced sequence $H_A(X) \to H_A(Y) \to 0$ is exact. For a torsion-free Abelian group A of finite tank, a group G is A-solvable if and only if there exists an A-balanced exact sequence $0 \to U \to A^{(I)} \to G \to 0$ with $S_A(U) = 0$ [2].

Theorem 1. *Let G be a torsion-free group of finite rank with finite type-set. If A is the cd-generator of G, then G is A-solvable if and only if $G = G(\tau_1) \oplus ... \oplus G(\tau_n)$.*

Proof. In the case that G is A-solvable, consider an A-balanced exact sequence $0 \to U \xrightarrow{\alpha} A^{(I)} \xrightarrow{\beta} G \to 0$ in which U is A-generated. It induces the exact sequence $0 \to H_A(U) \xrightarrow{H_A(\alpha)} H_A(A^{(I)}) \xrightarrow{H_A(\beta))} H_A(G) \to 0$ of right $E(A)$-modules. For every $i = 1, \dots, n$, let $U_i = S_{A_i}(U)$, $P_i = A_i^{(I)}$, and $G_i = S_{A_i}(G) = G(\tau_i)$. Since $E(A)$ is hereditary, U is A-projective, and, therefore, $U = U_1 + \dots + U_n$. Because $\mathrm{Hom}(A_i, A_j) = 0$ for $i \neq j$, we obtain $U_i \subseteq P_i$. Thus, $U = U_1 \oplus \dots \oplus U_n$. Furthermore, $\beta(P_i) \subseteq G_i$.

On the other hand, if $g \in G_i$, then there are $\phi_1, \dots, \phi_\ell \in \mathrm{Hom}(A_i, G)$ and $x_1, \dots, x_\ell \in A_i$ with $g = \phi_1(x_1) + \dots + \phi_\ell(x_\ell)$. Since the sequence is A-balanced, there are $\psi_1, \dots, \psi_\ell \in \mathrm{Hom}(A_i, A^{(I)})$ with $\beta\psi_j = \phi_j$ for all $j = 1, \dots, \ell$. Because of $\psi_j(A_i) \subseteq P_i$, we have $g \in \beta(P_i)$. Therefore, $G_i = \beta(P_i)$. Suppose there are $g_i \in G_i$ for $i = 1, \dots, n$ such that $g_1 + \dots + g_n = 0$. For every i, select $y_i \in P_i$ with $g_i = \beta(y_i)$. Since $\beta(y_1 + \dots + y_n) = 0$, we have

$y_1 + \ldots + y_n \in U = U_1 \oplus \ldots \oplus U_n$. Because of $U_i \subseteq P_i$, we obtain $y_i \in U_i \subseteq U$. Hence, $g_i = \beta(y_i) = 0$ for all i; and $G = G(\tau_1) \oplus \ldots \oplus G(\tau_n)$.

Conversely, it remains to show that groups of the form $G(\tau_i)$ is A-solvable. There exists an A_i-balanced exact sequence $0 \to U \to A_i^{(J)} \xrightarrow{\beta_i} G(\tau_i) \to 0$ in which U is A_i-, and hence also A-generated. If $j \neq i$ and $\phi : A_j \to G(\tau_i)$ is non-zero, then $0 \neq \phi(A_j) \subseteq G(\tau_i) \cap G(\tau_j)$, a contradiction. Hence, every map $\lambda : A \to G(\tau_i)$ satisfies $\lambda(\oplus_{j \neq i} A_j) = 0$. Let $\eta_i : A_i \to A_i^{(J)}$ be chosen such that $\beta_i \eta_i = \lambda | A_i$. Define $\eta : A \to A_i^{(J)}$ by $\eta(\oplus_{j \neq i} A_j) = 0$ and $\eta | A_i = \eta_i$. Then, $\beta_i \eta = \lambda$, and the last sequence is A-balanced. But then, $G(\tau_i)$ is A-solvable [2]. □

The last theorem also provides a criterion to determine the A-solvable group in case that $A = A_1 \oplus \ldots A_n$ where $A_1, \ldots, A_n$ are subgroups $\mathbb{Q}$ of pairwise incomparable types $\{\tau_1, \ldots, \tau_n\}$.

Corollary 2. *Let $A = A_1 \oplus \ldots A_n$ where $A_1, \ldots, A_n$ are subgroups $\mathbb{Q}$ of pairwise incomparable types. A torsion-free group G of finite rank is A-solvable if and only if $G = G(\tau_1) \oplus \ldots \oplus G(\tau_n)$.*

We now turn to the question to describe the A-solvable groups in case that A is a completely decomposable group such that the types of the A_i's are not necessarily incomparable. Consider subgroups $A_1, \ldots, A_n$, and B of $\mathbb{Q}$ such that $\tau_i = type(A_i)$ and $\tau = type(B)$ satisfy

i) τ_i and τ_j are incomparable for $i \neq j$, and

ii) $\tau \geq \tau_1, \ldots, \tau_n$.

Write $\tau_i = [(k_p^i)]$ for $i = 1, \ldots, n$ and $\tau = [(n_p)]$ where (k_p^i) and (n_p) are characteristics chosen in such a way that $k_p^i \leq n_p$ for all primes p and all $i = 1, \ldots, n$.

Lemma 3. *Let $A_1, \ldots, A_n$ and B be chosen as above. If G is a torsion-free group with $G = G(\tau)$ and $\alpha \in \mathrm{Hom}(A_i, G)$ for some $i \in \{1, ldots, n\}$, then there are homomorphisms $\sigma : A_i \to B$ and $\tau : B \to G$ with $\alpha = \tau\sigma$.*

Proof. We use the notation introduced before the statement of the lemma. Select $0 \neq a_i \in A_i$ such that $k_p^i = h_p^{A_i}(a_i)$ for all primes p. Then, $h_p^G(\alpha(a_i)) \geq k_p^i$ for all primes p. Moreover, there are primes $p_1, \ldots, p_m$ with $h_q^G(\alpha(a_i)) \geq n_q$ for all $q \neq p_1, \ldots, p_m$. For each $j = 1, \ldots, m$, we have $k_{p_j}^i \leq s_j = h_{p_j}^G(\alpha(a_i)) < n_{p_j} < \infty$. Select a non-zero $x \in B$ such that $h_p^B(x) = n_p$ for all primes p. We can find $b \in B$ such that $p_1^{n_{p_1} - s_1} \ldots p_m^{n_{p_m} - s_m} b = x$, i.e. $h_{p_j}^B(b) = s_j$ for $j = 1, \ldots, m$. Clearly, $h_p^B(b) = n_p$ for all $p \neq p_1, \ldots, p_m$.

For every non-zero $a \in A_i$, there are relatively prime integers r and s with $sa = ra_i$. If p is a prime dividing s, then $s = p^j s'$ and $(p, s') = 1$. Then,

$h_p^{A_i}(a_i) \geq j$ yields $h_p^B(b) \geq h_p^{A_i}(a_i) \geq j$. Hence, $sy = rb$ has a solution in B; and setting $\sigma(a) = y$ yields a map $\sigma : A_i \to B$ with $\sigma(a_i) = b$.

In the same way, let z be a non-zero element of B, and select relatively prime integers u and v with $vz = ub$. If p is a prime dividing v, say $v = p^t v'$ with $(p, v') = 1$, then $h_p^B(b) \geq t$. But then $h_p^G(\alpha(a_i)) \geq n_p = h_p^B(x) = h_p^B(b) \geq t$ unless $p \in \{p_1, \ldots, p_m\}$. For $k = 1, \ldots, m$, we have $h_{p_k}^G(\alpha(a_i)) = s_k = h_{p_k}^B(b) \geq t$. Hence, $vy = u\alpha(a_i)$ has a solution in G. Setting $\tau(z) = y$ defines a map $\tau : B \to G$ with $\tau(b) = \alpha(a_i)$. Clearly, $\alpha = \tau\sigma$. □

Theorem 4. *Let τ_1 and τ_2 be incomparable types, and $\tau_3 = sup(\tau_1, \tau_2)$. If $A_i \subseteq \mathbb{Q}$ with $type(A_i) = \tau_i$ for $i = 1, 2, 3$, then every A-generated torsion-free group G is A-solvable where $A = A_1 \oplus A_2 \oplus A_3$.*

Proof. Since a group G is A-solvable if all its finitely A-generated subgroups are A-solvable, one may assume that G is finitely A-generated.

Let $\pi : A^n \to G$ be an epimorphism, and consider $G_i = \pi(A_i^n)$. Then, $G = G_1 + G_2 + G_3$. Since $\tau_3 = sup(\tau_1, \tau_2)$, we obtain $G(\tau_3) = G(\tau_1) \cap G(\tau_2)$. Moreover, $G_i \subseteq S_{A_i}(G) = G(\tau_i)$, and $G_3 \subseteq G(\tau_3) \subseteq G(\tau_i)$ for all i. Hence, $G = G_1 + G(\tau_2)$ and

$$\begin{aligned} G(\tau_1)/G(\tau_3) &= G(\tau_1)/G(\tau_1) \cap G(\tau_2) \cong [G(\tau_1) + G(\tau_2)] \\ &= G/G(\tau_2) = [G_1 + G(\tau_2)]/G(\tau_2) \cong G_1/G_1 \cap G(\tau_2). \end{aligned}$$

Since G_1 is a torsion-free image of the homogeneous completely decomposable group A_1^n, it is isomorphic a direct summand of A_1^n, and hence $G_1 \cong A_1^s$ for some $s < \omega$. Because $G(\tau_2)$ is a pure subgroup of G, we obtain that $G_1 \cap G(\tau_2)$ is a pure subgroup of G_1, and hence a direct summand. Thus, $G(\tau_1)/G(\tau_3) \cong G_1/G_1 \cap G(\tau_2) \cong A_1^{t_1}$ for some $t_1 < \omega$. By Baer's Lemma [9, Lemma 86.4], $G(\tau_1) = U \oplus G(\tau_3)$ with $U \cong A_1^{t_1}$. In the same way, $G(\tau_2) = V \oplus G(\tau_3)$ with $V \cong A_2^{t_2}$ for some $t_2 < \omega$. Now, it is easy to see that $G = U \oplus V \oplus G(\tau_3)$. Since $U \oplus V$ is A-projective, it remains to show that $G(\tau_3)$ is A-solvable.

For this, consider an A_3-balanced sequence $0 \to W \to A_3^{(I)} \xrightarrow{\pi} G(\tau_3) \to 0$ in which $W = W(\tau_3)$ since $G(\tau_3)$ is A_3-solvable. It remains to show that this sequence is A-balanced since W is A-generated and $A_3^{(I)}$ is A-projective. Consider a map $\phi : A \to G(\tau_3)$, and define $\phi_i = \phi|A_i$ for $i = 1, 2, 3$. By Lemma 3, there are maps $\sigma_i : A_i \to A_3$ for $i = 1, 2$, and $\lambda_1, \lambda_2 : A_3 \to G(\tau_3)$ such that $\phi_i = \lambda_i\sigma_i$ for $i = 1, 2$. Since the sequence is A_3-balanced, there are $\psi_1, \psi_2, \psi_3 : A_3 \to A_3^{(I)}$ such that $\pi\psi_i = \lambda_i$ for $i = 1, 2$ and $\pi\psi_3 = \phi_3$. Define a map $\psi : A \to A_3^{(I)}$ by $\psi(a_1, a_2, a_3) = \psi_1\sigma_1(a_1) + \psi_2\sigma_2(a_2) + \psi_3(a_3)$ for all $a_i \in A_i$. Then, $\pi\psi(a_1, a_2, a_3) = \pi\psi_1\sigma_1(a_1) + \pi\psi_2\sigma_2(a_2) + \pi\psi_3(a_3) = \lambda_1\sigma_1(a_1) + \lambda_2\sigma_2(a_2) + \phi_3(a_3) = \phi_1(a_1) + \phi_2(a_2) + \phi_3(a_3) = \phi(a_1, a_2, a_3)$. □

Corollary 5. *There exist a torsion-free group A of rank 3 such that all torsion-free A-generated groups are A-solvable, although A is not flat as an $E(A)$-module.*

Proof. The existence of A is a direct consequence of Theorem 4. To see that this group is not flat as an $E(A)$-module, observe that $A_3 \cong A_1 + A_2$, and $A_1 \cap A_2$ has type strictly less that A_1 and A_2. By Ulmer's Theorem, A is not flat as an $E(A)$-module. □

The group A also has the smallest possible rank for such an example since every torsion-free group of rank 2 is flat as a module over its endomorphism ring [5].

Furthermore, a torsion-free group A of finite rank which is fully faithful as an $E(A)$-module is homogeneous completely decomposable if and only if $\mathbb{Q}$ is A-solvable [1]. Using the group A from Theorem 4, we obtain immediately:

Corollary 6. *There exists a torsion-free group of rank 3 which is not homogeneous completely decomposable such that $\mathbb{Q}$ is A-solvable.*

Finally, the author and Goeters showed in [4] that a torsion-free group A of finite rank is p-simple and a direct sum of irreducible Murley groups if $E(A)$ is hereditary and every reduced A-generated group is A-solvable. However, this result fails without the assumption that $E(A)$ is hereditary:

Corollary 7. *There exists a torsion-free group A of rank 3 such that all torsion-free A-generated groups are A-solvable, but A is not a finitely faithful S-group.*

Proof. An Abelian group is a finitely faithful S-group if $r_p(E(A)) = r_p(A)^2$ for all primes p. In Theorem 4, choose A_1 and A_2 such that $k_p, \ell_p < \infty$ for all primes p. Then, $r_p(A) = 3$. However, $E(A)$ has p-rank 5 since $\mathrm{Hom}(A_1, A_2) = \mathrm{Hom}(A_2, A_1) = 0$ and $\mathrm{Hom}(A_3, A_i) = 0$ for $i = 1, 2$. □

References

[1] Albrecht, U.: A-projective resolutions and an Azumaya theorem for a class of mixed abelian groups, Czechoslovak Math. J. 51(126) (2001), no. 1, 73–93.

[2] Albrecht, U.: Endomorphism rings, tensor products and Fuchs'Problem 47, Contemporary Mathemetics 130, (1992), 17-31.

[3] Albrecht, U.: Endomorphism rings of faithfully flat abelian groups, Results in Math., 17, (1990),179–201.

[4] Albrecht, U., and Goeters, H. P., Strong S-groups; Colloquium Mathematicum 80 (1999); 97 - 105.

[5] Arnold, D. M.: Finite Rank Torsion Free Abelian Groups and Rings, Lecture Notes in Mathematics, 931, Springer-Verlag, (1982).

[6] Arnold, D. M.: Endomorphism rings and subgroups of finite rank torsion-free Abelian groups, Rocky Mt. J. Math., 12 (1982), 241–256.

[7] Arnold, D.M., and Lady, L.; Endomorphism rings and direct sums of torsion-free abelian groups; Trans. Amer. Math. Soc. 211 (1975); 225 - 237.

[8] Arnold, D. M. and Murley, C. E.: Abelian groups A such that $\mathrm{Hom(A,-)}$ preserves direct sums of copies of A, Pac. Jour. of Math., 56 (1975), 7-20.

[9] Fuchs, L.: Infinite abelian groups I,II, Cambridge Univ. Press (1970, 1973).

[10] F. Ulmer; A flatness criterion in Grothendieck categories, Inv. Math. 19 (1973), 331-336.

[11] Warfield, R. B.: Extensions of torsion-free Abelian groups of finite rank, Arch. Math., 23 (1972), 145–150.

Received: April, 2009