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A Construction of the Maximal Matrix Anticongruence

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Abstract

This investigation is in Bishop's constructive mathematics. A construction (without of the axiom of choose) of the maximal anticongruence q on a semigroup S such that q^C is a matrix congruence on S and description of q-classes are given. For every element a of S a construction (without of the axiom of choose) of the strongly extensional maximal consistent ideal $A(a)$ of S, such that $a \bowtie A(a)$ and $A(a^2) = A(a)$, $A(aba) = A(a)$ and $A(abc) = A(ac)$ for each b and c, is given.

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1 Introduction and preliminaries

A. Setting and motivation. Our setting is Bishop's constructive mathematics (in sense book [1], [3] and [5]), mathematics developed with Constructive logic (or Intuitionistic logic ([13])) - logic without the Law of Excluded Middle $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive logic. Precisely, in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal logic. In Constructive logic 'Weak Law of Excluded Middle'$\neg P \vee \neg\neg P$ does not hold. It is interesting, in Constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove it without 'the crazy axiom'.

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Any notion in Bishop's constructive mathematics has positive defined symmetrical pair since Law of Excluded Middle does not hold in Constructive logic. Our intention is development of these symmetrical notions and their comparability with so-called the 'first notions' in Semigroup theory. As the first, semigroup is equipped with diversity relation compatible with the equality, and, the second, the semigroup operation is total extensional and strongly extensional function. Symmetrical relations to Green's relations are also interesting in Constructive algebra.

B. Set with apartness Let $(S, =, \neq)$ be a set (in the sense of books [1], [3], [5] and [13]), where "$=$" is an equality and "$\neq$" is a binary relation on S which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z$$

called *diversity relation* on S. Following Heyting, if the relation $\neq$ satisfies the following implication

$$x \neq z \Longrightarrow (\forall y \in S)(x \neq y \vee y \neq z),$$

we say that it is an *apartness*. Let Y be a subset of S and let $x \in S$. Following Bridges, by $x \bowtie Y$ we denote $(\forall y \in Y)(y \neq x)$ and by Y^C we denote subset $\{x \in S : x \bowtie Y\}$ - the strong complement of Y in S ([13]). The subset Y of S is *strongly extensional* ([13]) in S if and only if $y \in Y \Longrightarrow y \neq x \vee x \in Y$.

Example 1: (1) Let $\wp(S)$ be power-set of set S. If for subsets A, B of S we define $A \neq B$ if and only if $(\exists a \in A)\neg(aB)$ or $(\exists b \in B)\neg(b \in A)$, then the relation "$\neq$" is diversity relation on $\wp(S)$ but it is not an apartness.
(2) ([5]) The relation "$\neq$" defined on the set $\mathbf{Q}^N$ by

$$f \neq g \Longleftrightarrow (\exists k \in N)(\exists n \in N)(m \geq n \Longrightarrow |f(m) - g(m)| > k^{-1})$$

is an apartness on $\mathbf{Q}^N$. ♦

Let S be a set with apartness and let α, β be relations on S. The *filed product* ([6], [8]-[11]) of α and β is the relation defined by

$$\beta * \alpha = \{(x, z) \in S \times S : (\forall y \in S)((x, y) \in \alpha\ \vee\ (y, z) \in \beta)\}.$$

For natural $n > 2$, let ${}^n\alpha = \alpha * ... * \alpha$ (n factors). Put ${}^1\alpha = \alpha$. By $c(\alpha)$ we denote the intersection $c(\alpha) = \bigcap_{n \in N} {}^n\alpha$. The relation $c(\alpha)$ is a cotransitive relation on S, by Theorem 0.4 of the paper [11], called *cotransitive internal fulfillment* of the relation α.

A relation q on S is a *coequality relation* on S ([6], [8]-[12]) if and only if

$$q \subseteq \neq,\quad q^{-1} = q,\quad q \subseteq q * q.$$

For equality e and coequality q we say that they are *compatible* if and only if $e \circ q \subseteq q$ and $q \circ e \subseteq q$. In this case we can construct the following factor-sets: $S/(e,q) = \{ae : a \in S\}$, $S/(qC,q) = \{aqC : a \in S\}$ and $S/q = \{aq : a \in S\}$ with:

$$\begin{gathered} ae =_1 be \Longleftrightarrow (a,b) \in e,\ ae \neq_1 be \Longleftrightarrow (a,b) \in q; \\ aq^C =_1 bq^C \Longleftrightarrow (a,b) \in q^C,\ aq^C \neq bq^C \Longleftrightarrow (a,b) \in q \text{ and} \\ aq =_1 bq \Longleftrightarrow (a,b) \bowtie q,\ aq \neq_1 bq \Longleftrightarrow (a,b) \in q. \end{gathered}$$

Let us remind oneself of some standard notions and notations about functions: A function $\varphi : S \longrightarrow T$ is *strongly extensional* if $(\forall x, x' \in S)(\varphi(x) \neq_T \varphi(x') \Longrightarrow x \neq_S x')$; φ is an *embedding* if and only if $(\forall x, x' \in S)(x \neq_S x' \Longrightarrow \varphi(x) \neq_T \varphi(x'))$. If $\varphi : S \longrightarrow T$ is a strongly extensional function between sets with apartnesses, then the sets $Ker\varphi = \{(x,x') \in S \times S : \varphi(x) =_T \varphi(x')\}$ and $Antiker\varphi = \{(x,x') \in S \times S : \varphi(x) \neq_T \varphi(x')\}$ are compatible equality an coequality relation on S.

C. Semigroup with apartness Let $S = (S, =, \neq, \cdot)$ be a semigroup with apartness and where the semigroup operation is strongly extensional in the following sense

$$(\forall a, b, x, y \in S)((ay \neq by \Longrightarrow a \neq b) \wedge (xa \neq xb \Longrightarrow a \neq b)).$$

A subset T of S is a *consistent subset* of S ([2]) if and only if $(\forall x, y \in S)(xy \in T \Longrightarrow x \in T \wedge y \in T)$. Let T be a consistent subset of a semigroup S. T is a *filter* of S if T is a subsemigroup of S ([2]).

Example 2: Let S be the set Let S be the set

$$\left\{ \begin{pmatrix} 0\,0 \\ x\,x \end{pmatrix}, \begin{pmatrix} 0\,x \\ 0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0 \\ 0\,x \end{pmatrix}, \begin{pmatrix} x\,x \\ 0\,0 \end{pmatrix}, \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} : x \in R \wedge 0 \leq x \leq \frac{1}{2} \right\}.$$

The operation on S is the usual matrix multiplication. Then S is a semigroup with apartness. The set $Q = \{f \in S : f \neq 0\}$ is a consistent subset of S. ◆

Let q be a coequality relation on semigroup S. For q we say that it is *anticongruence* on S ([6], [8]-[12]) if and only if

$$(\forall a, b, x, y \in S)((ax, by) \in q \Longrightarrow (a,b) \in q \vee (x,y) \in q).$$

If q is anticongruence on semigroup S, then the strong complement q^C of q is a congruence on the semigroup S compatible with q. In this case we can construct semigroups $S/(q^C, q)$ and S/q with

$$aq^C =_1 bq^C \Longleftrightarrow (a,b) \in q^C,\ aq^C \neq_1 bq^C \Longleftrightarrow (a,b) \in q,\ (aq^C) \cdot (bq^C) = (ab)q^C$$

and

$$aq =_1 bq \Longleftrightarrow (a,b) \bowtie q,\ aq \neq_1 bq \Longleftrightarrow (a,b) \in q,\ aq \cdot bq = (ab)q.$$

Example 3: Let T be a strongly extensional subset of a semigroup S. Then:
(1) If T is a consistent subset of S, then the relation q on S, defined by

$$(a,b) \in q \Longleftrightarrow (a \neq b \wedge (a \in T \vee b \in T)),$$

is an anticongruence on S;
(2) The relation q on S defined by

$$(a,b) \in q \Longleftrightarrow (\exists x, y \in S^1)((xay \in T \wedge xby \bowtie T) \vee (xby \in T \wedge xay \bowtie T))$$

is an anticongruence on S. ♦

Semigroups with apartnesses were first defined and studied by A.Heyting. After that, P.T.Johnstone, J.C.Mulvey, F. Richman, D.A.Romano, W.Ruitenburg, A.S.Troelstra and D.van Dalen have worked on this topic. There are more general problems on semigroup with apartness in constructive algebra. In this paper we give a construction of a coequality relation q on a semigroup S with apartness by using of left and right principal consistent subsets of semigroups such that q is a matrix anticongruence on S. Besides, we will describe classes of q.

D. Goal of this paper. In the Classical Semigroup Theory, there several papers in which studied the Green relations. For example, in papers [4] and [7]. Various characterizations of the semigroups by Green's relations have been investigated by A. H. Cliford and G. B. Preston in their well-known book and in Bogdanicić - Ćirić's book [2]. Since in Constructive logic the 'Law of Excluded Middle' is not valid, in Bishop's constructive algebra symmetrical relations to Green's relations are also interesting. In his articles [6], [8]-[12] the author researches some special relations and subsets of semigroup with apartness generated by strongly extensional (left or right) consistent principal subsets. Particularly, in article [11] the author studied relations $c(s \cup s^{-1})$ and $c(s \cap s^{-1})$ and their classes, where $(a,b) \in s \Longleftrightarrow b \in C_{(a)} = \{x \in S^1 : x \bowtie SaS\}$. Some properties of relation $c(s)$ are given in article [12]. In article [6] the following results are given: Let a be an arbitrary element of semigroup $(S, =, \neq, \cdot, 1)$. Then the set $L_{(a)} = \{x \in S : x \bowtie Sa\}$ is a right consistent subset of S. The relation l on S, defined by $(a,b) \in l \Longleftrightarrow b \in cr(L_{(a)})$ is (Theorem 6) a consistent relation on S and the relation $c(l)$ is (Theorem 7) a quasi-antiorder on S such that that left class $A(a)$ is the maximal strongly extensional right potent semifilter of S with $a \bowtie A(a)$, and right class $B(a)$ is the maximal strongly extensional completely semiprime left ideal of S with $a \bowtie B(a)$. In article [9] the author investigated the quasi-antiorder $c(l)$, where l is a consistent relation on semigroup $(S, =, \neq, \cdot, 1)$ defined by $(a,b) \in l \Longleftrightarrow b \in L_{(a)}$. The following result are given:

(1) $c(l)$ is (Theorem 2.3) left compatible with the semigroup operation;
(2) left class $A(a)$ is (Theorem 3.1) the maximal strongly extensional right consistent subset of S with $a \bowtie A(a)$, and
(3) right class $B(a)$ is (Theorem 3.3) the maximal strongly extensional left ideal of S with $a \bowtie B(a)$.
In article [10] relation $(a, b) \in p \Longleftrightarrow b \in L_{(a)} \wedge a \in L_{(b)}$ are investigated: the relation $c(p)$ is (Theorem 3) right zero band anticongruence on S (An anticongruence q on semigroup S is right zero band anticongruence if and only if the factor-semigroup $S/(q^C, q)$ is a right zero band.) and the class $A(a)$ is (Theorem 5) the maximal strongly extensional right consistent left ideal of S with $a \bowtie A(a)$. In this article we investigate the following relation m, defined by

$$(a, b) \in m \Longleftrightarrow b \in L_{(a)} \wedge a \in L(b) \wedge a \in R(b) \wedge b \in R(a),$$

and relation $c(m)$. We prove (without the axiom of choose) that the relation $c(m)$ is the maximal matrix anticongruence on semigroup S and classes $A(a)$ ($a \in S$) of $c(m)$ are strongly extensional maximal consistent ideal of S such that $a \bowtie A(a)$ and $A(a^2) = A(a)$, $A(aba) = A(a)$ and $A(abc) = A(ac)$.

E. References For undefined notions and notations of semigroup theory we refer to [2], [4], [7] and of items in Constructive mathematics we refer to books [1], [3], [5], [13] and to papers [6], [8]-[12].

2 Matrix anticongruence

If q^C is a matrix congruence on a semigroup S ([2], [4], [7]), i.e. if

$$(\forall a, b \in S)((a, aba) \bowtie q)$$

we say that q is a *matrix anticongruence* on S. We have the following statement:

Lemma 1 ([2], Theorem 1.24) *The congruence* c *on a semigroup* S *is a matrix congruence if and only if* $(\forall x, y, z \in S)((x, x^2) \in c \wedge (xz, xyz) \in c)$.

Corollary 1.1 *A coequality relation* q *is a matrix anti-congruence on a semigroup* S *if and only if* $(x, xy) \bowtie q$ *and* $(x, yx) \bowtie q$ *for every* x, y *of* S.

Lemma 2 *Let* q *be a matrix anticongruence on a semigroup* S. *If* $a \in S$, *then* aq *is a strongly extensional consistent ideal of* S *such that* $a \bowtie aq$ *and* $a^2q = aq$, $abaq = aq$ *and* $abcq = acq$.

Proof. (i) It is clear that $a \bowtie aq$.

(ii) Let $x \in aq$, i.e. let $(a, x) \in q$ and let y be an arbitrary element of S. Then $(a, y) \in q \vee (y, x) \in q$. Thus $y \in aq \vee y \neq x$. So, the set aq is a strongly extensional subset of S.

(iii) Let $xy \in aq$, i.e. let $(a, xy) \in q$. Then $(a, x) \in q \vee (x, xy) \in q$ and $(a, y) \in q \vee (y, xy) \in q$. Thus, by Corollary 1.1, we have $y \in aq \wedge x \in aq$ because $(x, xy) \bowtie q$ and $(y, xy) \bowtie q$. So, the set aq is a consistent subset of S.

(iv) Let $x \in aq$, i.e. let $(a, x) \in q$ and let y be an arbitrary element of S. Then $(a, yx) \in q \vee (yx, x) \in q$ and $(a, xy) \in q \vee (xy, x) \in q$. Therefore, $xy \in aq$ and $yx \in aq$ because $(xy, x) \bowtie q$ and $(yx, x) \bowtie q$ by Corollary 1.1. So, the set aq is an ideal of S.

(v) Since $(a, a^2) \bowtie q$, then $aq = a^2q$. Let (u, v) be an arbitrary element of q. Then $(u, aba) \in q \vee (aba, a^2) \in q \vee (a^2, a) \in q \vee (a, v) \in q$ we conclude $u \neq aba \vee a \neq v$. Thus, $(aba, a) \bowtie q$. Similarly, we conclude that $(abc, ac) \bowtie q$. Finally, the set aq is a strongly extensional consistent ideal of S such that $a \bowtie aq$, $abaq = aq$ and $abcq = acq$. $\square$

We start with the following two theorems. Their proofs are technical:

Theorem 1 *Let a be an element of a semigroup S. The set $L_{(a)} = \{x \in S : x \bowtie Sa\}$ is a right consistent subset of S such that*:
(1) $a \bowtie L_{(a)}$;
(2) $L_{(a)} \neq \emptyset \Longrightarrow 1 \in L_{(a)}$;
(3) *If the element a is invertible, then* $L_{(a)} = \emptyset$;
(4) $(\forall x \in S)(L_{(a)} \subseteq L_{(xa)})$;
(5) $(\forall n \in N)(L_{(a)} \subseteq L_{(a^n)})$;
(6) $(\forall x \in S)\neg(xa \in L_{(a)})$;
(7) $(\forall n \in N)\neg(a^n \in L_{(a)})$.

Theorem 2 *Let a be an element of a semigroup S. The set $R_{(a)} = \{y \in S : y \bowtie aS\}$ is a left consistent subset of S such that*:
(8) $a \bowtie R_{(a)}$;
(9) $R_{(a)} \Longrightarrow 1 \in R_{(a)}$;
(10) *If a is an invertible element in S, then* $R_{(a)} = \emptyset$;
(11) $(\forall y \in S)(R_{(a)} \subseteq R_{(ay)})$;
(12) $(\forall n \in N)(R_{(a)}R_{(a^n)})$;
(13) $(\forall y \in S)\neg(ay \in R_{(a)})$;
(14) $(\forall n \in N)\neg(a^n \in R_{(a)})$.

We introduce the following notations:

$$(a, b) \in l \Longleftrightarrow b \in L_{(a)},\ (a, b) \in r \Longleftrightarrow b \in R_{(a)},$$
$$m = l \cap l^{-1} \cap r \cap r^{-1} \ .$$

These relations have the following properties:

Theorem 3 *Let a, b be elements of a semigroup S with apartness. The relation l has the following properties*:
(3.1) l is a consistent relation;
(3.2) $(a, b) \in l \Longrightarrow (\forall x \in S)((xa, b) \in l)$;

(3.3) $(a,b) \in l \Longrightarrow (\forall n \in N)(a^n, b) \in l)$;
(3.4) $(\forall x \in S)((a,xb) \in l \Longrightarrow (a,b) \in l)$;
(3.5) $(\forall n \in N)((a,b^n) \in l \Longrightarrow (a,b) \in l)$;
(3.6) $(\forall x \in S)\neg((a,xa) \in l)$;
(3.7) $(\forall n \in N)\neg((a,a^n) \in l)$;
(3.8) $(\forall y \in S)((ay,by) \in l \Longrightarrow (a,b) \in l)$.

Proof:

(1) $(a,b) \in l \Longleftrightarrow b \in L_{(a)}$
$\Longrightarrow b \neq a$
$\Longrightarrow (\forall s \in S)(b \neq s \vee s \neq a)$
$\Longrightarrow (\forall s \in S)((a,b) \neq (s,s))$.

(2) Let $(a,b) \in l$, i.e. let $b \bowtie Sa$. Then $b \bowtie Sxa$ for every x in S. So, $(xa,b) \in l$.

(3) Follows from (2).

(4) Let $(a,xb) \in l$, i.e. let $xb \bowtie Sa$. Then $xb \bowtie xSa$. Thus $b \bowtie Sa$. Therefore $(a,b) \in l$.

(5) Follows from (4).

(6) Suppose that $(a,xa) \in l$ for some $x \in S$. Then $xa \in L_{(a)}$ and $a \in L_{(a)}$ what is impossible. So, $(\forall x \in S)\neg((a,xa) \in l)$.

(7) Follows from (6).

(8) Let $(ay,by) \in l$, i.e. let $by \bowtie Say$. Thus $b \bowtie Sa$, i.e. $(a,b) \in l$. $\Box$

Theorem 4 *Let a, b be elements of a semigroup S with apartness. The relation r has the following properties*:
(4.1) *r is a consistent relation on S* ;
(4.2) $(a,b) \in r \Longrightarrow (\forall y \in S)((ay,b) \in r)$;
(4.3) $(a,b) \in r \Longrightarrow (\forall n \in N)((a^n,b) \in r)$;
(4.4) $(\forall y \in S)((a,by) \in r \Longrightarrow (a,b) \in r)$;
(4.5) $(\forall n \in N)((a,b^n) \in r \Longrightarrow (a,b) \in r)$;
(4.6) $(\forall y \in S)\neg((a,ay) \in r)$;
(4.7) $(\forall n \in N)\neg((a,a^n) \in r)$;
(4.8) $(\forall x \in S)((xa,xb) \in r \Longrightarrow (a,b) \in r)$.

Proof: The proof is similarly to theorem 3.

The next theorem is one of main results of this paper.

Theorem 5 *The relation $c(m)$ is a matrix anticongruence on S.*

Proof. (a)If (a,b) is an element of $c(m)$, then $(a,b) \in l$ and $(\forall s \in S)((a,b) \neq (s,s))$ by (3.1).So, the relation $c(m)$ is a consistent relation on S.

(b) (i) Let (a,b) be an arbitrary element of $c(m)$. Then $(a,b) \in m$ and $(b,a) \in m$ because m is a symmetric relation on S. If $(a,b) \in c(m)$, then $(a,b) \in {}^2m$, i.e. $(\forall s \in S)(a,s) \in m \vee (s,b) \in m)$. Thus $(\forall s \in S)((s,a) \in m \vee (b,s) \in m)$,

i.e. $(b,a) \in {}^2m$.
(ii) Suppose $(a,b) \in c(m) \Longrightarrow (b,a) \in {}^nm$.
(iii) Then $(a,b) \in c(m)$ implies that $(a,b) \in {}^{n+1}m$, i.e. $(\forall s \in S)((a,s) \in {}^nm \vee (s,b) \in m)$. Thus $(\forall s \in S)((s,a) \in {}^nm \vee (b,s) \in m)$, i.e. $(b,a) \in {}^{n+1}m$. Therefore, by induction, we have that $(b,a) \in \bigcap_{n\in N} {}^nm = c(m)$. So, the relation $c(m)$ is symmetric.

(c) The relation $c(m)$ is a cotransitive relation on S ([11, Theorem 2.4]).

(d) Let (u,v) be an arbitrary element of $c(m)$. Then $(u,a^2) \in c(m) \vee (a^2,a) \in c(m) \vee (a,v) \in c(m)$ holds and thus $u \neq a^2 \vee a \neq v$ by (a) of this proof and by (3.4). So, $(a^2,a) \bowtie c(m)$.

(e) Let (u,v) be an arbitrary element of $c(m)$. Then $(u,a) \in c(m) \vee (a,aba) \in c(m) \vee (aba,v) \in c(m)$ and thus $u \neq a \vee aba \neq v$ because $(a,aba) \in c(m)$ is impossible by (3.6). So, $(a,aba) \bowtie c(m)$.

(f) Let $a,b,y \in S$ such that $(ay,by) \in c(m)$. Then $(ay,a) \in c(m) \vee (a,b) \in c(m) \vee (b,by) \in c(m)$. Thus $(a,b) \in c(m)$ because $(ay,a) \in c(m)$ and $(b,by) \in c(m)$ are impossible. Analogously, we have that $(xa,xb) \in c(m)$ implies $(a,b) \in c(m)$. So, the relation $c(m)$ is a matrix anticongruence on S. □

3 c(m) -classes

By an element a of a semigroup S and for $n \in N$ we introduce the following notations

$$A(a) = \{x \in S : (a,x) \in c(m)\},\ A_n(a) = \{x \in S : (a,x) \in {}^nm\}.$$

By the following theorem we will present some basic characteristics of these sets:

Theorem 6 *Let a, b and c be elements of S. Then*:
(6.1) $A_1(a) = \{x \in L_{(a)} \cap R_{(a)} : a \in L_{(x)} \cap R_{(x)}\}$;
(6.2) $(\forall n \in N)(A_{n+1}(a) \subseteq A_n(a))$;
(6.3) $(\forall n \in N)(A_{n+1}(a) = \{x \in S : S = A_n(a) \cup A_1(x)\})$;
(6.4) $A(a) = \bigcap_{n\in N} A_n(a)$;
(6.5) $a \bowtie A(a)$;
(6.6) *The set $A(a)$ is a strongly extensional consistent ideal of S*;
(6.7) $A(a^2) = A(a)$;
(6.8) $A(aba) = A(a)$;
(6.9) $A(abc) = A(ac)$.

Proof: (2) $x \in A_{n+1}(a) \Longleftrightarrow (a,x) \in {}^{n+1}m$

$$\begin{aligned} &\Longleftrightarrow (\forall s \in S)((a,s) \in {}^nm \vee (s,x) \in m) \\ &\Longrightarrow (a,x) \in {}^nm \\ &\Longleftrightarrow x \in A_n(a). \end{aligned}$$

$$\begin{aligned}(3)\ x \in A_{n+1}(a) &\iff (\forall s \in S)((a,s) \in {}^n m \vee (s,x) \in m)\\ &\iff (\forall s \in S)(s \in A_n(a) \vee s \in A_1(x))\\ &\implies S = A_n(a) \cup A_1(x).\end{aligned}$$

(6) (i) Let $x \in A(a)$ and let y be an arbitrary element of S. Then $(a,x) \in c(m)$ and $(a,y) \in c(m)$ or $(y,x) \in c(m)$. Thus, $x \in A(a) \vee y \neq x$. So, the set $A(a)$ is a strongly extensional subset of S.

(ii) Suppose $xy \in A(a)$, i.e. $(a,xy) \in c(m)$. Then $(a,x) \in c(m) \vee (x,xy) \in c(m)$ and $(a,y) \in c(m) \vee (y,xy) \in c(m)$. By (3.6) and (4.6) we have $x \in A(a)$ and $y \in A(a)$. So, the set $A(a)$ is a consistent subset of S.

(iii) Let $x \in A(a)$, i.e. let $(a,x) \in c(m)$. Then $(a,xy) \in c(m) \vee (xy,x) \in c(m)$. Thus, $xy \in A(a)$ because $\neg((xy,x) \in c(m))$. Analogously, we have the implication $y \in A(a) \implies xy \in A(a)$. So, the set $A(a)$ is an ideal of S.

$$\begin{aligned}(7)\ x \in A(a) &\iff (a,x) \in c(m)\\ &\implies (a,a^2) \in c(m) \vee (a^2,x) \in c(m)\\ &\implies x \in A(a^2) \quad \text{(because } \neg((a,a^2) \in c(m)).\end{aligned}$$

Equalities (8) and (9) are provided analogously as (7). $\square$

As end of this analysis we will give a description of elements of semigroup $S/c(m)$

Theorem 7 *Let a, b and c be elements of S. Then the set $A(a)$ is the maximal strongly extensional consistent ideal of S such that $a \bowtie A(a)$ and $A(a^2) = A(a)$, $A(aba) = A(a)$ and $A(abc) = A(ac)$.*

Proof: Let T be a strongly extensional consistent ideal of S such that $a \bowtie T$. Then $a \bowtie StS$, i.e. then $a \in L_{(t)} \cap R_{(t)}$ for every t in T. Besides, for arbitrary element t of T we have $t \neq uav \vee uav \in T$ for every u, v of S because T is a strongly extensional subset of S. Thus $t \neq uav$ for every $u, v \in S$ because $uav \in T$ implies $a \in T$ what is impossible. So, $t \bowtie SaS$, i.e. $t \in L_{(a)} \cap R_{(a)}$. Therefore, $T \subseteq A_1(a)$. Assume that $T \subseteq A_n(a)$. Let t be an arbitrary element of T and let z be an arbitrary element of S. Then $t \neq uzv \vee uzv \in T$. As T is a consistent subset of S we have $t \bowtie SzS \vee z \in T \subseteq A_n(a)$. At the other hand, we have the implication $t \in T \implies StS \subseteq T$ because T is an ideal of S. Thus, $z \neq utv \vee z \in T \subseteq A_n(a)$ and $z \bowtie StS \vee z \in A_n(a)$. Finally, we have $z \in A_1(t) \cup A_n(a)$. Therefore, $S = A_1(t) \cup A_n(a)$, i.e. $T \subseteq A_{n+1}(a)$. By induction we have $T \subseteq A(a)$. So, the set $A(a)$ is the maximal strongly extensional consistent ideal of S such that $a \bowtie A(a)$. $\square$

For further, we need the following result:

Lemma 3 *Let q_1 and q_2 are two coequality relations on a set S and let S/q_1 and S/q_2 be their families of classes respectively. Then $q_1 \subseteq q_2$ if and only if $(\forall Z" \in S/q_2)(\exists Z' \in S/q_1)(Z' \subseteq Z")$.*

Finally, we have assertion that the relation $c(m)$ is the maximal matrix anticongruence on S:

Theorem 8 *The relation $c(m)$ is the maximal matrix anticongruence on a semigroup S with apartness.*

Proof: Let q be a matrix anticongruence on a semigroup S with apartness and let a be an arbitrary element of S. Then aq is a strongly extensional consistent ideal of S such that $a \bowtie aq$. Thus, $aq \subseteq A(a)$ by Theorem 7. Therefore, the relation $c(m)$ is the maximal matrix anticongruence on S by Lemma 3. $\Box$

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