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Proper Modules Over Profinite Ringoids

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Abstract

We define a notion of profinite ringoid and a suitable class of modules over these. We describe two functors which in the case of a connected profinite ringoid agree: the derived functors of these are the analogues of continuous cohomology of profinite ringoids. The resulting 'change of ringoid' formulae reduce its computation to continuous cohomology of profinite rings.

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1 Preliminaries

Although ringoids have been well studied in many branches of mathematics, the literature on their cohomology is somewhat hard to find. Motivated by their importance in algebraic topology, in this paper we outline a theory of profinite ringoids due to Baker [2] and the cohomology of proper modules over profinite ringoids which is intended to generalize standard ideas in Galois cohomology as described in Shatz [6]. Our ultimate motivation is the need to have 'change of rings theorems' for such cohomology, allowing reduction to cohomology of automorphism groups. Devinatz's paper [4] gives an indication of the kind of result required; an application in which the use of topological splittings of profinite ringoids is in sufficient because of the topology appears in [1].

2 Profinite Ringoids

Let $\mathcal{R}$ be a ringoid (i.e. a small category in which every morphism is invertible). The function $\mathrm{Obj}\mathcal{R} \to \mathrm{Mor}\mathcal{R}$ which sends each object to its identity morphism can be viewed as embedding $\mathrm{Obj}\mathcal{R}$ in to $\mathrm{Mor}\mathcal{R}$ so we can view $\mathcal{R}$ as consisting of the set of all its morphisms. Later we will use the notation

$$\mathcal{R}(*,x) = \bigcup_{y\in \mathrm{Obj}\mathcal{R}} \mathcal{R}(y,x) \subseteq \mathcal{R}.$$

Recall from Higgins [4] that a subgroupoid $\mathcal{N}$ of a groupoid $\mathcal{G}$ is normal if it has the following properties:

A) $\mathrm{Obj}\mathcal{N} = \mathrm{Obj}\mathcal{G}$;

B) for every morphism $x \xrightarrow{f} y$ in $\mathcal{G}$,

$$f\mathcal{N}(x,x) f^{-1} = \mathcal{N}(x,x).$$

We write $\mathcal{N} \lhd \mathcal{G}$ is a normal subgroupoid of $\mathcal{G}$.

Due to above definition, an ideal I of a ringoid $\mathcal{R}$ has the following properties:

1) $\mathrm{Obj}I = \mathrm{Obj}\mathcal{R}$

2) for every morphism $x \xrightarrow{f} y$ in $\mathcal{R}$,

$$f I(x,x) f^{-1} = I(x,x).$$

We can then form the quotient ringoid $\mathcal{R}/I$ whose objects are equivalence classes of objects of $\mathcal{R}$ under the relation $x \sim y \Leftrightarrow \exists x \xrightarrow{h} y$ in I. Similarly the morphisms are equivalance classes of morphisms of $\mathcal{R}$

$x \xrightarrow{f} y \ \sim \ x' \xrightarrow{f'} y' \Leftrightarrow \exists x \xrightarrow{p} x', y \xrightarrow{q} y'$ in I such that $f' = q\ f\ p^{-1}$.

Composition of equivalence classes is defined by

$$[y \xrightarrow{g} z][x \xrightarrow{f} y] = [x \xrightarrow{gf} z]$$

whenever these classes contain composable elements. This is well defined since given morphisms $x \xrightarrow{p} x', y \xrightarrow{q} y', y \xrightarrow{r} y', z \xrightarrow{s} z'$ in I,

$$(sgr^{-1})(qfp^{-1}) = sg(r^{-1}q)\,fp^{-1} = (sh)gfp^{-1}$$

where $h = g(r^{-1}q)g^{-1}$ is in $I(z,z)$. There is an evident quotient functor $\mathcal{R} \to \mathcal{R}/I$.

We define a ringoid $\mathcal{R}$ to be automorphism finite if for every $x \in \mathrm{Obj}\mathcal{R}$, $\mathcal{R}(x,x)$ is a finite ring. We define a ringoid $\mathcal{R}$ to be (automorphism) profinite if it is the inverse limit of automorphism finite ringoids,

$$\mathcal{R} \cong \varprojlim_{I\subset\mathcal{R}} \mathcal{R}/I.$$

Such a ringoid has a naturel topology in which basic open sets have the form

$$U(x \xrightarrow{f} y, I) = \{x \xrightarrow{g} y : gf^{-1} \in I(y,y)\},$$

where $f \in \mathrm{Mor}\mathcal{R}$ and $\mathcal{R}/I$ is automorphism finite.

A topological ringoid is a ringoid which is a topological space such that the partial composition $\mathcal{R} \underset{\mathrm{Obj}\mathcal{R}}{\times} \mathcal{R} \to \mathcal{R}$, inverse function $\mathcal{R} \to \mathcal{R}$, domain and codomain functions $\mathcal{R} \to \mathrm{Obj}\mathcal{R}$ are continuous, where $\mathrm{Obj}\mathcal{R}$ has the subspace topology. A profinite ringoid in the above sense is a topological ringoid.

Let $\mathcal{R}$ be a ringoid. Recall that $\mathcal{R}$ is connected if for every pair of objects x, y in $\mathcal{R}$ there is a morphism $x \xrightarrow{f} y$. More generally, $\mathcal{R}$ is a disjoint union of connected subringoids, which we refer to as the connected components of $\mathcal{R}$.

3 Proper Modules Over Profinite Ringoids

In order to define the cohomology of profinite ringoids we first need to define suitable a notion of module and we follow the ideas of Galois cohomology, accessibly described in Shatz [6], with Weibel [7] providing a more general cohomological discussion.

For a profinite ringoid $\mathcal{R}$, a proper $\mathcal{R}-$module (over a commutative unital ring $\mathbf{k}$) is a functor $\underline{M}:\mathcal{R} \to \mathrm{Mod}_{\mathbf{k}}$ in which for $x, y \in \mathrm{Obj}\mathcal{R}$, $m \in \underline{M}(x), f \in \mathcal{R}(x, y)$, the set

$$\mathrm{Stab}_{\mathcal{R}}(m, f, y) = \{g \in \mathcal{R}(x, y) : \underline{M}(g)m = \underline{M}(f)m\} \subseteq \mathcal{R}(x, y)$$

is open. This generalizes the notion of proper module for a profinite ring, for which stabilizers of points are of finite index. We will denote the category of all proper $\mathcal{R}-$ modules over $\mathbf{k}$ by $\mathrm{Mod}_{\mathbb{k},\mathcal{R}}$, where the morphisms are naturel transformations.

For later use we set

$$\underline{\mathbf{M}} = \bigoplus_{x \in \mathrm{Obj}\mathcal{R}} \underline{M}(x)$$

and view this as a topological space with the discrete topology. We then define a section of $\underline{M}$ to be a function $\Phi : \mathrm{Obj}\mathcal{R} \to \underline{\mathbf{M}}$ with the property that for all $x \in \mathrm{Obj}\mathcal{R}$,

$$\Phi(x) \in \underline{M}(x).$$

We will denote the set of all sections by $\mathrm{Sect}(\mathcal{R};\underline{M})$; this is a $\mathbf{k}-$module.

Proposition 2.1. Let $\mathcal{R}$ be a profinite ringoid and $\mathbf{k}$ a commutative unital ring.

a) $\mathrm{Mod}_{\mathbf{k},\mathcal{R}}$ is an abelian category, with structure inherited from that of $\mathrm{Mod}_{\mathbf{k}}$.

b) $\mathrm{Mod}_{\mathbf{k},\mathcal{R}}$ has sufficiently many injectives.

Proof: (a) is straightforward. For (b), suppose that $\underline{M}$ is a proper $\mathcal{R}-$module. Then we need to constract an embedding in to an injective object. We follow [6, chapter II § 2] in defining the functor $J_{\underline{M}}$ which for $x \in \mathrm{Obj}\mathcal{R}$ is given by

$$J_{\underline{M}}(x) = \{\Phi : \mathcal{R}(*, x) \to \underline{\mathbf{M}} : \Phi(g) \in \underline{M}(\mathrm{dom}\, g) \text{ and } \Phi \text{ continuous}\}$$

and for $f \in \mathrm{Mor}\mathcal{R}$,

$$J_{\underline{M}}(f) = \underline{M}(f)_*,$$

the map induced by composition with $\underline{M}(f)$. A modification of the argument of [6, chapter II § 2, Proposition 4] shows that $J_{\underline{M}}$ is a proper $\mathcal{R}-$module and furthermore there is an evident naturel transformation $j:\underline{M}\to J_{\underline{M}}$ for which

$$j:\varphi(x)\to J_{\underline{M}}(x);\, j(m)(g)=\varphi(g^{-1})m \qquad (\varphi\in\underline{M},\ x\in \mathrm{Obj}\mathcal{R}, m\in\varphi(x)).$$

It is now straigforward to verify the following adjunction formula which shold be compared with [6, chapter II § 2, Proposition 4] and [7, example 2.3.13]:

$$\mathrm{Mod}_{\mathbf{k},\mathcal{R}}(\underline{N},J_{\underline{M}})\cong\prod_{x\in \mathrm{Obj}\mathcal{R}}\mathrm{Mod}_{\mathbf{k}}(\underline{N}(x),\underline{M}(x)). \tag{3.1}$$

In particular if each of the $\underline{M}(x)$ is an injective $\mathbf{k}-$module, then $J_{\underline{M}}$ is an injective $\mathcal{R}-$module. ■

4 Continuous Cohomology Of Profinite Ringoids

We will consider two types of functor $\mathrm{Mod}_{\mathbf{k},\mathcal{R}}\to\mathrm{Mod}_{\mathbf{k}}$. The first is a sort of fixed point consruction

$$(\ )^{\mathcal{R}}:\underline{M}\to\underline{M}^{\mathcal{R}}=\{\Phi\in \mathrm{Sect}(\mathcal{R},\underline{M}):\forall f\in\mathcal{R},\ \underline{M}(f)\Phi(\mathrm{dom}\, f)=\Phi(\mathrm{codom}\, f)\}.$$

This functor is left exact, so has rigth derived functors $\mathcal{R}^{n}(\ )^{\mathcal{R}}$ for $n\geq 0$, which can be viewed as the continuous cohomology of $\mathcal{R}$, $\mathbf{H}_c^n(\mathcal{R};)$.

For the second type of functor, we fix $x_0\in\mathrm{Obj}\mathcal{R}$ and define a kind of 'localization at x_0',

$$(\ )_{x_0}:\underline{M}\to\underline{M}(x_0)^{\mathcal{R}(x_0,x_0)}$$

obtained by restricting to $\underline{M}(x_0)$ and taking the fixed points under the action of the automorphism group of x_0. Again this is left exact and has right derived functors $\mathrm{R}^n(\ )_{x_0}$ which we denote by $\mathbf{H}^n_{x_0}(\mathcal{R};)$.

Proposition 3.1. If the profinite ringoid $\mathcal{R}$ is connected, then for any $x_0\in\mathrm{Obj}\mathcal{R}$, there is a natural equivalence of functors

$$(\ )_{x_0}\cong(\ )^{\mathcal{R}}.$$

Hence there are natural equivalences of functors

$$\mathbf{H}_c^n(\mathcal{R};)\cong\mathbf{H}^n_{x_0}(\mathcal{R};) \qquad (n\geq 0).$$

Proof: For the first part, we produce a $\mathbf{k}-$isomorphism $F:\underline{M}_{x_0}\xrightarrow{\cong}\underline{M}^{\mathcal{R}}$. For $m\in\underline{M}_{x_0}$, we define $F(m)=\Phi_m$ by

$$\Phi_m(x)=\underline{M}(f)m \qquad (x\in\mathrm{Obj}\mathcal{R})$$

where we choose any $f\in\mathcal{R}(x_0,x)$; this choice does not affect the outcome since for a second choice $g\in\mathcal{R}(x_0,x),\ f^{-1}g\in\mathcal{R}(x_0,x_0)$ and therefore

$$\underline{M}(g)m = \underline{M}(f)\underline{M}(f^{-1}g)m = \underline{M}(f)m.$$

This is easily verified to be an isomorphism.

The second part is an immediate consequence of the first by a standart result on δ – functors, see for example [7, chapter 2]. ■

This reduce the calculation of the continuous cohomology $\mathbf{H}_c^*(\mathcal{R};\underline{M})$ of a proper module $\underline{M}$ to continuous ring cohomology, allowing application of the standart tecniques described in [6].

Theorem 3.2. If $\mathcal{R}$ is a connected profinite ringoid and $\underline{M}$ a proper $\mathcal{R}$ – module over $\mathbf{k}$, then for any $x_0 \in \operatorname{Obj}\mathcal{R}$

$$\mathbf{H}_c^*(\mathcal{R};\underline{M}) \cong \mathbf{H}_c^*(\mathcal{R}(x_0,x_0);\underline{M}(x_0)).$$

In the general case we have the following.

Theorem 3.3. If $\mathcal{R}$ is a profinite ringoid and $\underline{M}$ a proper $\mathcal{R}$ – module over $\mathbf{k}$, then for each $n \geq 0$,

$$\mathbf{H}_c^n(\mathcal{R};\underline{M}) \cong \prod_C \mathbf{H}_c^n(\mathcal{R}(x_C,x_C);\underline{M}(x_C)),$$

where C ranges over the connected components of $\mathcal{R}$ and x_C is a chosen object of C.

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