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On Additive Groups of Involution Rings with Chain Conditions

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Abstract

Various structures of additive groups of rings with involution are studied under chain conditions on *-ideals and *-biideals. The situations are analyzed when the additive groups are torsion free, non-nil torsion free, torsion, non-nil torsion, and in general, mixed.

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1. Introduction

Study of additive groups of rings was initiated by R.A.Beaumont in [5] and was followed by others as L. Redei and T. Szele [12], R.A. Beaumont and H.S. Zuckerman [6], L. Fuchs [9], A. Kertesz [10], and S. Feigelstock [7]. In particular, Feigelstock treated this subject more deeply. Theory of involution appeared in this subject in [3] during a study of embedding on rings with involution. Our aim here is to extend and investigate this subject using rings with involution under chain conditions. Mostly, we follow and extend the work of [7] to involution rings.

In [3],Aburawash gave the following characterization of the additive groups of involution rings with DCC on*-biideals.

Proposition 1.1 ([2], Lemma 1). *The additive group G of a ring R with involution satisfying DCC for *-biideals can be decomposed in the form*

$$G \cong \bigoplus_{\alpha} Q^{+} \bigoplus_{finite} \mathbb{Z}_{p_i^{\infty}} \bigoplus_{\beta} \mathbb{Z}_{p_j^{k_j}}$$

with $p_j^{k_j} \mid m$ *where* α, β *are arbitrary cardinals,* p_i, p_j *are primes, and* m *is a fixed positive integer. Moreover, if* R *has the identity, this decomposition can be reduced to*

$$G \cong \underset{\alpha}{\oplus} Q^+ \underset{\beta}{\oplus} \mathbb{Z}_{p_j^{k_j}}$$

We begin our work by proving that the converse of this Proposition holds in particular case. As a result we get a concrete characterization of the additive group of such rings. Further, we analyze different situations in which the additive group G is torsion free, non-nil torsion free, torsion, non-nil torsion, and in general, when G is mixed.

In the sequel we assume that all rings are associative (may not possess identity). The underlying additive group of a ring R is denoted by $G = R^+$.

An additive group G is said to be *nil*, in the sense of Feigelstock and Schlussel [8], in case the only ring R with $G = R^+$ is the zero ring ($R^2 = 0$). G is termed as *non-nil* in case there is a ring R, with $G = R^+$ such that $R^2 \neq 0$.

A ring R (an additive abelian group G) together with a unary operation $*$ is said to be a *ring (a group) with involution* in case for all $a, b \in R$,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad and \quad (ab)^* = b^* a^*$$

(respectively, for all $a, b \in G$, $(a^*)^* = a$, *and* $(a+b)^* = a^* + b^*$).

An ideal I of R (a subgroup H of G) which is closed under involution; that is $I^{(*)} = \{a^* \in R \mid a \in I\} \subseteq I$, is termed as **-ideal* ($I \lhd^* R$) (respectively, **-subgroup* ($H \leq^* G$)).

The *principal ideal* (**-ideal*) of an involution ring generated be the element a will be denoted by $\langle a \rangle$ ($\langle a \rangle^*$) and the *principal subgroup* (**-subgroup*) of an involution group generated be the element a will be denoted by (a) ($(a)^*$).

A subring A of R is said to be a *biideal* of R if $ARA \subseteq A$ and a **-biideal* if ;in addition, it is closed under the involution of R.

A is called a *principal *-biideal* (see [11]), if

$$A = \langle a \rangle_{bi}^* = \mathbb{Z}a + \mathbb{Z}a^* + aRa + a^*Ra + aRa^* + a^*Ra^*.$$

A ring with involution $*$ is a *principal *-biideal ring* if each *-biideal is a principal *-biideal.

The following auxiliary lemmas are straightforward and can be verified by elementary ring theoretic arguments.

Lemma 1.2. *Every field has no propoer biideals.*

Proof. Let B be a nonzero biideal of a field F, then for every $a \in F$ and $b \in B$, we have $a = bab^{-1} \in BFB \subseteq B$, whence $F = B$.■

2. Additive group of involution rings with chain conditions

First, we prove Proposition 4.3.1 in [7], when the involution ring has dcc on *-ideals.

Proposition 2.1. *A group G is the additive group of an involution ring satisfying dcc on *-ideals if and only if*

$$G \cong \bigoplus_{\alpha} \mathbb{Q}^+ \bigoplus_{finite} \mathbb{Z}_{p_i^\infty} \bigoplus_{\beta} \mathbb{Z}_{p_j^{k_j}}$$

with $p_j^{k_j} \mid m$, where α, β are arbitrary cardinals, p_i, p_j are primes, and m is a fixed positive integer.

Proof.Let R be an involution ring satisfying dcc for *-ideals. Following a similar proof to that of Proposition 4.3.1 in [7], we get

$$R^+ \cong \bigoplus_{\alpha} \mathbb{Q}^+ \bigoplus_{finite} \mathbb{Z}_{p_i^\infty} \bigoplus_{\beta} \mathbb{Z}_{p_j^{k_j}}.$$

Now, let

$$G \cong \bigoplus_{\alpha} \mathbb{Q}^+ \bigoplus_{finite} \mathbb{Z}_{p_i^\infty} \bigoplus_{\beta} \mathbb{Z}_{p_j^{k_j}}.$$

with $p_j^{k_j} \mid m$, where α, β are arbitrary cardinals, p_i, p_j are primes, and m is a fixed positive integer. Let F be a field with

$$F^+ \cong \bigoplus_{\alpha} \mathbb{Q}^+$$

and S be a zero ring with

$$S^+ = \bigoplus_{finite} \mathbb{Z}_{p_i^\infty}.$$

Now

$$\bigoplus_{\beta} \mathbb{Z}_{p_j^{k_j}} = \bigoplus_{j=1}^{n} \bigoplus_{k=1}^{n_j} \bigoplus_{\beta_j} \mathbb{Z}_{p_j^k} ,$$

p_j a prime and $p_j^k \mid m$, $j = 1, \ldots, n, k = 1, \ldots, n_j$. For β_j finite, let T_{jk} be the sum of β_j copies of the ring of integers modulo ${p_j}^k$. For β_j infinite, by [9, Lemma 122.3], a commutative ring with unity T_{jk} can be constructed so that

$$T_{jk}^+ = \bigoplus_{j} \mathbb{Z}_{p_j^k} ,$$

and the only (leftt) ideals in T_{jk} are T_{jk}, $pT_{jk}, \cdots$, $p^{n_j} T_{jk} = 0$. Put

$$T = \bigoplus_{j=1}^{n} \bigoplus_{k=1}^{n_j} T_{jk}.$$

Then the ring $R = F \oplus S \oplus T$ is commutative and possesses only finitely many ideals [7, page 51]. Hence R with the identity involution has finitely many *-ideals. ■

Secondly, we give the converse of Lemma 1 posed by Aburawash in [3] in a particular case.

Proposition 2.2. *The additive group G of a ring R with involution satisfying dcc for *-biideals can be decomposed in the form*

$$G \cong \underset{\alpha}{\oplus}\mathbb{Q}^+ \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty} \underset{\beta}{\oplus} \mathbb{Z}_{p_j^{k_j}}$$

with $p_j^{k_j} \mid m$ where α, β are arbitrary cardinals, p_i, p_j are primes, and m is a fixed positive integer. Mmoreover, if

$$G \cong \underset{\alpha}{\oplus}\mathbb{Q}^+ \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty} \underset{finite}{\oplus} \mathbb{Z}_{p_j^{k_j}} \underset{\beta}{\oplus} \mathbb{Z}_{p_j}$$

*with $p_j^{k_j} \mid m, \, , j = 1, \ldots, n, k = 1, \ldots, n_j$, where α, β are arbitrary cardinals, p_i, p_j are primes, and m is a fixed positive intege*r, then G is *the additive group of a ring R with involution satisfying dcc for *-biideals.*

Proof. Let R be an involution ring satisfying the dcc for *-biideals. Then by [3, Lemma 1],

$$R^+ \cong \underset{\alpha}{\oplus}\mathbb{Q}^+ \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty} \underset{\beta}{\oplus} \mathbb{Z}_{p_j^{k_j}}.$$

Now, let

$$G \cong \underset{\alpha}{\oplus}\mathbb{Q}^+ \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty} \underset{finite}{\oplus} \mathbb{Z}_{p_j^{k_j}} \underset{\beta}{\oplus} \mathbb{Z}_{p_j},$$

with $p_j^{k_j} \mid m$, $j = 1, \ldots, n, k = 1, \ldots, n_j$, where α, β are arbitrary cardinals, p_i, p_j are primes, and m is a fixed positive integer. Let F be a field with

$$F^+ \cong \underset{\alpha}{\oplus}\mathbb{Q}^+$$

and S be a zero ring with

$$S^+ = \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty}.$$

Now

$$\underset{finite}{\oplus} \mathbb{Z}_{p_j^{k_j}} = \overset{n}{\underset{j=1}{\oplus}} \overset{n_j}{\underset{k=1}{\oplus}} \underset{\beta_j}{\oplus}\mathbb{Z}_{p_j^k} \,,$$

p_j a prime and $p_j^k \mid m,\ j = 1, \ldots, n,\ k = 1, \ldots, n_j$. Let T_{jk} be the sum of β_j copies of the ring of integers modulo ${p_j}^k$.Put

$$T = \bigoplus_{j=1}^{n} \bigoplus_{k=1}^{n_j} T_{jk}.$$

For $\underset{\beta}{\oplus}\mathbb{Z}_{p_j}$, by [7, Theorem 4.1.1], a filed A can be constructed so that

$$A^+ = \underset{\beta}{\oplus}\mathbb{Z}_{p_j}.$$

F and A have finitely many biideals by Lemma 1.2. Since S is a zero ring, each biideal is an ideal, whence S has finitely many biideals.T is finite so it has finitely many biideals.Then the ring $R = F \oplus S \oplus T \oplus A$ is commutative and possesses only finitely many biideals . Hence R with the identity involution has finitely many *-biideals. ■

Theorem 2.3. *Let G be a torsion free group. Then the following are equivalent:*

(1) *G is the additive group of a ring with involution possessing only finitely many *-biideals.*

(2) *G is the additive group of a ring with involution satisfying DCC for *-biideals.*

(3) *$G \cong \underset{\alpha}{\oplus} Q^+$, for some cardinal α.*

Proof : (1) $\Longrightarrow$ (2) is clear.

(2) $\Longrightarrow$ (3): Let R be an involution ring satisfying DCC for *-biideals, with $R^+ = G$. For every prime p, and positive integer n,

$$p^n R \supseteq p^{n+1} R,$$

where each $p^n R$ is a *-ideal by [4, Corollary. 2.2]. Hence there exists a positive integer n such that

$$p^n R = p^{n+1} R.$$

Therefore, for $a \in G$, there exists $b \in G$ such that $p^n a = p^{n+1} b$, or $p^n(a - pb) = 0$. Since G is torsion free, $a = pb$, and so G is p-divisible for any prime p. Hence G is divisible, and

$$G \cong \underset{\alpha}{\oplus} Q^+ ,$$

by [7, Proposition 1.1.3].

(3) $\Longrightarrow$ (1): Since $\underset{\alpha}{\oplus} Q^+$ is the additive group of some field, by [7, Theorem 4.1.3],G is the additive group of some ring with the identity involution satisfying DCC for *-biideals. ■

Corollary 2.4. *Let G be a non-nil torsion free group. Then the following are equivalent:*

(1) *Every involution ring R with $R^+ = G$ and $R^2 \neq 0$ possesses only finitely many *-biideals.*

(2) *Every involution ring R with $R^+ = G$ and $R^2 \neq 0$ satisfies DCC for *-biideals.*

(3) $G \cong Q^+$.

Proof : (1) $\Longrightarrow$ (2) is clear and (3) $\Longrightarrow$ (1) is by [7, Corollary 2.2.5].

(2) $\Longrightarrow$ (3): Suppose that every involution ring R with $R^+ = G$ and $R^2 \neq 0$ satisfies DCC for *-biideals. So by Theorem 2.3,

$$G \cong \bigoplus_{\alpha} Q^+$$

and

$$G \cong Q^+ \oplus H.$$

Let S be a zero ring with $S^+ = H$. The ring direct sum $R = Q \oplus S$ with the identity involution satisfies $R^+ \cong G$, and $R^2 \neq 0$. If $H \neq 0$, choose $0 \neq a \in H$. The infinite chain of *-biideals in R,

$$\langle a \rangle_{bi}^{*} \supset \langle 3a \rangle_{bi}^{*} \supset \left\langle 2^2 a \right\rangle_{bi}^{*} \supset \cdots$$

is properly descending, a contradiction. Hence $H = 0$ and $G \cong Q^+$.■

Remark: In the above proof one may note that if $G \cong \bigoplus_{finite} Q^+$, by [2, Proposition 13], then (2) and (1) of Corollary 2.4 are satisfied.

Theorem 2.5. *Let G be a torsion group. The following are equivalent:*

*(1) G is the additive group of a ring with involution possessing only finitely many *-ideals.*

*(2) G is the additive group of a ring with involution satisfying ascending chain condition for *-ideals.*

(3) $G = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j}$, *$p_i$ a prime, m, n_j positive integers, α_j an arbitrary cardinal, $i = 1, \cdots, m$, $j = 1, \cdots, n_j$.*

Proof. *(1)$\Longrightarrow$(2)* is obvious.

(2)$\Longrightarrow$(3). Suppose there exists infinitely many distinct primes $\{p_i\}_{i=1}^{\infty}$for which $G_{p_i} \neq 0$. Let R be a ring with involution with $R^+ = G$ such that

R satisfies the ascending chain condition for *-ideals. The infinite chain of *-ideals in R,

$$G_{p_1} \subseteq \overset{2}{\underset{i=1}{\oplus}} G_{p_i} \subset ... \subset \overset{k}{\underset{i=1}{\oplus}} G_{p_i} \subset \cdots$$

is properly ascending, a contradiction. Hence $G_p \neq 0$ for only a finite set of primes $\{p_i\}_{i=1}^{m}$. For every $p \in \{p_i\}_{i=1}^{m}$,

$$G_p[p] \subseteq G_p[p^2] \subseteq ... \subseteq G_p[p^k] \subseteq ...$$

is an ascending chain of *-ideals in R, each contained in G_p. Hence there exists a positive integer n_i such that

$$G_{p_i} = G_{p_i}[p_i^{n_i}],$$

$i = 1, \ldots .m$. Therefore by [7, page 53],

$$G_{p_i} = \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j},$$

α_j is a cardinal number, and so

$$G = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j}.$$

(3)⟹(1). Let

$$T = \overset{n}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} T_{ij},$$

where T_{ij} is as defined in the proof of Proposition 2.1. The ring T is commutative with unity, hence it has the identity involution and possesses only finitely many *-ideals. .■

Lemma 2.6. *Let G be a torsion group. The following are equivalent:*

*(1) G is the additive group of a ring with involution possessing only finitely many *-biideals.*

*(2) G is the additive group of a ring with involution satisfying ascending chain condition for *-biideals.*

(3) $G = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{finite}{\oplus} \mathbb{Z}_{p_i^j} \underset{\beta}{\oplus} \mathbb{Z}_{p_l}$, *$p_i$, p_l primes, m, n_j, n positive integers, β an arbitrary cardinal, $i = 1, \cdots, m$, $j = 1, \cdots, n_j, l = 1, ..., n$.*

Proof. *(1) ⟹(2)* is obvious.

(2)⟹(3). G is the additive group of a ring with involution satisfying ascending chain condition for *-ideals, then by lemma 2.5,

$$G = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j},$$

p_i *a prime,* m, n_j *positive integers,* α_j *an arbitrary cardinal,* $i = 1, \cdots, m$, $j = 1, \cdots, n_j$. Hence G satisfies *(3).*

(3)⟹(1). Let G be of the form (3). Let T_{jk} be the sum of β_j copies of the ring of integers modulo ${p_j}^k$. Put

$$T = \overset{n}{\underset{j=1}{\oplus}} \overset{n_j}{\underset{k=1}{\oplus}} T_{jk},$$

then T is finite and has finitely many *-biideals. Let $K = \underset{\beta}{\oplus} \mathbb{Z}_{p_i}$, and F be a field on K , by [7, Theorem 4.1.1], hence F has finitely many *-biideal. Thus $R = T \oplus F$ has finitely many *-biideals and $R^+ = G$. ■

Theorem 2.7. *Let G be a torsion group. The following are equivalent:*

(1) G is bounded.

*(2) G is a principle *-ideal ring group.*

*(3) G is the additive group of a involution ring satisfying ascending chain condition for *-ideals.*

*(4) G is the additive group of a involution ring possessing only finitely many *-ideals.*

Proof. *(1)⟹(2)* follows from [3, Theorem 4.10].

(2)⟹(3) by the proof of [3, Theorem 4.10].

(3)⟹(4) is deduced by Theorem 2.5.

(4)⟹(1) follows from Theorem 2.5. ■

Corollary 2.8. *Let G be a non-nil torsion group. The following are equivalent:*

*(1) Every involution ring R with $R^+ = G$ and $R^2 \neq 0$, possesses only finitely many *-biideals.*

(2) $G = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j}$, *$p_i$ a prime, m, n_j positive integers, α_j a finite cardinal, $i = 1, \ldots, m,\ j = 1, 2, \ldots, n_j$ (that is G is finite).*

Proof. *(1)⟹(2).* Suppose that G satisfies *(1).* It may be assumed that G is of the form (3) of Lemma 2.6. Suppose that α_j is an infinite cardinal for some index j. Then

$$G = \overset{\infty}{\underset{j=1}{\oplus}} (a_j) \oplus H\ ,$$

$|a_j| = p, p$ a prime. Let R_j be a ring isomorphic to the ring of integers modulo p, with $R_j^+ = (a_j),\ j = 1, 2, \cdots$, and let S be the zero ring with $S^+ = H$. The ring direct sum

$$R = \overset{\infty}{\underset{j=1}{\oplus}} R_j \oplus S$$

is a commutative ring, hence R with the identity involution is an involution ring satisfies $R^+ = G$, and $R^2 \neq 0$.Moreover

$$R_j \overset{*}{\lhd} R$$

for $j = 1, 2, \cdots$, a contradiction.

(2)⟹(1) is obvious. ■

Theorem 2.9. *Let G be a torsion group. The following are equivalent.*

*(1) G is the additive group of an involution ring satisfying dcc for *-ideals.*

(2) $G = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j} \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty}$, p_i a prime, m, n_j positive integers, α_j an arbitrary cardinal, $i = 1, \ldots, m$, $j = 1, \ldots, n_j$.

Proof. *(1)⟹(2).* Let R be a ring with involution satisfying dcc for *-ideals with $R^+ = G$. By Theorem 2.1,

$$G \cong \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j} \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty},$$

p_i a prime, m, n_j positive integers, α_j an arbitrary cardinal, $i = 1, \ldots, m, j = 1, \ldots, n_j$.

(2)⟹(1). If G satisfies Condition *(2)*, then by the proof of Theorem 2.1, G is the additive group of a commutative involution ring satisfying dcc for *-ideals. ■

Corollary 2.10. *Let G be a torsion group. G is the additive group of an involution ring with trivial annihilator satisfying dcc for *-ideals if and only if G satisfies one, and hence all, of the equivalent conditions in Theorem 2.7.*

Proof. If G is the additive group of a ring with involution having trivial annihilators, then G is reduced, by [7, Lemma 2.2.1]. If, in addition, G is the additive group of a ring with involution satisfying dcc for *-ideals, then G is bounded, by Theorem 2.9. Conversely, if G satisfies the equivalent conditions of Theorem 2.7, then the ring constructed in proving the implication *(3)⟹(1)* of Theorem 2.5, is a ring with involution having trivial annihilator and satisfying dcc for *-ideals. ■

Theorem 2.11. *Let G be a torsion group. The following are equivalent.*

*(1) G is the additive group of an involution ring satisfying dcc for *-biideals.*

(2) $G = \underset{i=1}{\overset{m}{\oplus}}\underset{j=1}{\overset{n_j}{\oplus}}\underset{finite}{\oplus}\mathbb{Z}_{p_i^{k_j}}\underset{l=1}{\overset{n}{\oplus}}\underset{\alpha_i}{\oplus}\mathbb{Z}_{p_l}\underset{finite}{\oplus}\mathbb{Z}_{p_i^\infty}, p_i, p_l$ *primes,* m, n_j, n *positiveintegers,*α_j *an arbitrary cardinal,* $i = 1, \cdots, m,\ j = 1, \cdots, n_j, l = 1, ..., n.$

Proof. Let R be a ring with involution satisfying dcc on *-ideals with $R^+ = G$. By Proposition 2.2,

$$G \cong \underset{i=1}{\overset{m}{\oplus}}\underset{j=1}{\overset{n_j}{\oplus}}\underset{\alpha_j}{\oplus}\mathbb{Z}_{p_i^j}\underset{finite}{\oplus}\mathbb{Z}_{p_i^\infty},$$

p_i a prime, m, n_j positive integers, α_j an arbitrary cardinal, $i = 1, \ldots., m, j = 1, \ldots, n_j$, whence the form of *(2)* follows.

(2)⟹(1). If G satisfies Condition *(2)*, then by the proof of Proposition 2.2, G is the additive group of a commutative involution ring satisfying dcc for *-biideals.■

Corollary 2.12. *Let* G *be a non-nil torsion group. The following are equivalent:*

(1) Every involution ring R *with* $R^+ = G$*, and* $R^2 \neq 0$ *satisfies the dcc for *-biideals.*

(2) $G = \underset{i=1}{\overset{m}{\oplus}}\underset{j=1}{\overset{n_j}{\oplus}}\underset{\alpha_j}{\oplus}\mathbb{Z}_{p_i^j}\underset{finite}{\oplus}\mathbb{Z}_{p_i^\infty}$, p_i *a prime,* n, n_j *positive integers,* α_j *a finite cardinal,* $i = 1, \ldots, m,\ \ j = 1, 2, \ldots, n_j$.

Proof. *(1)⟹(2).* Suppose that G satisfies Condition *(1).* It may be assumed that G is of the form of Proposition 2.2. Suppose that α_j is an infinite cardinal for some index j. Then

$$G = \overset{\infty}{\underset{k=1}{\oplus}}(a_k)\underset{i=1}{\overset{m}{\oplus}}\underset{j=1}{\overset{n_j}{\oplus}}\underset{\beta_j}{\oplus}\mathbb{Z}_{p_i^j}\underset{finite}{\oplus}\mathbb{Z}_{p_i^\infty},$$

$|a_k| = p^j$, p a prime, β_j is a finite cardinal, and $i = 1, \ldots .m, j = 1, \ldots n_j$. Let T_{ij} be the sum of β_j copies of the ring of integers modulo p_i^j. Put

$$T = \underset{i=1}{\overset{m}{\oplus}}\underset{j=1}{\overset{n_j}{\oplus}} T_{ij}.$$

and let S be the zero ring with

$$S^+ = \underset{finite}{\oplus}\mathbb{Z}_{p_i^\infty}.$$

Let R_k be a ring isomorphic to the ring of integers modulo p, with $R_k^+ = (a_k), k = 1, 2, \ldots..$The ring direct sum

$$R = \overset{\infty}{\underset{k=1}{\oplus}} R_k \oplus T \oplus S$$

with the identity involution is an involution ring satisfying $R^+ = G$ and $R^2 \neq 0$. Further

$$R_k \overset{*}{\lhd} R$$

for $k = 1, 2, \ldots.,$ a contradiction.

(2)⟹(1). Suppose Condition (2) is satisfied. Then

$$G = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j} \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty},$$

p_i a prime, m, n_j positive integers, α_j a finite cardinal, $i = 1, \ldots, m, j = 1, \ldots, n_j$. Let R be any ring with involution such that $R^+ = G$ and $R^2 \neq 0$. Let

$$H = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j}$$

and

$$K = \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty}.$$

Since K is the maximal divisible subgroup of G, K is an ideal in R and $KH = HK = 0$, by [7, Lemma 2.2.1]. Hence H is an ideal in R, consequently $R = S \oplus T$ and $T^+ = K$, $S^+ = H$,. By [7, Theorem 2.1.1], K is nil, so $H^2 \neq 0$, by Corollary 2., every involution ring S with $S^+ = H$ possesses only finitely many *-biideals. Hence from the proof of Proposition 2.2, T has only finitely many *-biideals. ■

Theorem 2.13. *Let G be a group. The following are equivalent:*

*(1) G is the additive group of an involution ring possessing only finitely many *-ideals.*

(2) $G \cong \underset{\alpha}{\oplus} \mathbb{Q}^+ \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j}$, p_i a prime, m, n_j non-negative integers, α, α_j arbitrary cardinals, $i = 1, \ldots, m, j = 1, \ldots, n_j$.

Proof. *(1)⟹(2).* Let R be an involution ring possessing only finitely many *-ideals, with $R^+ = G$. Then $\bar{R} = R/G_t$ is a ring possessing only finitely many *-ideals, with $\bar{R}^+$ torsion free. Therefore

$$\bar{R}^+ \cong \underset{\alpha}{\oplus} \mathbb{Q}^+,$$

where α is an arbitrary cardinal, by Theorem 2.3. Let I be a *-ideal in G_t, $0 \neq a \in I, x \in R$. Since G/G_t is divisible, there exists $y \in G, b \in G_t$ such that $x = |a\,|y + b$. Clearly

$$ax = ab \text{ and } xa = ba.$$

Hence I is a *-ideal in R and G_t is of the form of Condition *(3)* in Theorem 2.5. By [7, Proposition 1.1.2],

$$G = H \oplus G_t,$$

where H is torsion free. It is clear that

$$H \cong G/G_t \cong \underset{\alpha}{\oplus} \mathbb{Q}^+.$$

(2)⟹(1). Let G be a group satisfying condition *(2)*. Let F be a field such that

$$F^+ = \underset{\alpha}{\oplus} \mathbb{Q}^+$$

and let R be the ring constructed in proving the implication *(3)⟹(1)* for Theorem 2.5. The ring direct sum $S = F \oplus R$ with the identity involution possesses only finitely many *-ideals, and $S^+ \cong G$. ■

Corollary 2.14. *Let G be a group. G is the additive group of an involution ring R with trivial annihilator and satisfying dcc for *-ideals if and only if G satisfies one, and hence both of the equivalent conditions of Theorem 2.13.*

Proof. Let G the additive group of an involution ring satisfying dcc on *-ideals,hence by Proposition 2.1,

$$G \cong \underset{\alpha}{\oplus} \mathbb{Q}^+ \underset{finite}{\oplus} \mathbb{Z}_{p_i^{\infty}} \underset{\beta}{\oplus} \mathbb{Z}_{p_i^{k_j}}$$

and by [10, Lemma 57.8] , $\mathbb{Z}_{p^\infty}$ is contained in the annihilator of the ring R. Hence

$$G \cong \underset{\alpha}{\oplus} \mathbb{Q}^+ \underset{\beta}{\oplus} \mathbb{Z}_{p_i^{k_j}}.$$

Conversely, if

$$G \cong \underset{\alpha}{\oplus} \mathbb{Q}^+ \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j}$$

p_i a prime, m, n_j non-negative integers. By Theorem 2.13, G is the additive group of an involution ring satisfying dcc for *-ideals and with trivial annihilator.■

Theorem 2.15. *Let G be a group. The following are equivalent:*

*(1) G is the additive group of an involution ring possessing only finitely many *-biideals.*

(2) $G \cong \underset{\alpha}{\oplus} \mathbb{Q}^+ \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{finite}{\oplus} \mathbb{Z}_{p_i^j} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i}$, p_i a prime, m, n_j non-negative integers, α, α_j arbitrary cardinals, $i = 1, \ldots, m, j = 1, \ldots, n_j$.

Proof. *(1)⟹(2).* Let R be an involution ring possessing only finitely many *-biideals, with $R^+ = G$. Then $\bar{R} = R/G_t$ is a ring possessing only finitely many *-biideals, with $\bar{R}^+$ torsion free. Therefore

$$\bar{R}^+ \cong \underset{\alpha}{\oplus}\mathbb{Q}^+,$$

where α is an arbitrary cardinal, by Theorem 2.3. Let I be a *-biideal in G_t, $0 \neq a_1, a_2 \in I, x \in R$. Since G/G_t is divisible, there exists $y \in G, b \in G_t$ such that $x = |a_1|y + b$. Clearly

$$a_1 x a_2 = a_1 b a_2.$$

Hence I is a *-biideal in R and G_t is of the form of Condition *(3)* in Lemma 2.6. By [7, Proposition 1.1.2],

$$G = H \oplus G_t,$$

where H is torsion free. It is clear that

$$H \cong G/G_t \cong \underset{\alpha}{\oplus}\mathbb{Q}^+.$$

(2)⟹(1). Let G be a group satisfying condition *(2).* Let F be a field such that

$$F^+ = \underset{\alpha}{\oplus}\mathbb{Q}^+$$

and let R be the ring constructed in proving the implication *(3)⟹(1)* for Lemma 2.6. The ring direct sum $S = F \oplus R$ with the identity involution possesses only finitely many *-biideals, and $S^+ \cong G$. ■

Corollary 2.16. *Let G be a non-nil group. The following are equivalent*

*(1) Every involution ring R with $R^+ = G$, and $R^2 \neq 0$ possesses only finitely many *-biideals.*

(2) $G \cong \underset{\alpha}{\oplus}\mathbb{Q}^+ \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j}$, p_i a prime, m, n_j non-negative integers. α, α_j finite cardinals, $i = 1, \ldots, m$, $j = 1, 2, \ldots, n_j$.

Proof. *(1)⟹(2).* Suppose that G satisfies Condition *(1).* It may be assumed that G is of the form *(2)* of Theorem 2.15. Suppose that α_j is an infinite cardinal for some index j. Then

$$G = \overset{\infty}{\underset{i=1}{\oplus}}(a_i) \underset{\alpha}{\oplus} \mathbb{Q}^+ \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{finite}{\oplus} \mathbb{Z}_{p_i^j},$$

$|a_i| = p$, p a prime. Let R_i be a ring isomorphic to the ring of integers modulo p, with

$$R_i^+ = (a_i),\ i = 1, 2, ...,$$

Let F be a field with

$$F^+ \cong \underset{\alpha}{\oplus}\mathbb{Q}^+$$

Let T_{ij} be the sum of β_j copies of the ring of integers modulo p_i^j. Put

$$T = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} T_{ij}.$$

$$R = \overset{\infty}{\underset{i=1}{\oplus}} R_i \oplus F \oplus T$$

with the identity involution is an involution ring satisfying $R^+ = G$ and $R^2 \neq 0$. Moreover,

$$R_i \overset{*}{\lhd} R$$

for $i = 1, 2, \ldots..$, a contradiction. Now, suppose that $\alpha$is an infinite cardinal. Let

$$\underset{\alpha}{\oplus}\mathbb{Q} = H$$

and let T_{ij} be sum the of α_j copies of the ring of integers modulo p_i^j. If S is the zeroring such that $S^+ = H$, then the ring direct sum

$$R = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} T_{ij} \oplus S$$

is an involution ring with the identity involution satisfying $R^+ = G$, and $R^2 \neq 0$. Choose $0 \neq a \in H, < n!a >_{bi}$ is a *-biideals in R for every positive integer n, a contradiction.

(2)⟹(1). Let

$$G \cong \underset{\alpha}{\oplus}\mathbb{Q}^+ \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j}$$

p_i a prime, m, n_j non-negative integers. α, α_j finite cardinals, $i = 1, \ldots., m, j = 1, 2, \ldots, n_j$. Let

$$H = \underset{\alpha}{\oplus}\mathbb{Q}^+,$$

H is the maximal divisible subgroup of G and let

$$K = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus}\mathbb{Z}_{p_i^j},$$

K is the maximal torsion subgroup of G, hence H and K are ideals in every ring R with $R^+ = G$. If R is an involution ring with $R^+ = G$, $R^2 \neq 0$,

$$R = R_o \oplus R_t,$$

such that R_o is a divisible *-ring and R_t is a torsion *-ring. Now we have three cases.

(i) $R_o^2 \neq 0$, $R_t^2 \neq 0$. By the remark after Corollary 2.4 and Corollary 2.8, R_o and R_t possess finitely many *-biideals. Hence R possesses finitely many *-biideals.

(ii) $R_o^2 \neq 0$, $R_t^2 = 0$. Then by the remark after Corollary 2.4, R_o possesses only finitely many *-biideals. By [7, Corollary 4.3.11], R possesses only finitely many ideals. But R_t is a zero ring, it possesses only finitely many *-biideals, hence R possesses only finitely many *-biideals.

(iii) $R_o^2 = 0$, $R_t^2 \neq 0$. For this case, we use Corollary 2.8 and [7, Corollary 4.3.11] with the same technique as in (ii).■

Corollary 2.17.*Let G be a non-nil group. The following are equivalent:*

*(1) Every ring R with $R^+ = G$, and $R^2 \neq 0$ satisfies dcc for *-biideals.*

(2) $G \cong \underset{\alpha}{\oplus} \mathbb{Q}^+ \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j} \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty}$, p_i a prime, m, n_j non-negative integers, α, α_j finite cardinals, $i = 1, \ldots .m, j = 1, \ldots .n_j$.

Proof. *(1)$\Longrightarrow$(2).* By Proposition 2.2,

$$G \cong \underset{\alpha}{\oplus} \mathbb{Q}^+ \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j} \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty}$$

p_i a prime, m, n_j non-negative integers, α, α_j are arbitrary cardinals, $i = 1, \ldots .m$, $j = 1, \ldots n_j$. Let

$$H = \overset{m}{\underset{i=1}{\oplus}} \overset{n_j}{\underset{j=1}{\oplus}} \underset{\alpha_j}{\oplus} \mathbb{Z}_{p_i^j} \underset{finite}{\oplus} \mathbb{Z}_{p_i^\infty}$$

and

$$K = \underset{\alpha}{\oplus} \mathbb{Q}^+.$$

Suppose S and T are the rings constructed in the proof of *(1)$\Longrightarrow$(2)* of Corollaries 2.12 and 2.4, respectively, such that $S^+ = H$ and $T^+ = K$. Hence $R = S \oplus T$, with $R^+ = G$, $S^2 \neq 0$ and $T^2 \neq 0$, does not satisfy dcc on *-biideals, a contradiction, so α_j, α_i are finite cardinals.

Conversely, let

$$G = \underset{\alpha}{\oplus} \mathbb{Q}^+ \oplus H$$

and

$$H = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n_j} \bigoplus_{\alpha_j} \mathbb{Z}_{p_i^j} \bigoplus_{finite} \mathbb{Z}_{p_i^\infty},$$

p_i a prime, m, n_j non-negative integers, α, α_j are finite cardinals, $i = 1, \ldots .m, j = 1, \ldots n_j$. Since H is a torsion subgroup of G, H is a *-ideal in R ,such that $R^+ = G$.

If $H^2 \neq 0$, by Corollary 2.12, H satisfies dcc for *-biideals. We have the following two cases.

(i) $(R/H)^2 \neq 0$, then by the remark after Corollary 2.4, R/H satisfies dcc for *-biideals, and by [1, Proposition 5], every involution ring R with $R^+ = G$, $R^2 \neq 0$, R satisfies dcc for *-biideals.

(ii) If $(R/H)^2 = 0$, then by [7, Corollary 4.3.14], R/H satisfies dcc for ideals, so R/H satisfies dcc for *-biideals,and hence by the above argument, R satisfies dcc for *-biideals.

If $H^2 = 0$, by [7, Corollary 4.3.14], H satisfies dcc for *-biideals. Again we have two cases: (i) $(R/H)^2 \neq 0$ and (ii) $(R/H)^2 = 0$. The proof follows by similar arguments as above. ■

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