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Some Generating Functions of Jacobi Polynomials with New Parameter Lie-Group

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Abstract

Making suitable interpretations to the index and the parameters of Jacobi polynomial $P_n^{(\alpha,\beta)}(\mathrm{x})$ in order to derive the elements of Lie-algebra, we have considered five parameters Lie-algebra for this polynomial which does not seen to appear before. By means of this group the theoretic method some new generating functions for Jacobi polynomials are obtained from which several special generating functions can be easily derived .

Keywords: Jacobi polynomials, generating functions

1. INTRODUCTION:

The Jacobi polynomials $P_n^{(\alpha,\beta)}(\mathrm{x})$, defined by

$$P_n^{(\alpha,\beta)}(\mathrm{x}) = \frac{(1+\alpha)_{\mathrm{n}}}{n!}\,{}_2F_1\left[\begin{matrix} -n, 1+\alpha+\beta+\mathrm{n}; \\ 1+\alpha; \end{matrix} \frac{1-x}{2}\right] \quad \text{..........................(1.1)}$$

is the solution of ordinary differential equation:

$$(1-x^2)\frac{d^2v}{dx^2}+[\beta-\alpha-(2+\alpha+\beta)x]\frac{dv}{dx}+n(1+\alpha+\beta+n)v=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

Various generating functions for Jacobi polynomials were derived by E. Feldheim (1943), W. A. Alsalam (1964), M. K. Das (1972) and others.

The object of the present paper is to derive some generating functions of Jacobi polynomials by interpreting n, α, β simultaneously with the help of Weisner's group theoretic method (Mcbride 1971).

Here the following generating functions are derived for $P_n^{(\alpha,\beta)}(x)$ by finding a set of infinitesimal operators A_i(i=1,2,3,4,5) constituting a Lie-algebra:

$$P_n^{(\alpha,\beta)}\left[\frac{yx-t_1}{y-t_1}\right]=\sum_{P=0}^{n}\frac{1}{P!}\frac{1}{2^P}(\alpha+\beta+n+1)_P \times P_{n-P}^{(\alpha+p,\beta+P)}(x)(t_1)^P \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.3)$$

$$[1-t_2y(1+x)]^{1+\alpha+n}[1+t_2y(1-x)]^{\beta}\ P_n^{(\alpha,\beta)}[x+t_2y(1-x^2)]=\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}(n+1)_k P_{n+k}^{(\alpha-k.\beta-k)}(x)(t_2)^k \quad \ldots\ldots\ldots\ldots(1.4)$$

$$\begin{aligned}&(1-t_3(1+x))^{1+\alpha+n}(1+t_3(1-x))^{\beta}\left(1+\frac{1}{yt_3w_1}(1+yt_3(1-x))\right)^{n}\\ &\times P_n^{(\alpha,\beta)}\left[\left(yt_3(1-yt_3(1+x))\right)\left(x+yt_3(1-x^2)\right)+\frac{1}{w_1}\left(1+yt_3(1-x)\right)\right]\\ &=\sum_{k=0}^{\infty}\sum_{p=0}^{n-k}\frac{(-1)^k}{k!}\frac{(-1/w_1)^p}{p!}2^{k-p}(\alpha+\beta+n+1)_p(n-p+1)_k\\ &\times P_n^{(\alpha+p-k,\beta+p-k)}(x)(t_3)^{k-p}\end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.5)$$

2. Group theoretic method

Replacing d/dx by ∂/∂x, α by $y\frac{\partial}{\partial y}$, β by $z\frac{\partial}{\partial z}$, n by $t\frac{\partial}{\partial t}$ and v by u(x, y, z, t) we get the following partial differential equation

$$(1-x^2)\frac{\partial^2u}{\partial x^2}+z(1-x)\frac{\partial^2u}{\partial z\partial x}-y(1+x)\frac{\partial^2u}{\partial y\partial x}+ty\frac{\partial^2u}{\partial t\partial y}+tz\frac{\partial^2u}{\partial t\partial z}+t^2\frac{\partial^2u}{\partial t^2}-2x\frac{\partial u}{\partial t}+2t\frac{\partial u}{\partial t}=0 \quad \ldots\ldots\ldots(2.1)$$

Thus $u_1(x,y,z,t)= P_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n$ is a solution of the differential equation (2.1). We now seek linearly independent differential operators A, B and C each of the form

$$A_i = A_i^{(1)}\partial/\partial x + A_i^{(2)}\partial/\partial y + A_i^{(3)}\partial/\partial z + A_i^{(4)}\partial/\partial t + A_i^0 \quad \text{..............................(2.2)}$$

Such that :

$$A\left[P_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n\right] = a_n P_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n \quad \text{...(2.3)}$$

$$B\left[P_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n\right] = b_n P_{n-1}^{(\alpha,\beta)}(x)y^\alpha z^\beta t^{n-1} \quad \text{..(2.4)}$$

$$C\left[P_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n\right] = c_n P_{n+1}^{(\alpha,\beta)}(x)y^\alpha z^\beta t^{n+1} \quad \text{..(2.5)}$$

Where a_n, b_n and c_n are coefficients involving α,β, and n[cf. SRivatstatava and Manocha 1984]. This necessitates the bringing into use of the recurrence relations [cf. Rainville (1960)]

$$\frac{d}{dx}\left[P_n^{(\alpha,\beta)}(x)\right] = \frac{1}{x-1}\left[nP_n^{(\alpha,\beta)} - (\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x)...\right] \quad \text{.....................(2.6)}$$

And

$$\frac{d}{dx}\left[P_n^{(\alpha,\beta)}(x)\right] = \frac{1}{\left(1-x^2\right)}[(1+\alpha+\beta+n)(x+1)-2\beta]P_n^{(\alpha,\beta)} - 2(n+1)P_{n+1}^{(\alpha,\beta-1)}(x) \quad \text{.....(2.7)}$$

With the help of (2.6) and (2.7), it follows form (2.3),(2.4) and (2.5) that

$$\left.\begin{aligned}
A_1 &= y\,\partial/\partial y \\
A_2 &= z\,\partial/\partial z \\
A_3 &= t\,\partial/\partial t \\
A_4 &= (x-1)z\,t^{-1}\,\partial/\partial x - z\,\partial/\partial t \\
A_5 &= (1-x^2)\,z^{-1}t\,\partial/\partial x - (x+1)z^{-1}\,t\,y\,\partial/\partial y - (x-1)t\,\partial/\partial z - (1+x)z^{-1}\,t^2\partial/\partial t - (1+x)z^{-1}t
\end{aligned}\right\} \quad (2.8)$$

which satisfy the following rules:

$$\left.\begin{aligned}
A_1\left[P_n^{(\alpha,\beta)}(x)\,y^\alpha z^\beta t^n\right] &= \alpha P_n^{(\alpha,\beta)}(x)\,y^\alpha z^\beta t^n \\
A_2\left[P_n^{(\alpha,\beta)}(x)\,y^\alpha z^\beta t^n\right] &= \beta P_n^{(\alpha,\beta)}(x)\,y^\alpha z^\beta t^n \\
A_3\left[P_n^{(\alpha,\beta)}(x)\,y^\alpha z^\beta t^n\right] &= nP_n^{(\alpha,\beta)}(x)\,y^\alpha z^\beta t^n \\
A_4\left[P_n^{(\alpha,\beta)}(x)\,y^\alpha z^\beta t^n\right] &= -(\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x)\,y^\alpha z^{\beta+1} t^{n-1}
\end{aligned}\right\} \quad (2.9)$$

$$A_5\left[P_n^{(\alpha,\beta)}(\mathrm{x})\, y^{\alpha} z^{\beta} t^{n}\right] = -(n+1)P_{n+1}^{(\alpha,\beta-1)}(\mathrm{x})\,\mathrm{y}^{\alpha}\,\mathrm{z}^{\beta-1}\,\mathrm{t}^{\mathrm{n+1}}$$

3. Lie Algebra

Now we shall find the commutator relations by using commutator notation with

[A,B]u=(AB-BA)u

$[A_1, A_2] = 0$; $[A_2, A_3] = 0$; $[A_3, A_4] = -A_4$

$[A_1, A_3] = 0$; $[A_2, A_4] = 0$; $[A_3, A_5] = A_5$

$[A_1, A_4] = 0$;

$[A_1, A_5] = 0$; $[A_2, A_5] = -A_5$; $[A_4, A_5] = 2(A_1+A_2)$

So we see from the above commutator relations that set of operators $\{1, A_i, i = 1, 2, 3, 4, 5\}$ generating a lie Algebra.

Now the partial differential operator L, given by:

$$L = (1-x^2)\frac{\partial^2 u}{\partial x^2} + z(1-x)\frac{\partial^2 u}{\partial z\partial x} - y(1+x)\frac{\partial^2 u}{\partial y\partial x} + ty\frac{\partial^2 u}{\partial y\partial t} + tz\frac{\partial^2 u}{\partial t\partial z} + t^2\frac{\partial^2 u}{\partial t^2} - 2x\frac{\partial u}{\partial x} + 2t\frac{\partial u}{\partial t}$$

Which can be express as:

$(x-1)L=A_5A_4-2A_3(A_1+A_3)$ ……………………………………………….(3.1)

It can be easy verified that the operator A_i (i= 1, 2, 3, 4, 5) Commute with (x-1)L,

i.e., $[(x-1)L,A_i]=0$ …………………………………………..………….(3.2)

The extended form of the group generated by A_i (i = 1, 2, 3, 4, 5) are given by

$$e^{a_1A_1}u(x,y,z,t) = u(\mathrm{x}, \mathrm{e}^{a_1}\,\mathrm{y}, \mathrm{z}, \mathrm{t})$$

$$e^{a_2A_2}u(x,y,z,t) = u(\mathrm{x}, \mathrm{y}, \mathrm{e}^{a_2}\mathrm{z}, \mathrm{t})$$

$$e^{a_3A_3}u(x,y,z,t) = u(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{e}^{a_3}\,\mathrm{t})$$

$$e^{a_4A_4}u(x,y,z,t) = u(\frac{\mathrm{tx - a_4 z}}{t - \mathrm{a_4 z}}, \mathrm{y}, \mathrm{z}, \mathrm{t - a_4 z})$$

$$e^{a_5A_5}u(x,y,z,t)=\left(\frac{z-t(1+x)a_5}{z}\right)\left[\frac{xz+t(1-x^2)a_5}{z},\frac{yz-yt(1+x)a_5}{z},z+t(1-x)a_5,\frac{tz-t^2(1+x)a_5}{z}\right]$$

where a_i(i=1,2,3,4,5) are constants.
Thus we have

$$e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}u(\text{x, y, z, t})=\left(\frac{z-t(1+x)a_5}{z}\right)\text{u}\,(\xi,\eta.\rho,\theta)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.3)$$

Where

$$\xi=\frac{(tz-t^2(1+x)a_5)(xz+t(1-x^2)a_5)-a_4z(z+t(1-x)a_5)}{tz^2-t^2(1+x)a_5-a_4z(z+t(1-x)a_5)}$$

$$\eta=e^{a_1}y\left(\frac{z-t(1+x)a_5}{z}\right)$$

$$\rho=e^{a_2}\left(z+t(1-x)a_5\right)$$

$$\theta=e^{a_5}\left(\frac{z-t(1+x)a_5}{z}\right)\left[t-\frac{a_4z^2(z+t(1-x)a_5}{z-t(1+x)a_5}\right]$$

4. Generating Functions

From the (2,1), u(x,y,z,t)= $P_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n$ is a solution of the systems.

Lu = 0	Lu = 0	Lu = 0	Lu = 0
$(A_1-\alpha)u = 0$;	$(A_2-\beta)u = 0$;	$(A_3-n)u = 0$;	$(A_1 + A_2 + A_3 - n - \beta - \alpha)u = 0$

From (3,2) we easily get

$$S((x-1)L)P_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n=((x-1)L)SP_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n=0$$

Where

$$S=e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}$$

There fore the transformations $S\left[P_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n\right]$ is annulled by L.

By putting $a_1 = a_2 = a_3 = 0$ in (3.3) we get

$$e^{a_5A_5}e^{a_4A_4}\left[P_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n\right]=y^\alpha\left(\frac{z-t(1+x)a_5}{z}\right)^{1+\alpha+n}(z+t(1-x)a_5)^\beta\left[t-\frac{a_4z\;(z+t(1-x)a_5}{z-t(1+x)a_5}\right]$$

$$\times P_n^{(\alpha,\beta)}\frac{(tz-t^2(1+x)a_5)(xz+t(1-x^2)a_5)-a_4z^2(z+t(1-x)a_5)}{tz^2-t^2z(1+x)a_5-a_4z^2(z+t(1-x)a_5)}\;\ldots\ldots\ldots\ldots.(4.1)$$

If we change the order of $e^{a_5A_5}e^{a_4A_4}$ we shall get the relation different from (4.1)

But

$$e^{a_5A_5}e^{a_4A_4}\left[P_n^{(\alpha,\beta)}(x)y^\alpha z^\beta t^n\right]=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(a_5)^k}{k!}\frac{(a_4)^p}{p!}(\alpha+\beta+n+1)_p(-1)^k(2)^{(k-p)}(n-p+1)_k$$

$$\times P_{n-p+k}^{(\alpha+p-k,\beta+p-k)}(x) y^{\alpha+p-k} z^{\beta+p-k} t^{n-p+k} \quad \ldots\ldots\ldots\ldots..(4.2)$$

Equating the result (4.1) and (4.2) we get

$$y^{\alpha}\left(\frac{z-t(1+x)a_5}{z}\right)^{1+\alpha+n}(z+t(1-x)a_5)^{\beta}\left[t-\frac{a_4 z\ (z+t(1-x)a_5}{z-t(1+x)a_5}\right]$$
$$\times P_n^{(\alpha,\beta)}\frac{(tz-t^2(1+x)a_5)(xz+t(1-x^2)a_5)-a_4 z^2(z+t(1-x)a_5)}{tz^2-t^2 z(1+x)a_5-a_4 z^2(z+t(1-x)a_5)}$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(a_5)^k}{k!}\frac{(a_4)^p}{p!}(\alpha+\beta+n+1)_p(-1)^k(2)^{(k-p)}(n-p+1)_k$$
$$\times P_{n-p+k}^{(\alpha+p-k,\beta+p-k)}(x) y^{\alpha+p-k} z^{\beta+p-k} t^{n-p+k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.3)$$

Now we shall consider the following cases:-

Case1:
Letting a_4=1 , a_5=0 and writing t_1=z/t in (4.3) we get

$$P_n^{(\alpha,\beta)}\left[\frac{yx-t_1}{y-t_1}\right]=\sum_{P=0}^{n}\frac{1}{P!}\frac{1}{2^P}(\alpha+\beta+n+1)_P \times P_{n-P}^{(\alpha+p,\beta+P)}(x)(t_1)^P \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4.4)$$

Which is (1.3)

Case2:
Let a_5=1 , a_4=0 and writing t_2=t/z in (4,3) we get

$$[1-t_2 y(1+x)]^{1+\alpha+n}[1+t_2 y(1-x)]^{\beta}\ P_n^{(\alpha,\beta)}[x+t_2 y(1-x^2)]=\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}(n+1)_k P_{n+k}^{(\alpha-k.\beta-k)}(x)(t_2)^k$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.5)$$

Which is (1.4)

Case 3:
Finally substituting a_5 = 1 , a_4 = -1/w_1 and writing t/yz = t_3 in (4.3)

We get

$$(1-t_3(1+x))^{1+\alpha+n}\ (1+t_3(1-x))^{\beta}\ (1+\frac{1}{yt_3 w_1}(1+yt_3(1-x)))^{n}$$
$$\times P_n^{(\alpha,\beta)}\left[\left(yt_3\left(1-yt_3(1+x)\right)\right)\left(x+yt_3(1-x^2\right)+\frac{1}{w_1}\left(1+yt_3(1-x)\right)\right]$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n-k}\frac{(-1)^k}{k!}\frac{(-1/w_1)^p}{p!}2^{k-p}(\alpha+\beta+n+1)_p(n-p+1)_k \quad \ldots\ldots\ldots\ldots\ldots\ldots..(4.6)$$
$$\times P_n^{(\alpha+p-k,\beta+p-k)}(x)(t_3)^{k-p}$$

Which is (1.5)

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References

Al-Salam, W.A.(1964) : operational Representations for the Laguerre and other polynomials. Duke Math, Jour. 31,127.

Das, M.K. (1972) : some properties of special functions derived from the theory of continuous transformation group. Proceedings of the American Mathematical society. 35,2 (565).

Feldheim,E. (1943) : Relations entre les polynomial de Jacobi, laguerre of Hermite. Acta Math , 75 , 117 – 198, 120.

Mcbride, E. B. (1971) : obtaining generating functions, springer-verlage. Berlin.

Rainville, E. D.(1960) : Special functions, MacMillan, New York. Reprinted by Chelsa Publ. Co., Bronx, New York (1971)

Srivatstava H.M. and Manocha H.L. (1984): A Treatise on Generating Functions, Ellis Horwood ltd., Italsted press, John Willy, Chichester.

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