# Identification of the Source Term and the Diffusion Coefficient of a Degenerated/Singular Parabolic Equations

**Hamed Ould Sidi**

URMCD Research Unit, Faculty of Legal, Economic and Social Sciences

University of Nouakchott al Aasriya, Mauritania

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# Contents

# Preface

In this book we will study an inverse diffusion problem for degenerate and singular parabolic equation. In modeling a phenomenon using partial differential equations (PDEs), the physical / mechanical / biological parameters involved are not necessarily well known. However, the resolution of these equations, which is the object of the direct problem, can only be done if all the data of the system are identified (initial and boundary conditions, coefficients involved in the equations, spatial domain ...). If this is not the case, additional information, via experimental measures for example, is then necessary to their determination. The mathematical notion of the inverse problem consists in the possibility of finding the value of a parameter from partial measurements (localized, during a given time, possibly repeated) on the solution of the system considered.

It should first be emphasized that the inverse problems are generally not well posed in the sense of Hadamard [51]. A mathematical model of physical phenomena is considered a well-posed problem if it has the following properties:

1. Existence of a solution;

2. Uniqueness of the solution;

3. Stability (the solution is depends continuously on the data).

The demonstration of these properties is the objective of the study of inverse problems in the field of PDE analysis. Of course, the issues linked to these properties go beyond purely theoretical interest and they influence the possible results of an approximate or numerical resolution of an inverse problem. Indeed, the next logical step, which more specifically concerns studies for application purposes, consists in studying the actual identification (or reconstruction) of the parameters. For example, regardless of the numerical method used for a reconstruction problem, since the data actually available will be noisy, the lack of stability would prevent a satisfactory approach to the solution of the reverse problem. Just as the lack of uniqueness would require a priori information in order to be able to make a choice. We can also point out that from the point of view of the techniques used and the problems encountered around In PDEs, inverse problems have many points in common with questions of controllability. Data assimilation, or identification of the diffusion coefficient or source term is an essential procedure, is used for numerical weather prediction [52], atmospheric modeling, ocean circulation [54], and simulation digital in river hydraulics [55]. The uniqueness of the solution to an inverse source or parameter determination problem in a PDE is often demonstrated first, as will appear in the partial bibliography that we will give here. The stability of the opposite problem is, however, the stated objective, which is more difficult to achieve. In both cases, the most widely used techniques are based on energy inequalities in weighted Sobolev spaces, called Carleman inequalities and introduced into [56] (to demonstrate uniqueness properties for a Cauchy problem of an elliptic operator). These inequalities, which have been generalized among others by Hrmander [57] for large classes of differential operators, have many applications in the study of PDEs, and not only in inverse problems. For example, for the demonstration of properties of unique continuation (see [58], [59], [60], [61], [62].)

In the first chapter we study an inverse source problem for degenerate/singular parabolic equations with degeneracy and singularity occurring in the interior of the spatial domain. Based on Carleman estimates, we establish Lipschitz stability for the source term provided that additional measurement data are given on a suitable interior sub-domain.

The second chapter is devoted to the numerical solution for the same equation treated in the first chapter, the problem is reformulated in a least-squares framework, which leads to a non-convex minimization problem that is solved using a Tikhonov regularization. With the aim of showing that the minimization problem and the direct problem are well-posed, we prove that the solutions behavior changes continuously with the source term. And we prove the differentiability of the functional J, which gives the existence of the gradient of J, that is computed using the adjoint state method. Finally, to show the convergence of the descent method, we prove that the gradient of J is Lipschitz continuous. Also we present some numerical experiments to show the performance and stability of the proposed approach

The third chapter deals with the identification of the diffusion coefficient of a degenerate / singular parabolic problem. The mathematical formulation of the model leads to a non-convex minimization problem. First, we prove that the problem admits at least one solution. To solve this inverse problem, one proposes two approaches: if one has a preliminary knowledge of the diffusion coefficient (the draft state) on all the grid of analysis, one proposes an approach based on the regularization type Tikhonov generalized and in the numerical part one gives results which show the impact of the draft error on the reconstruction of the solution. However, in reality we only have partial knowledge of the draft state, which handicaps the regularization method, from which we propose a new approach based on hybrid algorithms (marry the genetic algorithm with a gradient-type descent method) and we minimize the initial problem without any regularization. Finally, we give a comparison between the two approaches.

# Chapter 1

# Lipschitzian stability fo a degenerate/singular parabolic equations

This chapter is concerned with the question of Lipschitz stability results in inverse source problems for linear degenerate heat equations with singular potentials. More precisely, we consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - (au_x)_x - \dfrac{\lambda}{b(x)} u = h(t,x), & (t,x) \in Q_T, \\ Bu(0) = Bu(1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where $Bu(x) = u(t,x)$ or $Bu(x) = u_x(t,x)$ for all $t \in [0,T]$, $u_0 \in L^2(0,1)$, $T > 0$ fixed and $Q_T := (0,T) \times (0,1)$. Moreover, we assume that the constant $\lambda$ satisfy suitable assumptions described below and the functions $a$ and $b$ degenerate at the same interior point $x_0$ of the spatial domain $(0,1)$ (for the precise assumptions we refer to section 1.1). For $t_0 \in (0,T)$ given, let $Q_{t_0}^T = (t_0,T) \times (0,1)$ and $T' := \frac{T+t_0}{2}$. For a given $C_0 > 0$, we denote by $S(C_0)$ the space

$$S(C_0) := \{f \in H^1(t_0,T;L^2(0,1)) : |f_t(t,x)| \le C_0|f(T',x)|, \text{ a.e. } x \in (0,1)\}.$$

In the mathematical literature, null controllability and inverse problems of uniformly parabolic equations is very popular and widely studied, (see [36, 37, 38, 42, 45, 46, 51, 52, 53, 61]) and the references therein. The main result in these papers is the development of suitable Carleman estimates, which are crucial tools to obtain observability inequalities and Lipschitz stability for term sources, initial data, potentials and diffusion coefficients. More recently, there have been many papers dealing with the Lipschitz stability for purely degenerate or purely singular parabolic operators using global Carleman estimates, (see for instance [40, 43, 47, 57, 58, 59, 60] ). In contrast, in both fields of controllability and inverse problems very few results are known for degenerate/singular parabolic equations, even though this class of operators occurs in interesting theoretical and applied problems.

For this reason, the present paper is devoted to the study of an inverse source problem for the operator defined in (1.1), that couples a general degenerate diffusion coefficient with a general singular potential with degeneracy and singularity at the interior of the space domain. In particular, this unifies the results of [39] and [59] in the purely degenerate operator and in the purely singular one, respectively. More precisely, we will follow the approach introduced in [52] for the treatment of uniformly parabolic problems which is based on the use of global Carleman estimates. For this purpose, we use and extend some recent Carleman estimates for degenerate/singular equations obtained by Fragnelli and Mugnai [48]. As a consequence, we prove a stability estimate of Lipschitz type in determining the term source by the observation data at a given time $T'$.

We point out the fact that, in [48] the authors consider the well-posedness of problem (1.1) but only with Dirichlet boundary conditions. Due to Neumann boundary conditions, a key point in this work is the study of well-posedness. The result is obtained applying a new form of Hardy-Poincaré type inequalities.

The aim of this paper is to identify the source term $h$ from measurements of the solution $u$ at time $T'$. It is worthwhile to point out that we do not need the supplement distributed measurements. To our knowledge, this paper is the first one concerning Lipschitz stability results in inverse problems for degenerate/singular parabolic equations such as (1.1).

For fixed $T > T' > 0$, the main result is as follows.

**Theorem 1** *Let $C_0 > 0$. Then, there exists $C = C(T, t_0, x_0, C_0) > 0$ such that, for all $h \in S(C_0)$ and $u_0 \in L^2(0,1)$,*

$$\|h\|^2_{L^2(Q^T_{t_0})} \leq C\|(au_x)_x(T',\cdot) + \frac{\lambda}{b(\cdot)}u(T',\cdot)\|^2_{L^2(0,1)}. \tag{1.2}$$

The main novelty in the present problem is the inverse problem associated to a parabolic equation presenting both a degenerate diffusion coefficient and a singular potential with degeneracy and singularity inside the spatial domain. (For controllability result related to this problem with Dirichlet boundary conditions, we refer to [48], where the null controllability property is proved via an observability inequality for the adjoint equation).

The paper is organized in the following way: in Section 2, we give some preliminary results, such as Hardy-Poincaré inequalities, that will be useful for the rest of the paper. In Section 3, we study well-posedness of the problem and we give a regularization of the time derivative of the solutions applying the previous inequalities. In Section 4, we prove our main result.

A final comment on the notation: in the rest of the paper we will write, for shortness, (Dbc) or (Nbc) in place of Dirichlet boundary conditions or Neumann ones, respectively.

## 1.1 Preliminary results

In this part of the paper we give different weighted Hardy-Poincaré inequalities that will be very important for the rest of the paper. At first we recall some Hardy-Poincaré inequalities in the case of (Dbc) and we prove them in the case of Neumann ones.

The ways in which $a$ and $b$ degenerate at $x_0$ can be quite different, and for this reason, following [48], to establish our results, we give the following definitions and assumptions:

**Assumptions 2** *Double weakly degenerate case (WWD) There exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ in $[0,1] \setminus \{x_0\}$, $a, b \in C^1([0,1] \setminus \{x_0\})$ and there exists $K, L \in (0,1)$ such that $(x-x_0)a' \le Ka$ and $(x-x_0)b' \le Lb$ a.e. in $[0,1]$.*

**Assumptions 3** *Weakly strongly degenerate case (WSD) There exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a > 0$ in $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\})$, $b \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$ , $\exists\ K \in (0,1), L \in [1,2)$ such that $(x-x_0)a' \le Ka$ and $(x-x_0)b' \le Lb$ a.e. in $[0,1]$.*

**Assumptions 4** *Strongly weakly degenerate case (SWD) There exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a > 0$ in $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, $b \in C^1([0,1] \setminus \{x_0\})$, $\exists K \in [1,2)$, $L \in (0,1)$ such that $(x-x_0)a' \le Ka$ and $(x-x_0)b' \le Lb$ a.e. in $[0,1]$ and, if $K > \frac{4}{3}$, then $\exists\ \gamma \in (0,K]$ such that $\dfrac{a}{|x-x_0|^\gamma}$ is nonincreasing on the left of $x = x_0$ and nondecreasing on the right of $x = x_0$.*

For the wellposedness of the problem (1.1), as in [48], a key ingredient in the proof takes the form of special Hardy-Poincaré inequalities. In order to deal with these inequalities we consider different classes of weighted Hilbert spaces, which are suitable to study the three different situations given above, namely the (WWD), (WSD), and (SWD) cases. Thus, we introduce

$$\mathcal{K}_a := \begin{cases} H^1_a(0,1) := \left\{ u \in W^{1,1}_0(0,1) \ :\ \sqrt{a}u_x \in L^2(0,1) \right\} \text{ if (Dbc) hold,} \\ \mathcal{H}^1_a(0,1) := \left\{ u \in W^{1,1}(0,1) \ :\ \sqrt{a}u_x \in L^2(0,1) \right\} \text{ if (Nbc) are in force,} \end{cases}$$

and

$$\mathcal{K}_{a,b}(0,1) := \left\{ u \in \mathcal{K}_a(0,1) \ :\ \frac{u}{\sqrt{b}} \in L^2(0,1) \right\}$$

endowed with the inner products

$$\langle u, v \rangle_{\mathcal{K}_a} := \int_0^1 au'v'\,dx + \int_0^1 uv\,dx,$$

and

$$\langle u, v \rangle_{\mathcal{K}_{a,b}} := \int_0^1 au'v'\,dx + \int_0^1 uv\,dx + \int_0^1 \frac{uv}{b}\,dx,$$

respectively.

For our further results, it is important to remind the following fundamental Hardy-Poincaré inequality in the case of (Dbc):

**Lemma 5** *[48, Proposition 2.14] If one among Hypotheses 2-4 holds with $K + L \leq 2$, then there exists a constant $C > 0$ such that for all $u \in \mathcal{K}_{a,b}(0,1)$ we have*

$$\int_0^1 \frac{u^2}{b(x)}\, dx \leq C \int_0^1 a(x)|u'|^2\, dx. \tag{1.3}$$

In order to analyze the well-posedness of the problem (1.1) with Neumann boundary conditions, we will also need the analogous Hardy-Poincaré inequalities stated before for the case of (Dbc). To this aim we have a new approach: first, we use a reflection procedure and then we employ the Hardy-Poincaré inequalities with (Dbc).

In particular, we have the following result in the Neumann case, paying the price of introducing a zero order term in the inequality.

**Lemma 6** *If one among Hypotheses 2-4 holds with $K + L \leq 2$, then there exist two positive constants $\tilde{C}, C > 0$ such that for all $u \in \mathcal{K}_{a,b}(0,1)$ we have*

$$\int_0^1 \frac{u^2}{b(x)}\, dx \leq \tilde{C} \int_0^1 u^2\, dx + C \int_0^1 a(x)|u'|^2\, dx. \tag{1.4}$$

**Proof.** The proof of this last inequality is obtained using a cut-off argument and reflection procedure. To this aim, since $x_0 \in (0,1)$, we choose $\alpha, \beta > 0$ such that $\alpha + \beta < x_0$, $1 + \alpha + \beta < 2 - x_0$, and consider a smooth function $\xi : [-1,2] \to$ such that

$$\begin{cases} \xi \equiv 1, & x \in [0,1], \\ \xi \equiv 0, & x \in [-1,-\alpha] \cup [1+\alpha, 2]. \end{cases} \tag{1.5}$$

Now, consider

$$U(t,x) := \begin{cases} u(t, 2-x), & x \in [1,2], \\ u(t,x), & x \in [0,1], \\ u(t,-x), & x \in [-1,0]. \end{cases} \tag{1.6}$$

Let us also define the following functions

$$\tilde{a}(x) := \begin{cases} a(2-x), & x \in [1,2], \\ a(x), & x \in [0,1], \\ a(-x), & x \in [-1,0] \end{cases} \quad \text{and} \quad \tilde{b}(x) := \begin{cases} b(2-x), & x \in [1,2], \\ b(x), & x \in [0,1], \\ b(-x), & x \in [-1,0]. \end{cases} \tag{1.7}$$

Next, set $V := \xi U$. Using the the fact that $\xi$ is supported in $[-\alpha, 1+\alpha]$, observe that $V(-\alpha-\beta) = V(1+\alpha+\beta) = 0$. Thus, we can apply the analogue of Lemma 5 on

$(-\alpha-\beta, 1+\alpha+\beta)$ in place of $(0,1)$, obtaining that there exists a positive constant $C$, such that $V$ satisfies,

$$\int_{-\alpha-\beta}^{1+\alpha+\beta} \frac{V^2}{\tilde{b}(x)}\,dx \leq C\int_{-\alpha-\beta}^{1+\alpha+\beta} \tilde{a}(x)|V'|^2\,dx.$$

By definition of $\xi$, $U$ and $V$, we have

$$\begin{aligned}\int_0^1 \frac{u^2}{b(x)}\,dx = \int_0^1 \frac{V^2}{b(x)}\,dx &\leq \int_{-\alpha-\beta}^{1+\alpha+\beta} \frac{V^2}{\tilde{b}(x)}\,dx \\ &\leq C\int_{-\alpha-\beta}^{1+\alpha+\beta} \tilde{a}(x)|V'|^2\,dx.\end{aligned}$$

Using the fact that $\xi_x$ is supported in $[-\alpha, 0]\cup[1, 1+\alpha]$, it follows

$$\begin{aligned}&\int_{-\alpha-\beta}^{1+\alpha+\beta} \tilde{a}(x)|V'|^2\,dx \leq 2\int_{-\alpha-\beta}^{1+\alpha+\beta} \tilde{a}(x)\left[\xi_x^2U^2+\xi^2U_x^2\right]\,dx\\ &\leq \tilde{c}\left(\int_{-\alpha}^{0}+\int_{1}^{1+\alpha}\right)U^2\,dx + c\int_{-\alpha}^{1+\alpha}\tilde{a}U_x^2\,dx\\ &\leq \tilde{c}\int_{-1}^{2}U^2\,dx + c\int_{-1}^{2}\tilde{a}U_x^2\,dx,\end{aligned}$$

where $\tilde{c}$ and $c$ are some universal positive constants.

Hence, there exist two positive constants $\tilde{C}$ and $C$ such that

$$\begin{aligned}\int_0^1 \frac{u^2}{b(x)}\,dx &\leq \tilde{C}\int_{-1}^{2}U^2\,dx + C\int_{-1}^{2}\tilde{a}U_x^2\,dx\\ &\leq \tilde{C}\int_0^1 u^2\,dx + C\int_0^1 au_x^2\,dx.\end{aligned}$$

■

In the rest of the paper we will denote by $C^*$ and $\tilde{C}^*$ the best constants that appear in (1.3) or (1.4).

## 1.2 Well-posedness and regularity of the internal degenerate/singular problem

In order to study well-posedness of problem (1.1) and in view of Lemma 5 and 6, we introduce the space

$$\mathcal{K} := \mathcal{K}_{a,b}(0,1),$$

where the Hardy-Poincaré inequality (1.3) and (1.4) holds.

We underline that, from Lemma 5 and 6, the standard norm $\|.\|^2_{\mathcal{K}}$ is equivalent to

$$\|.\|^2_{\sim} := \begin{cases} \int_0^1 a(u')^2dx \text{ if (Dbc) hold,} \\ \int_0^1 u^2dx + \int_0^1 a(u')^2dx \text{ if (Nbc) are in force,} \end{cases}$$

From now on, we make the following assumptions on $a$, $b$ and $\lambda$:

**Assumptions 7** *1. One among the Hypotheses 2, 4 or 4 holds true with $K + L \leq 2$;*
*2. We also assume that*

$$\lambda \in \big(0, \frac{1}{C^*}\big). \tag{1.8}$$

Using the lemmas given in the previous section one can prove the next inequalities, which are crucial to prove well-posedness.

**Proposition 8** *Assume that (7) holds. Then there exist two positive consatnts $k$ and $\Lambda$ such that for all $u \in \mathcal{K}$*

$$\int_0^1 a(x)|u'|^2 - \lambda\frac{u^2}{b(x)} + ku^2\,dx \geq \Lambda\|u\|^2_{\sim}. \tag{1.9}$$

(Let us mention that, when (Dbc) holds, the result is proved in [48, Proposition 2.18] with $k$ is simply equal to 0). **Proof.** Assume that (Nbc) are satisfied. By Lemma 6, we can write

$$\int_0^1 a(x)|u'|^2 - \lambda\frac{u^2}{b(x)}\,dx \geq \int_0^1 a(x)|u'|^2\,dx - \lambda\tilde{C}^*\int_0^1 u^2\,dx - \lambda C^*\int_0^1 a(x)|u'|^2\,dx.$$

Thus

$$\int_0^1 a(x)|u'|^2 - \lambda\frac{u^2}{b(x)} + 2\lambda\tilde{C}^*u^2\,dx \geq \Lambda\|u\|^2_{\sim},$$

where $\Lambda := \min\{\lambda\tilde{C}^*, (1 - \lambda C^*)\} > 0$. Hence the result holds with $k := 2\lambda\tilde{C}^*$. ■

Now, let us go back to problem (1.1), recalling the following Let $u_0 \in L^2(0,1)$ and $h \in L^2(Q_T)$. A function $u$ is said to be a (weak) solution of (1.1) if

$$u \in L^2(0,T;\mathcal{K}) \cap H^1([0,T];\mathcal{K}')$$

and it satisfies (1.1) in the sense of $\mathcal{K}'$- valued distributions. Note that, by [54, Lemma 11.4], any solution belongs to $C([0,T];L^2(0,1))$. Finally, we introduce the Hilbert space

$$\mathcal{W} := \begin{cases} H^2_{a,b}(0,1) := \{u \in H^1_a(0,1) : au' \in H^1(0,1) \text{ and } A_1u \in L^2(0,1)\}, \text{ if (Dbc) hold,} \\ \mathcal{H}^2_{a,b}(0,1) := \{u \in \mathcal{H}^1_a(0,1) : au' \in H^1(0,1) \text{ and } A_2u \in L^2(0,1)\}, \text{ if (Nbc) are in force,} \end{cases}$$

where

$$A_iu := (au')' + \frac{\lambda}{b}u,\ i = 1,2 \text{ with}$$

$$\begin{cases} D(A_1) := H^2_{a,b}(0,1) \text{ if (Dbc) hold and} \\ D(A_2) := \{u \in \mathcal{H}^2_{a,b}(0,1) : u'(0) = u'(1) = 0\} \text{ if (Nbc) are in force.} \end{cases}$$

As in [49, Lemma 2.2], using the fact that $u'(0) = u'(1) = 0$ for all $u \in D(A_2)$, one can prove the following formula of integration by parts which is a crucial tool for the rest of the paper:

**Lemma 9** *For all $(u, v) \in D(A_i) \times \mathcal{K}$ one has*

$$\int_0^1 (au')'v dx = -\int_0^1 au'v'dx. \tag{1.10}$$

Hence, the next result holds thanks to the theory of semigroups.

**Proposition 10** *The following assertions hold.*

*(i) The operator $(A_i, D(A_i))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(0, 1)$. Moreover, the semigroup is analytic.*

*(ii) For all $u_0 \in D(A_i)$ and $h \in H^1(0, T; L^2(0, 1))$, the problem (1.1) has a unique solution*

$$u \in C\big([0, T]; D(A_i)\big) \cap C^1([0, T]; L^2(0, 1)).$$

*Moreover, if $u_0 \in L^2(0, 1)$, then for all $\varepsilon \in (0, T)$ there holds*

$$u \in C\big([\varepsilon, T]; D(A_i)\big) \cap C^1([\varepsilon, T]; L^2(0, 1)).$$

*(iii) For all $u_0 \in L^2(0, 1)$ and for all $h \in L^2(0, T; L^2(0, 1))$, problem (1.1) has a unique solution $u \in C\big([0, T]; L^2(0, 1)\big) \cap L^2(0, T; \mathcal{K})$ such that for all $\varepsilon \in (0, T)$ there holds*

$$u \in L^2\big(\varepsilon, T; D(A_i)\big) \cap H^1\big(\varepsilon, T; L^2(0, 1)\big).$$

*Moreover, if $h \in H^1(0, T; L^2(0, 1))$ and $\varepsilon \in (0, T)$, we have*

$$u \in H^1\big([\varepsilon, T]; D(A_i)\big) \cap H^2([\varepsilon, T]; L^2(0, 1)).$$

**Proof.**
**If Dirichlet boundary conditions hold:** The proof of statement $(i)$ can be found in [48], whereas statements $(ii)$ and $(iii)$ are a consequence of $(i)$ and [41, Proposition 3.3 and Proposition 3.8].
**If Neumann boundary conditions hold:** For the well-posedness of (1.1) with (Nbc), it suffices to show that for suitable $k \geq 0$ the operator $-(A_2 - kI)$ is self-adjoint and positive. Assume that Hypothesis 7 holds and consider the constant $k \geq 0$ that is given by Proposition 8. We will proceed as in [48] proving that $(A_2 - kI, D(A_2))$ is nonpositive, self-adjoint and hence m-dissipative by [44, Corollary 2.4.8].

**$A_2 - kI$ is nonpositive.** By Proposition 8 and Lemma 9, it follows that, for any $u \in D(A_2)$ we have

$$\begin{aligned}
-\langle (A_2 - kI)u, u\rangle_{\mathbb{H}} &= -\int_0^1 \big((au_x)_x + \frac{\lambda}{b}u\big)u dx + \int_0^1 ku^2 dx, \\
&= \int_0^1 \big(au_x^2 - \frac{\lambda}{b}u^2 + ku^2\big)dx, \\
&\geq \Lambda \|u\|_\sim^2 \geq 0,
\end{aligned}$$

which proves the result.
**$A_2 - kI$ is self-adjoint.** Clearly, it is sufficient to show that $A_2$ is self-adjoint. Let $T : L^2(0,1) \to L^2(0,1)$ be the mapping defined in the following usual way: to each $f \in L^2(0,1)$ associate the weak solution $u = T(f) \in \mathcal{K}$ of

$$\int_0^1 \big(au'v' - \frac{\lambda}{b}uv + kuv\big)\, dx = \int_0^1 fv\, dx,$$

for every $v \in \mathcal{K}$. Note that $T$ is well defined by Lax-Milgram Lemma via Proposition 8, which also implies that $T$ is continuous. Now, it is easy to see that $T$ is injective and symmetric. Thus it is self adjoint. As a consequence, $A_2 = T^{-1} : D(A_2) \to L^2(0,1)$ is self-adjoint (for example, see [56, Proposition A.8.2]).

■

However, to study the inverse problem, we need to state some further regularity properties of the time derivative of the solution.

In fact, a straightforward consequence of (1.3) and of (1.4) is the next result.

**Lemma 11** *Let $u_0 \in D(A_i)$ and $h \in H^1(0,T;L^2(0,1))$ be given. Consider $u$ the corresponding solution of* (1.1). *Then the function $z := u_t$ belongs to $L^2(0,T;\mathcal{K})$ and is the solution of the variational problem*

$$\begin{cases}
z \in \{v \in L^2(0,T;\mathcal{K})\colon\ v_t \in L^2(0,T;\mathcal{K}')\}, \\
\langle z_t(t), w\rangle_{L^2(0,1)} + \langle \sqrt{a}z_x(t), \sqrt{a}w_x\rangle_{L^2(0,1)} - \lambda\langle \frac{z(t)}{\sqrt{b}}, \frac{w}{\sqrt{b}}\rangle_{L^2(0,1)} \\
=< h_t(t), w >_{L^2(0,1)},\ \forall\ \ w \in \mathcal{K}, \\
z(0) = A_i u_0 + h(0).
\end{cases} \tag{1.11}$$

**Proof.** The proof follows the one of [40, Lemma 2.2], but it is different for the presence of the singular potential.

Let $u_0 \in D(A_i)$, we observe that the function $z := u_t$ satisfies in the sense of distribution, the following equation

$$z_t - A_i z = h_t,\ i = 1,2.$$

Thus, in order to verify that $z$ satisfies (1.11), it remains to prove that $z$ belongs to $L^2(0,T;\mathcal{K})$. To this aim, we use the method of differential quotients (see for instance

[55]).

Let us consider $0 < \delta < \dfrac{T}{2}$, $t \in (\delta, T-\delta)$ and $-\delta < s < \delta$. Then

$$\begin{cases} u_t(t+s) - A_i u(t+s) = h(t+s), \\ u_t(t) - A_i u(t) = h(t). \end{cases} \tag{1.12}$$

Moreover, setting $u^s(t) = \dfrac{u(t+s)-u(t)}{s} \in D(A_i)$, for all $t \in (\delta, T-\delta)$, and $h^s(t) = \dfrac{h(t+s)-h(t)}{s}$, we deduce from (1.12) that

$$u_t^s(t) - A_i u^s(t) = h^s(t). \tag{1.13}$$

Multiplying (1.13) by $u^s(t)$ and integrating by parts with respect to $x$, one gets, for all $t \in (\delta, T-\delta)$,

$$\langle u_t^s(t), u^s(t)\rangle_{L^2(0,1)} + \|\sqrt{a} u_x^s(t)\|_{L^2(0,1)}^2 = \langle h^s(t), u^s(t)\rangle_{L^2(0,1)} + \lambda \int_0^1 \frac{\big(u^s(t)\big)^2}{b} dx.$$

Integrating over $(\delta, T-\delta)$, this leads to

$$\begin{aligned} \frac{1}{2}\|u^s(T-\delta)\|_{L^2(0,1)}^2 + \int_\delta^{T-\delta}\int_0^1 a(u_x^s(t))^2 dxdt = \frac{1}{2}\|u^s(\delta)\|_{L^2(0,1)}^2 + \int_\delta^{T-\delta}\int_0^1 h^s(t)u^s(t)dxdt \\ + \lambda \int_\delta^{T-\delta}\int_0^1 \frac{\big(u^s(t)\big)^2}{b} dxdt. \end{aligned} \tag{1.14}$$

Yet,

$$\int_\delta^{T-\delta}\int_0^1 h^s(t)u^s(t)dxdt \le \frac{1}{2}\int_\delta^{T-\delta}\int_0^1 (h^s(t))^2 dxdt + \frac{1}{2}\int_\delta^{T-\delta}\int_0^1 (u^s(t))^2 dxdt.$$

Since $h \in H^1(0,T;L^2(0,1))$, applying Jensen inequality, we get

$$\|h(t+s)-h(t)\|_{L^2(0,1)}^2 \le s\int_t^{t+s} \|h_t(\tau)\|_{L^2(0,1)}^2 d\tau = s^2 \int_0^1 \|h_t(t+sy)\|_{L^2(0,1)}^2 dy.$$

It follows that

$$\begin{aligned} \int_\delta^{T-\delta}\int_0^1 (h^s(t))^2 dxdt &\le \int_\delta^{T-\delta}\int_0^1 \|h_t(t+sy)\|_{L^2(0,1)}^2 dydt \\ &= \int_0^1\int_\delta^{T-\delta} \|h_t(t+sy)\|_{L^2(0,1)}^2 dtdy \\ &= \int_0^1\int_{\delta+ys}^{T-\delta+ys} \|h_t(\eta)\|_{L^2(0,1)}^2 d\eta dy, \quad \text{ with } \eta = t+sy \\ &\le \int_0^1\int_0^T \|h_t(\eta)\|_{L^2(0,1)}^2 d\eta dy = \int_0^T \|h_t(\eta)\|_{L^2(0,1)}^2 d\eta. \end{aligned} \tag{1.15}$$

Analogously, one has

$$\int_{\delta}^{T-\delta}\int_0^1 (u^s(t))^2 dxdt \leq \int_0^T \|u_t(\eta)\|^2_{L^2(0,1)}d\eta. \tag{1.16}$$

Moreover, using the definition of $u^s$ and the fact that $u \in C^1([0,T];L^2(0,1))$, one has

$$\|u^s(\delta)\|^2_{L^2(0,1)} \leq \sup_{t\in[0,T]} \|u_t(t)\|^2_{L^2(0,1)}. \tag{1.17}$$

At this stage, we distinguish the two cases (Dbc) and (Nbc) in order to take into account the different Hardy-Poincaré inequalities satisfied by $u^s$.

**If Dirichlet boundary conditions hold:** Thanks to the Hardy-Poincaré inequality (5) we have

$$\int_{\delta}^{T-\delta}\int_0^1 \frac{\big(u^s(t)\big)^2}{b}dxdt \leq C^* \int_{\delta}^{T-\delta}\int_0^1 a(u^s_x(t))^2 dxdt. \tag{1.18}$$

From (1.15)-(1.17) and using (1.18), (1.14) turns into

$$\begin{aligned}
&(1-\lambda C^*)\int_{\delta}^{T-\delta}\int_0^1 a(u^s_x(t))^2 dxdt\\
&\leq \frac{1}{2}\sup_{t\in[0,T]}\|u_t(t)\|^2_{L^2(0,1)} + \frac{1}{2}\int_0^T \|u_t(\eta)\|^2_{L^2(0,1)}d\eta\\
&+\frac{1}{2}\int_0^T \|h_t(\eta)\|^2_{L^2(0,1)}d\eta.
\end{aligned} \tag{1.19}$$

**If Neumann boundary conditions hold:** The proof in this case is similar to the previous one. In fact, thanks to the Hardy-Poincaré inequality (6) we have

$$\int_{\delta}^{T-\delta}\int_0^1 \frac{\big(u^s(t)\big)^2}{b(x)}\,dxdt \leq \tilde{C}^* \int_{\delta}^{T-\delta}\int_0^1 \big(u^s(t)\big)^2\,dxdt + C^*\int_{\delta}^{T-\delta}\int_0^1 a(x)(u^s_x(t))^2\,dxdt.$$

Then (1.14) becomes

$$\begin{aligned}
&\frac{1}{2}\int_{\delta}^{T-\delta}\int_0^1 \big(u^s(t)\big)^2\,dxdt + (1-\lambda C^*)\int_{\delta}^{T-\delta}\int_0^1 a(u^s_x(t))^2 dxdt\\
&\leq \frac{1}{2}\sup_{t\in[0,T]}\|u_t(t)\|^2_{L^2(0,1)} + (1+\lambda\tilde{C}^*)\int_0^T \|u_t(\eta)\|^2_{L^2(0,1)}d\eta\\
&+\frac{1}{2}\int_0^T \|h_t(\eta)\|^2_{L^2(0,1)}d\eta.
\end{aligned} \tag{1.20}$$

Using the assumption on $\lambda$ one can prove the next inequality which is valid in all situations, namely the (Dbc) and (Nbc) cases

$$\|u^s(t)\|^2_{\sim} \leq C(T)\Big(\sup_{t\in[0,T]}\|u_t(t)\|^2_{L^2(0,1)} + \int_0^T \|h_t(\eta)\|^2_{L^2(0,1)}d\eta\Big) =: M, \,\forall t\in(\delta,T-\delta)$$

Therefore, the quantity $\|u^s(t)\|^2_\sim$ is bounded by a positive constant which does not depend on $s$. So there exists a subsequence $u^s$ that weakly converges to some $v \in L^2(\delta, T-\delta; \mathcal{K})$ as $s \longrightarrow 0$. Yet, $u^s$ converges to $z$ in $\mathcal{D}'(\delta, T-\delta; \mathcal{K})$. Thus $z = v \in L^2(\delta, T-\delta; \mathcal{K})$, and

$$\|z\|^2_{L^2(\delta,T-\delta;\mathcal{K})} \leq \liminf_{s\longrightarrow 0} \|u^s\|^2_{L^2(\delta,T-\delta;\mathcal{K})} \leq M.$$

Since $M$ does not depend on $\delta$, $z \in L^2(0,T;\mathcal{K})$. ■

## 1.3 Lipschitz Stability Results in Inverse Source Problems

As we said in the introduction of this paper, our method is based on the global Carleman estimate derived in [48]. Therefore we recall this fundamental tool in the following section before proving Theorem 1 in section 1.3.2.

### 1.3.1 Carleman estimate

In this subsection, as in [40] for the purely degenerate case, the main ingredient to obtain Lipschitz stability is Carleman estimates for degenerate/singular equations. For null controllability of a singular parabolic equation with interior degeneracy, Carleman estimates were obtained in [48]. For inverse problems, these estimates are not sufficient, and one needs also some additional estimates on the term $u$ with a special weight and the derivative term $u_t$.

We proceed here as in [50]: we define the following time and space weight functions

$$\begin{aligned} &\theta(t) := \frac{1}{[(t-t_0)(T-t)]^4}, \quad \eta(t) := T + t_0 - 2t, \\ &\psi(x) := c_1 \left[\int_{x_0}^{x} \frac{y-x_0}{a(y)} dy - c_2\right] \quad \text{and} \quad \varphi(t,x) := \theta(t)\psi(x), \end{aligned} \tag{1.21}$$

with $c_2 > \max\left\{\frac{(1-x_0)^2}{a(1)(2-K)}, \frac{x_0^2}{a(0)(2-K)}\right\}$ and $c_1 > 0$. For this choice, it is easy to prove that $-c_1c_2 \leq \psi(x) < 0$ and $\eta$ is positive if $0 < t < T'$ and negative if $T' < t < T$.

Now we are ready to state the Carleman estimate related to (1.1) with (Dbc).

**Theorem 12** *Assume one among Hypotheses 2-4 with $K + L \leq 2$ and let $T > 0$. Then there exist two positive constants $C$ and $s_0$ such that the solution $u$ of* (1.1) *with (Dbc) in $L^2(\varepsilon, T; \mathcal{W}) \cap H^1([\varepsilon, T], \mathcal{K})$ satisfies, for all $s \geq s_0$,*

$$\begin{aligned} &\int_{Q_{t_0}^T} \Big(s\theta a(x)u_x^2 + s^3\theta^3\frac{(x-x_0)^2}{a(x)}u^2 + s\theta^{3/2}|\eta\psi|u^2 + \frac{1}{s\theta}u_t^2\Big)e^{2s\varphi}\,dx\,dt \\ &\leq C\Big(\int_{Q_{t_0}^T} h^2e^{2s\varphi}\,dx\,dt + sc_1\int_{t_0}^{T}\Big[\theta a(x-x_0)u_x^2e^{2s\varphi}\Big]_{x=0}^{x=1}dt\Big). \end{aligned} \tag{1.22}$$

**Proof.** Let $u$ be the solution of (1.1). For $s > 0$, the function $w = e^{s\varphi}u$ satisfies

$$\underbrace{-(aw_x)_x - s\varphi_t w - s^2 a\varphi_x^2 w - \frac{\lambda}{b}w}_{L_s^+ w} + \underbrace{w_t + 2sa\varphi_x w_x + s(a\varphi_x)_x w}_{L_s^- w} = \underbrace{he^{s\varphi}}_{h_s}.$$

Moreover, $w(t_0, x) = w(T, x) = 0$. This property allows us to apply the Carleman estimates established in [48] to $w$ with $Q_{t_0}^T$ in place of $(0, T) \times (0, 1)$ obtaining

$$\begin{aligned}&\int_{t_0}^T \|L_s^+ w\|^2 + \|L_s^- w\|^2\,dt + \int_{Q_{t_0}^T}\Big(s\theta a(x) w_x^2 + s^3\theta^3\frac{(x-x_0)^2}{a(x)}w^2\Big)\,dx\,dt\\ &\le C\Big(\int_{Q_{t_0}^T} h^2 e^{2s\varphi}\,dx\,dt + s\int_{t_0}^T \Big[\theta a^2 (w_x)^2\psi'\Big]_{x=0}^{x=1} dt\Big).\end{aligned} \tag{1.23}$$

The operators $L_s^+$ and $L_s^-$ are not exactly the ones of [48]. However, one can prove that the Carleman estimates do not change. Following [40, Theorem 3.1], one can check that

$$\begin{cases} \int_{Q_{t_0}^T} s\theta^{3/2}|\eta\psi|w^2\,dx\,dt \le C\Big(\int_{Q_{t_0}^T} h^2 e^{2s\varphi}\,dx\,dt + s\int_{t_0}^T \Big[\theta a^2 (w_x)^2\psi'\Big]_{x=0}^{x=1} dt\Big) \\ \text{and} \\ \int_{Q_{t_0}^T} \frac{1}{s\theta}w_t^2\,dx\,dt \le C\Big(\int_{Q_{t_0}^T} h^2 e^{2s\varphi}\,dx\,dt + s\int_{t_0}^T \Big[\theta a^2 (w_x)^2\psi'\Big]_{x=0}^{x=1} dt\Big). \end{cases} \tag{1.24}$$

Thus, a straightforward consequence of (1.23) and of (1.24) is the following inequality

$$\begin{aligned}&\int_{Q_{t_0}^T}\Big(s\theta a w_x^2 + s^3\theta^3\frac{(x-x_0)^2}{a}w^2 + s\theta^{3/2}|\eta\psi|w^2 + \frac{1}{s\theta}w_t^2\Big)\,dx\,dt\\ &\le C\Big(\int_{Q_{t_0}^T} h^2 e^{2s\varphi}\,dx\,dt + s\int_{t_0}^T \Big[\theta a^2 (w_x)^2\psi'\Big]_{x=0}^{x=1} dt\Big).\end{aligned} \tag{1.25}$$

Recalling the definition of $w$, we have $u = e^{-s\varphi}w$ and $u_x = -s\theta\psi' e^{-s\varphi}w + e^{-s\varphi}w_x$. Thus, substituting in (1.25), Theorem 12 follows. ■

## 1.3.2 Proof of Lipschitz Stability in Inverse Source Problems

The object of this subsection is to prove Theorem 1 which recovers a source term $h$ from the knowledge of $(au_x)_x(T', .)$, where $T' < T$ is fixed.

As it is by now classical, having Carleman estimates in hand, for proving Theorem 1 we will follow the method developed in [43, 60, 61] to obtain the Lipschitz stability for the source term $h$.

**Proof of Theorem 1.** Let $u_0 \in L^2(0, 1)$ and $h \in S(C_0)$ and $t_0 \in (0, T)$. We set $z := u_t$. According to Lemma 11, $z$ is the solution of (1.11) on $(t_0, T)$. Observe that, according to Proposition 12 $(iii)$, the solution $u$ of the problem (1.1) belongs to

$$H^1\big([t_0, T]; D(A_i)\big) \cap H^2([t_0, T]; L^2(0, 1)).$$

In particular, $u(t_0) \in D(A_i)$ and $z \in L^2\big([t_0,T];D(A_i)\big) \cap H^1([t_0,T];L^2(0,1))$.

We divide the proof into two steps and we recall that $h \in H^1(t_0,T,L^2(0,1))$.

**Step 1.**(Carleman estimate without observations). Such estimates are obtained by studying an auxiliary problem, introduced with suitable cut-off function and reflection procedure. To this aim, as in the proof of Lemma 6, we choose $\alpha, \beta > 0$ such that $\alpha+\beta < x_0$, $1+\alpha+\beta < 2-x_0$, and let $\xi : [-1,2] \to$ be the function defined as in (1.5). Now, for all $t \in [t_0,T]$ we consider the function

$$W(t,x) := \begin{cases} z(t,2-x), & x \in [1,2], \\ z(t,x), & x \in [0,1], \\ z(t,-x), & x \in [-1,0], \end{cases} \tag{1.26}$$

where $u$ solves (1.1). Then the boundary condition on $W$ are

$$\begin{cases} BW(t,-1) = Bu_t(t,1) = 0, & \forall t \in [t_0,T], \\ BW(t,2) = Bu_t(t,0) = 0, & \forall t \in [t_0,T], \end{cases}$$

since $Bu(t,0) = Bu(t,1) = 0$. Thus we can apply Lemma 11 considering as initial data $u(t_0)$ and obtaining that $W$ satisfies the following problem

$$\begin{cases} W_t - (\tilde{a}W_x)_x - \dfrac{\lambda}{\tilde{b}}W = \tilde{h}, & (t,x) \in (t_0,T) \times (-1,2), \\ BW(t,-1) = BW(t,2) = 0, & t \in (t_0,T), \end{cases} \tag{1.27}$$

where the functions $\tilde{a}(x)$, $\tilde{b}(x)$ are defined as in (1.7) and

$$\tilde{h}(t,x) := \begin{cases} h_t(t,2-x), & (t,x) \in (t_0,T) \times [1,2], \\ h_t(t,x), & (t,x) \in (t_0,T) \times [0,1], \\ h_t(t,-x), & (t,x) \in (t_0,T) \times [-1,0]. \end{cases} \tag{1.28}$$

Now, set $Z := \xi W$. Then, $Z$ solves the problem

$$\begin{cases} Z_t - (\tilde{a}Z_x)_x - \dfrac{\lambda}{\tilde{b}}Z = H, & (t,x) \in (t_0,T) \times (-\alpha-\beta, 1+\alpha+\beta), \\ Z(t,-\alpha-\beta) = Z(t,1+\alpha+\beta) = 0, & t \in (t_0,T), \end{cases}$$

with $H := \xi\tilde{h} + (\tilde{a}\xi_x W)_x + \tilde{a}\xi_x W_x$. Observe that $Z_x(t,-\alpha-\beta) = Z_x(t,1+\alpha+\beta) = 0$ and, by the assumption on $a$ and the fact that $\xi_x$ is supported in $[-\alpha,0] \cup [1,1+\alpha]$, $H \in L^2((t_0,T) \times (-\alpha-\beta, 1+\alpha+\beta))$. Now, define $\tilde{\varphi}(t,x) := \theta(t)\tilde{\psi}(x)$, where

$$\tilde{\psi}(x) := \begin{cases} \psi(2-x) = c_1 \left[ \displaystyle\int_{2-x_0}^{x} \frac{t-2+x_0}{\tilde{a}(t)}dt - c_2 \right], & x \in [1,2], \\ \psi(x), & x \in [0,1], \\ \psi(-x) = c_1 \left[ \displaystyle\int_{-x_0}^{x} \frac{t+x_0}{\tilde{a}(t)}dt - c_2 \right], & x \in [-1,0]. \end{cases} \tag{1.29}$$

Thus, we can apply the analogue of Theorem 12 on $(t_0, T) \times (-\alpha-\beta, 1+\alpha+\beta)$ in place of $(0,T)\times(0,1)$ and with weight $\tilde{\varphi}$, obtaining that there exist two positive constants $C$ and $s_0$ ($s_0$ sufficiently large), such that $Z$ satisfies, for all $s \geq s_0$,

$$\begin{aligned}
&\int_{t_0}^{T}\int_{-\alpha-\beta}^{1+\alpha+\beta}\left(s\theta\tilde{a}Z_x^2 + s^3\Theta^3\frac{(x-x_0)^2}{\tilde{a}}Z^2 + s\theta^{3/2}|\eta\tilde{\psi}|Z^2 + \frac{1}{s\theta}Z_t^2\right)e^{2s\tilde{\varphi}}\,dxdt\\
&\leq C\left(\int_{t_0}^{T}\int_{-\alpha-\beta}^{1+\alpha+\beta}H^2e^{2s\tilde{\varphi}}dxdt + sc_1\int_{t_0}^{T}\left[\tilde{a}\theta e^{2s\tilde{\varphi}}(x-x_0)Z_x^2dt\right]_{x=-\alpha-\beta}^{x=1+\alpha+\beta}\right)\\
&= C\int_{t_0}^{T}\int_{-\alpha-\beta}^{1+\alpha+\beta}H^2e^{2s\tilde{\varphi}}dxdt.
\end{aligned}\tag{1.30}$$

Using again the fact that $\xi_x$ is supported in $[-\alpha, 0]\cup[1, 1+\alpha]$, it follows

$$\begin{aligned}
&\int_{t_0}^{T}\int_{-\alpha-\beta}^{1+\alpha+\beta}H^2e^{2s\tilde{\varphi}}dxdt = \int_{t_0}^{T}\int_{-\alpha-\beta}^{1+\alpha+\beta}(\xi\tilde{h} + (\tilde{a}\xi_xW)_x + \tilde{a}\xi_xW_x)^2e^{2s\tilde{\varphi}}dxdt\\
&\leq C\left(\int_{t_0}^{T}\int_{-\alpha-\beta}^{1+\alpha+\beta}\tilde{h}^2e^{2s\tilde{\varphi}}dxdt + \int_{t_0}^{T}\left(\int_{-\alpha}^{0}+\int_{1}^{1+\alpha}\right)(W^2+\tilde{a}W_x^2)e^{2s\tilde{\varphi}}dxdt\right)\\
&\leq C\left(\int_{t_0}^{T}\int_{0}^{1}h_t^2e^{2s\varphi}dxdt + \int_{t_0}^{T}\left(\int_{0}^{\alpha}+\int_{1-\alpha}^{1}\right)(z^2+az_x^2)e^{2s\varphi}dxdt\right).
\end{aligned}$$

Thus (1.30) becomes

$$\begin{aligned}
I :=& \int_{t_0}^{T}\int_{-\alpha-\beta}^{1+\alpha+\beta}\left(s\theta\tilde{a}Z_x^2 + s^3\Theta^3\frac{(x-x_0)^2}{\tilde{a}}Z^2 + s\theta^{3/2}|\eta\tilde{\psi}|Z^2 + \frac{1}{s\theta}Z_t^2\right)e^{2s\tilde{\varphi}}\,dxdt\\
\leq& C\left(\int_{t_0}^{T}\int_{0}^{1}h_t^2e^{2s\varphi}dxdt + \underbrace{\int_{t_0}^{T}\left(\int_{0}^{\alpha}+\int_{1-\alpha}^{1}\right)(z^2+az_x^2)e^{2s\varphi}dxdt}_{J}\right)
\end{aligned}$$

Now we show that $J$ can be absorbed by $I$. Let $\epsilon > 0$ be small enough.

So, since

$$\begin{cases}
\inf_{t\in[t_0,T]}\theta(t) > 0,\\
\inf_{x\in[0,\alpha]\cup[1-\alpha,1]}a(x) > 0,\\
\text{and} \inf_{x\in[0,\alpha]\cup[1-\alpha,1]}\dfrac{(x-x_0)^2}{a(x)} > 0,
\end{cases}$$

one has for $s$ large enough

$$\begin{aligned}
&\int_{t_0}^T \left(\int_0^\alpha + \int_{1-\alpha}^1\right)(z^2 + az_x^2)e^{2s\varphi}dxdt \\
&\le \epsilon \int_{t_0}^T \left(\int_0^\alpha + \int_{1-\alpha}^1\right)\left[s\theta a z_x^2 + s^3\theta^3 \frac{(x-x_0)^2}{a} z^2\right] e^{2s\varphi}\, dxdt \\
&\le \epsilon \int_{t_0}^T \int_{-\alpha-\beta}^{1+\alpha+\beta} \left[s\theta \tilde{a} Z_x^2 + s^3\Theta^3 \frac{(x-x_0)^2}{\tilde{a}} Z^2\right] e^{2s\tilde{\varphi}}\, dxdt.
\end{aligned}$$

Therefore, by choosing $\epsilon$ small and $s$ large enough, we see that $J$ can be absorbed by $I$.

Hence, we conclude that, for s large enough, and for a positive constant $C$:

$$\begin{aligned}
&\int_{t_0}^T \int_{-\alpha-\beta}^{1+\alpha+\beta} \left(s\theta\tilde{a}Z_x^2 + s^3\Theta^3\frac{(x-x_0)^2}{\tilde{a}}Z^2 + s\theta^{3/2}|\eta\tilde{\psi}|Z^2 + \frac{1}{s\theta}Z_t^2\right) e^{2s\tilde{\varphi}}\, dxdt \\
&\le C \int_{t_0}^T \int_0^1 h_t^2 e^{2s\varphi} dxdt.
\end{aligned} \tag{1.31}$$

The purpose of the first step is then accomplished.

**Step 2.** Proceeding as in [39, Theorem 1.5], one can prove the following

**Lemma 13** *There exists a constant $C > 0$ such that*

$$\int_{t_0}^T \int_0^1 h_t^2 e^{2s\varphi} dxdt \le \frac{C}{\sqrt{s}} \int_0^1 h^2(T',x)e^{2s\varphi(T',x)}dx, \tag{1.32}$$

*recalling that $T' = (T + t_0)/2$.*

Furthermore, since $Z$ satisfies

$$Z(T',x) = u_t(T',x) = (au_x)_x(T',x) + \frac{\lambda}{b(x)}u(T',x) + h(T',x) \text{ for a.e. } x \in [0,1],$$

it follows that

$$\begin{aligned}
\int_0^1 h^2(T',x)e^{2s\varphi(T',x)}dx \le & C\|(au_x)_x(T',x) + \frac{\lambda}{b(x)}u(T',x)\|^2_{L^2(0,1)} \\
& + C\int_0^1 Z^2(T',x)e^{2s\varphi(T',x)}.
\end{aligned} \tag{1.33}$$

In the next step, we provide some estimate of the last term of the right hand side in the above inequality. In fact, using the same computations of [39, Theorem 1.5], one can find $C > 0$ such that

$$\begin{aligned}
&\int_0^1 Z^2(T',x)e^{2s\varphi(T',x)}dx \\
\le & C\int_{t_0}^T \int_{-\alpha-\beta}^{1+\alpha+\beta} \left(s\theta\tilde{a}Z_x^2 + s^3\Theta^3\frac{(x-x_0)^2}{\tilde{a}}Z^2 + s\theta^{3/2}|\eta\tilde{\psi}|Z^2 + \frac{1}{s\theta}Z_t^2\right) e^{2s\tilde{\varphi}}\, dxdt.
\end{aligned} \tag{1.34}$$

Consequently, Combining (1.31), (1.32) and (1.34), we deduce that

$$\int_0^1 Z^2(T',x)e^{2s\varphi(T',x)}dxdt \leq \frac{C}{\sqrt{s}}\int_0^1 h^2(T',x)e^{2s\varphi(T',x)}dx, \tag{1.35}$$

for a positive constant $C$. Hence, by (1.35) and (1.33), taking s large enough, we get

$$\gamma\int_0^1 h^2(T',x)dxdt \leq C\|(au_x)_x(T',x) + \frac{\lambda}{b(x)}u(T',x)\|^2_{L^2(0,1)},$$

where

$$\gamma = \min_{x\in[0,1]} e^{2s\varphi(T',x)},$$

which together with

$$|h(t,x)| \leq |h(T',x)| + \int_{T'}^t |h_t(s,x)|ds \leq C|h(T',x)|$$

proves the claim. ■

# Chapter 2

# Numerical approach and simulations for a degenerate/singular parabolic equations

In this chapter, we develop a numerical approach to compute the unknown source term of (2.1), and we give illustrations of the obtained results through simulations.

## 2.1 Numerical approach

As the theoretical stability is guaranteed by Theorem 1, in this subsection we study the inverse source problem from the numerical viewpoint. To this end, let us define our inverse problem which we use in computations.

**Inverse Source Problem (ISP).** Let $u$ be the solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} - (a u_x)_x - \dfrac{\lambda}{b(x)} u = h(t,x), & (t,x) \in Q_T, \\ u(0,t) = u(1,t) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases} \tag{2.1}$$

Determine the source term $h(t,x)$ from the measured data at the final time $u(T,.)$.

**Remark 14** *It should be mentioned that we do not need the supplement distributed measurements to obtain the numerical solution of the inverse problem.*

Numerically, we treat Problem (ISP) by interpreting its solution as a minimizer of the following least squares functional with the Tikhonov regularization

$$\min_{h \in \mathcal{U}} J(h), \quad J(h) = \frac{1}{2}\|u(T,.) - \tilde{u}\|^2_{L^2(0,1)} + \frac{\varepsilon}{2}\|h - h_0\|^2_{L^2(Q)}, \tag{2.2}$$

where $\tilde{u} \in L^2(0,1)$ is the observation data with noise, $h_0$ is the initial guess for $h$, $\varepsilon > 0$ stands for the regularization parameter and $\mathcal{U}$ is the set of admissible unknown sources defined in the following way

$$\mathcal{U} := \{h \in H^1(0,T;L^2(0,1)) : \|h\|_{H^1(0,T;L^2(0,1))} \leq r, r > 0\}. \tag{2.3}$$

Evidently, the set $\mathcal{U}$ is a bounded, closed, and convex subset of $H^1(0,T;L^2(0,1))$.

Firstly in this section, with the aim of showing that the minimization problem and the direct problem are well-posed, we prove that the solution's behavior changes continuously with the source term, for this we prove the Lipschitz continuity of the input-output operator $F : h \longrightarrow u$, where $u$ is the weak solution of (2.1) with term source $h$. Secondly, we prove the differentiability of the functional $J$, which gives the existence of the gradient of $J$, that is computed using the adjoint state method. Finally, to show the convergence of the descent method, we prove that the gradient of $J$ is Lipschitz continuous, this gives that $\lim\limits_{k\to\infty} \parallel \nabla J(h^k) \parallel_{L^2(\Omega)} = 0$ and $\big(J(h^k)\big)_k$ is a monotone decreasing sequence , where $(h^k)_k$ is the sequence of iterations obtained by the Landweber iteration algorithm $h^{k+1} = h^k - t_k \nabla J(h^k)$ and $t_k$ is chosen by the inaccurate linear search by the Armijo-Goldstein Rule. Also we present some numerical experiments to study the performance of this approach.

We are now going to show the existence of minimizers to the problem (2.2). To do so, we need the following lemma. Assume Hypothesis 1.8. Let $u$ be the weak solution of (2.1) corresponding to a given source term $h$. Then, the input-output operator $F$ : $H^1(0,T;L^2(0,1)) \to C([0,T];L^2(0,1)) \cap L^2(0,T;\mathcal{H})$ defined as $F(h) := u$ is Lipschitz continuous. **Proof.** First, take $u_0 \in D(A)$. Then, let the source term $h$ be perturbed by a small amount $\delta h$ such that $h+\delta h \in \mathcal{U}$. Consider $\delta u = u^\delta - u$, where $u^\delta$ is the weak solution of (2.1) with source term $h^\delta := h + \delta h$. Then $\delta u \in C^1\big([0,T];L^2(0,1)\big) \cap C\big(0,T;D(A)\big)$ satisfies the following sensitivity problem:

$$\begin{cases} \partial_t \delta u - \partial_x\big(a\partial_x \delta u\big) - \dfrac{\lambda}{b}\delta u = \delta h(t,x), & (t,x) \in Q, \\ \delta u(0) = \delta u(1) = 0, & t \in (0,T), \\ \delta u(0,x) = 0, & x \in (0,1). \end{cases} \tag{2.4}$$

Let $v(t,x)$ be a smooth function. From equation (2.4) and by the Gauss Green identity [**?**, Lemma 2.21], we have

$$\int_0^1 \partial_t \delta u v \, dx + \int_0^1 \Big(a\partial_x \delta u v_x - \frac{\lambda}{b}\delta u v\Big)\, dx = \int_0^1 \delta h v \, dx.$$

We take $\delta u$ as a mutual test function for $v$ to deduce

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\delta u)^2 \, dx + \int_0^1 \Big(a(\partial_x \delta u)^2 - \frac{\lambda}{b}(\delta u)^2\Big)\, dx = \int_0^1 \delta h \delta u \, dx.$$

Then, using Lemma 8, by the Cauchy-Schwarz inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\|\delta u(t)\|^2_{L^2(0,1)} + \Lambda\int_0^1 a(\partial_x \delta u(t))^2\,dx \le \frac{1}{2}\|\delta u(t)\|^2_{L^2(0,1)} + \frac{1}{2}\|\delta h(t)\|^2_{L^2(0,1)}, \tag{2.5}$$

for every $t \le T$, from which

$$\frac{d}{dt}\|\delta u(t)\|^2_{L^2(0,1)} \le \|\delta u(t)\|^2_{L^2(0,1)} + \|\delta h(t)\|^2_{L^2(0,1)}.$$

Applying Gronwall's inequality, we obtain

$$\begin{aligned}\|\delta u(t)\|^2_{L^2(0,1)} &\le e^T\Big(\|\delta u(0)\|^2_{L^2(0,1)} + \|\delta h\|^2_{L^2(Q)}\Big)\\ &= e^T\|\delta h\|^2_{L^2(Q)},\end{aligned} \tag{2.6}$$

for every $t \le T$. From (2.5) and (2.6), we immediately get

$$\int_0^T \|\sqrt{a}\delta u_x(t)\|^2_{L^2(0,1)}dt \le C_T\|\delta h\|^2_{L^2(Q)}, \tag{2.7}$$

for every $t \le T$ and some universal constant $C_T > 0$. Thus, by (2.6) and (2.7), we obtain

$$\sup_{t\in[0,T]}\|\delta u(t)\|^2_{L^2(0,1)} + \int_0^T \|\delta u(t)\|^2_{\mathcal{H}}dt \le C_T\|\delta h\|^2_{L^2(Q)},$$

from which it follows that

$$\|\delta u\|^2_{C([0,T];L^2(0,1))} + \|\delta u\|^2_{L^2(0,T;\mathcal{H})} \le C\|\delta h\|^2_{H^1(0,T;L^2(0,1))},$$

if $u_0 \in D(A)$. Since $D(A)$ is dense in $L^2(0,1)$, the same inequality holds if $u_0 \in L^2(0,1)$. This completes the proof Lemma 2.1. ■

An immediate consequence of Lemma 2.1 is the following result Assume Hypothesis 1.8. Then, the functional $J$ is continuous on $\mathcal{U}$ and there exists a minimizer $h^\star \in \mathcal{U}$ of $J(h)$, i.e.

$$J(h^\star) = \min_{h\in\mathcal{U}} J(h).$$

. Let $u$ the weak solution of (2.1) with source term $h$. The input-output operator $F : H^1(0,T;L^2(0,1)) \to C([0,T];L^2(0,1)) \cap L^2(0,T;\mathcal{H})$, $F(h) = u$ is G-derivable.

The most important issue in numerical solutions of inverse problems is the Lipschitz continuity of the gradient, which ensures the convergence of the method of descent, for that we have the follows result

Let $h$ and $\delta h$, such that $h + \delta h \in \mathcal{U}$, than $\nabla J$ is Lipschitz continuous

$$\| \nabla J(h+\delta h) - \nabla J(h) \|_{L^2(Q)} \leqslant L_1 \| \delta h \|_{H^1(0,T;L^2(0,1))} \tag{2.8}$$

with the Lipschitz constant $L_1 > 0$.

**Proof of Proposition 2.1.** Let $\delta h$ be a small variation such that $h + \delta h \in \mathcal{U}$, we define the function

$$F'(h) : \delta h \in \mathcal{U} \to \delta u, \tag{2.9}$$

where $\delta u$ is the solution of the variational problem

$$\begin{gathered}\int_\Omega \partial_t(\delta u)v\,dx + \int_\Omega (a(x)\partial_x(\delta u)\partial_x v - \frac{\lambda}{b(x)}\delta uv)\,dx = \int_\Omega \delta hvdx \quad \forall v \in H_0^1(\Omega)\\ \delta u(0,t) = \delta u(1,t) = 0 \quad \forall t \in ]0,T[\\ \delta u(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{2.10}$$

We set

$$\phi(h) = F(h + \delta h) - F(h) - F'(h)\delta h. \tag{2.11}$$

We want to show that

$$\phi(h) = o(\delta h). \tag{2.12}$$

We easily verify that the function $\phi$ is the solution of variational problem

$$\begin{gathered}\int_\Omega \partial_t\phi v\,dx + \int_\Omega \left(a(x)\partial_x\phi\partial_x v - \frac{\lambda}{b(x)}\phi v\right)dx = \int_\Omega (\delta h - (\delta h)^2)vdx \quad \forall v \in H_0^1(\Omega)\\ \phi(0,t) = \phi(1,t) = 0 \quad \forall t \in ]0,T[\\ \phi(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{2.13}$$

In the same way as that used in the proof of continuity, we deduce

$$\|\phi\|^2_{C([0,T];L^2(0,1))} + \|\phi\|^2_{L^2(0,T;\mathcal{H})} \le C\|\delta h - (\delta h)^2\|^2_{H^1(0,T;L^2(0,1))},$$

Hence, the function $F(h) = u$ is G-derivable and we deduce the existence of the gradient of the functional $J$. ■

Before starting the demonstration of Proposition 2.1, we compute the gradient of $J$ using the adjoint state method.

We define the Gâteaux derivative of $u$ at $h$ in the direction $f \in L^2(\Omega\times]0,T[)$, by

$$\hat{u} = \lim_{s\to 0} \frac{u(h + sf) - u(h)}{s}, \tag{2.14}$$

$u(h + sf)$ is the weak solution of (2.1) with source term $h + sf$, and $u(h)$ is the weak solution of (2.1) with source term $h$.

We compute the Gâteaux (directional) derivative of (2.1) at $h$ in some direction $f \in L^2(\Omega\times]0,T[)$, and we get the so-called tangent linear model:

$$\begin{gathered}\partial_t\hat{u} - A\hat{u} = f\\ \hat{u}(0,t) = \hat{u}(1,t) = 0 \quad \forall t \in ]0,T[\\ \hat{u}(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{2.15}$$

We introduce the adjoint variable $P$, and we integrate,

$$\int_0^1\int_0^T \partial_t \hat{u} P\, dt\, dx - \int_0^1\int_0^T A\hat{u}P\, dx = \int_0^1\int_0^T fP\, dt\, dx, \tag{2.16}$$

$$\int_0^1 \Big([\hat{u}P]_0^T - \int_0^T \hat{u}\partial_t P\, dt\Big)\, dx - \int_0^T \langle A\hat{u}, P\rangle_{L^2(\Omega)} dt = \langle f, P\rangle_{L^2(\Omega\times]0,T[)}, \tag{2.17}$$

$$\begin{aligned}&\int_0^1 [\hat{u}(T)P(T) - \hat{u}(0)P(0)]dx - \int_0^T \langle \hat{u}, \partial_t P\rangle_{L^2(\Omega)} dt - \int_0^T \langle A\hat{u}, P\rangle_{L^2(\Omega)} dt \\ &= \langle f, P\rangle_{L^2(\Omega\times]0,T[)}.\end{aligned} \tag{2.18}$$

Let us take $P(x=0) = P(x=1) = 0$, then we may write $\langle \hat{u}, AP\rangle_{L^2(\Omega)} = \langle A\hat{u}, P\rangle_{L^2(\Omega)}$. With $P(T) = 0$ we may now rewrite (2.18) as

$$\int_0^T \langle \hat{u}, \partial_t P + AP\rangle_{L^2(\Omega)} dt = -\langle f, P\rangle_{L^2(\Omega\times]0,T[)}$$

this gives

$$\begin{gathered}\int_0^T \langle \hat{u}, \partial_t P + AP\rangle_{L^2(\Omega)} dt = -\langle f, P\rangle_{L^2(\Omega\times]0,T[)} \\ P(x=0) = P(x=1) = 0, \quad P(T) = 0.\end{gathered} \tag{2.19}$$

The discretization in time of (3.29), using the Rectangular integration method, gives

$$\begin{gathered}\sum_{j=0}^{M+1} \langle \hat{u}(t_j), \partial_t P(t_j) + AP(t_j)\rangle_{L^2(\Omega)} \Delta t = \langle -P, f\rangle_{L^2(\Omega\times]0,T[)} \\ P(x=0) = P(x=1) = 0, \quad P(T) = 0.\end{gathered} \tag{2.20}$$

With

$$t_j = j\Delta t, \quad j \in \{0, 1, 2, \dots, M+1\},$$

where $\Delta t$ is the step in time and $T = (M+1)\Delta t$.

The Gâteaux derivative of $J$ at $h$ in the direction $f \in L^2(\Omega)$ is given by

$$\hat{J}(f) = \lim_{s\to 0} \frac{J(h+sf) - J(h)}{s}.$$

After some computations, we arrive at

$$\hat{J}(f) = \langle u(T) - \tilde{u}, \hat{u}(T)\rangle_{L^2(\Omega)} + \langle \varepsilon(h - h_0), f\rangle_{L^2(\Omega\times]0,T[)}. \tag{2.21}$$

The adjoint model is

$$\begin{gathered}\partial_t P(T) + AP(T) = \frac{1}{\Delta t}(u(T) - \tilde{u}), \quad \partial_t P(t_j) + AP(t_j) = 0 \quad \forall t_j \neq T \\ P(x=0) = P(x=1) = 0 \quad \forall t_j \in ]0; T[ \\ P(T) = 0.\end{gathered} \tag{2.22}$$

From equations (3.30), (3.31) and (3.32), the gradient of $J$ is given by

$$\frac{\partial J}{\partial h} = -P + \varepsilon(h - h_0). \tag{2.23}$$

Problem (3.32) is retrograde, we make the change of variable $t \longleftrightarrow T - t$.

**Proof of Proposition 2.1.** We have $\nabla J(h) = -P_1 + \varepsilon(h - h_0)$ with $P_1$ is the solution of the adjoint model( with change of variable $t_j \longleftrightarrow T - t_j$)

$$\begin{cases} \begin{cases} \partial_t P_1(0) - AP_1(0) = \dfrac{1}{\Delta t}(\tilde{u} - u_1(T)) \\ \partial_t P_1(t_j) - AP_1(t_j) = 0 \qquad \forall t_j \neq 0 \end{cases} \\ P_1(x,t) = 0 \quad \forall x \in \partial\Omega, \quad \forall t \in ]0;T[ \\ P_1(x,0) = 0. \end{cases}$$

where $u_1$ is the weak solution of (2.1) with source term $h$, and $\nabla J(h + \delta h) = -P_2(T) + \varepsilon(h + \delta h - h_0)$ with $P_2$ is the solution of the adjoint model ( with change of variable $t_j \longleftrightarrow T - t_j$)

$$\begin{cases} \begin{cases} \partial_t P_2(0) - AP_2(0) = \dfrac{1}{\Delta t}(\tilde{u} - u_2(T)) \\ \partial_t P_2(t_j) - AP_2(t_j) = 0 \qquad \forall t_j \neq 0 \end{cases} \\ P_2(x,t) = 0 \ \forall x \in \partial\Omega, \quad \forall t \in ]0;T[ \\ P_2(x,0) = 0. \end{cases}$$

where $u_2$ is the weak solution of (2.1) with source term $h + \delta h$.

Let $\delta P = P_1 - P_2$, we easily verify that $\delta P$ is the solution of the variational problem

$$\begin{cases} \displaystyle\int_\Omega \partial_t \delta P v \, dx + \int_\Omega \left(a(x)\partial_x \delta P \partial_x v - \frac{\lambda}{b(x)}\delta P v\right) dx = \frac{1}{\Delta t}\int_\Omega (u_2(T) - u_1(T))1_0 \quad \forall v \in H_0^1(\Omega) \\ \delta P(x,t) = 0 \quad \forall (x) \in \partial\Omega \quad , \forall t \in ]0;T[ \\ \delta P(x,0) = 0 \quad \forall (x) \in \Omega. \end{cases} \tag{2.24}$$

Hence, $\delta P$ is weak solution of (2.1) with $h = (u_2(T) - u_1(T))1_0$. We apply the estimate in proposition 10, we obtain and there exists a positive constant $C$ such that

$$\sup_{t\in[0,T]} \|\delta P(t)\|^2_{L^2(0,1)} + \int_0^T \|\delta P(t)\|^2_{\mathcal{H}} dt \leq C\|(u_2(T) - u_1(T))1_0\|^2_{L^2(Q)}, \tag{2.25}$$

then

$$\| \delta P \|^2_{L^2(0,T;\mathcal{H})} \leqslant C\left(\| (u_2(T) - u_1(T))1_0 \|^2_{L^2(Q)}\right), \tag{2.26}$$

and

$$\| \delta P \|^2_{C([0,T];L^2(0,1))} \leqslant C\left(\| (u_2(T) - u_1(T))1_0 \|^2_{L^2(Q)}\right), \tag{2.27}$$

the constant C depending only on $\Omega$ and T.

we showed above the Lipschitz continuity of the input-output operator

$$\begin{aligned} F: H^1(0,T;L^2(0,1)) &\longrightarrow C([0,T];L^2(0,1)) \cap L^2(0,T;\mathcal{H}) \\ h &\longmapsto u \end{aligned} \tag{2.28}$$

from where

$$\| (u_2(T)-u_1(T))1_0 \|^2_{L^2(Q)} \leqslant C \| \delta h \|^2_{H^1(0,T;L^2(0,1))} \tag{2.29}$$

therefore

$$\| \delta P \|^2_{L^2(0,T;\mathcal{H})} \leqslant C \| \delta h \|^2_{H^1(0,T;L^2(0,1))}, \tag{2.30}$$

and

$$\| \delta P \|^2_{C([0,T];L^2(0,1))} \leqslant C \| \delta h \|^2_{H^1(0,T;L^2(0,1))}, \tag{2.31}$$

We have

$$\begin{aligned} \| \nabla J(u_0+\delta u_0) - \nabla J(u_0) \|_{L^2(Q)} &= \| \delta P + \varepsilon \delta h \|_{L^2(Q)} \\ &\leqslant \| \delta P \|_{L^2(Q)} + \| \varepsilon \delta h \|_{L^2(Q)} \ . \end{aligned} \tag{2.32}$$

therefore

$$\| \nabla J(h+\delta h) - \nabla J(h) \|_{L^2(Q)} \leqslant (\sqrt{C}+\varepsilon) \| \delta h \|_{H^1(0,T;L^2(0,1))} \ . \tag{2.33}$$

This completes the proof of the theorem. ■

## 2.2 Algorithm and simulations

In this section, a numerical algorithm on the basis of the conjugate gradient method is designed to treat the inverse problem and some numerical experiments are also performed.

The main steps for descent method at each iteration are:

- Calculate $u^k$ solution of (2.1) with source term $h^k$
- Calculate $P^k$ solution of the adjoint problem
- Calculate the descent direction $d_k = -\nabla J(h^k)$
- Find $t_k = \underset{t>0}{argmin} \ J(h^k + t d_k)$
- Update the variable $h^{k+1} = h^k + t_k d_k$.

The algorithm ends when $\left|\nabla J(h)\right| < \mu$, where $\mu$ is a given small precision.

The value $t_k$ is chosen by the inaccurate linear search by the Armijo-Goldstein Rule as follows:

Let $\alpha_i, \beta \in [0,1[$ and $\alpha > 0$

if $J(h^k + \alpha_i d_k) \leq J(h^k) + \beta \alpha_i d_k^T d_k$, $t_k = \alpha_i$ and stop.

if not, $\alpha_i = \alpha\alpha_i$.

In this section, we are going to reconstruct the source term in all three cases : the purely singular case $\alpha = 0$, purely degenerate case $\lambda = 0$, and degenerate-singular case.

For the simulations, in all the tests below we take $a(x) =\mid x - x_0 \mid^\alpha$, $b(x) =\mid x - x_0 \mid^\beta$, step in space $N = 100$ and step in time $M = 100$.

**The purely singular case** $\alpha = 0$

For this test we take $\beta < 2$ and $\lambda < 0$ (example $\beta = \dfrac{1}{2}$ and $\lambda = -1$ )

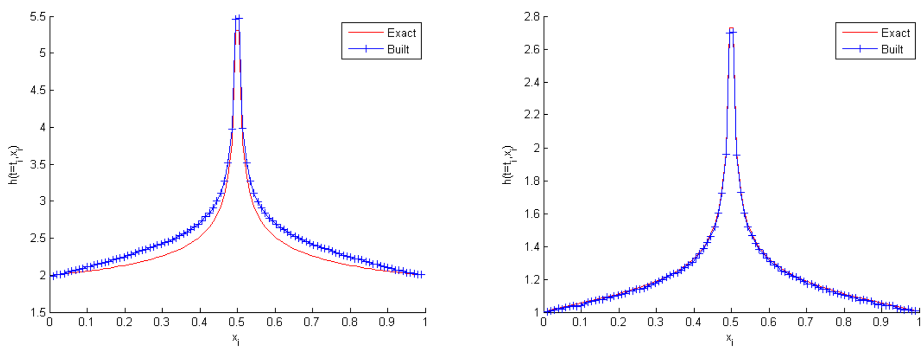


Figure 2.1: Graph of $h(t = t_0, .)$ (left). And $h(t = t_{100}, .)$ (right).

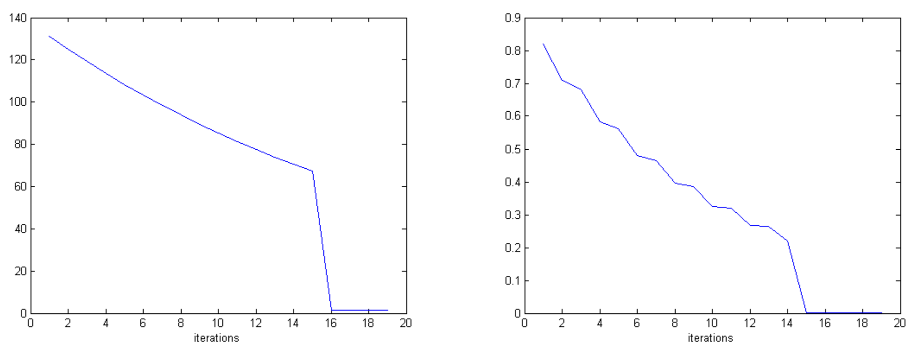


Figure 2.2: Graph of $\|\nabla J\|_2$ (left). And of $J$ (right).

**The purely degenerate case** $\lambda = 0$

For this test we take $0 < \alpha < 2$ (example $\alpha = \dfrac{1}{2}$)

**The degenerate and singular case**

For this test we take $0 < \alpha$, $\lambda < 0$ and $\beta < 2 - \alpha$ (example $\beta = \alpha = \dfrac{1}{2}$ and $\lambda = -1$)

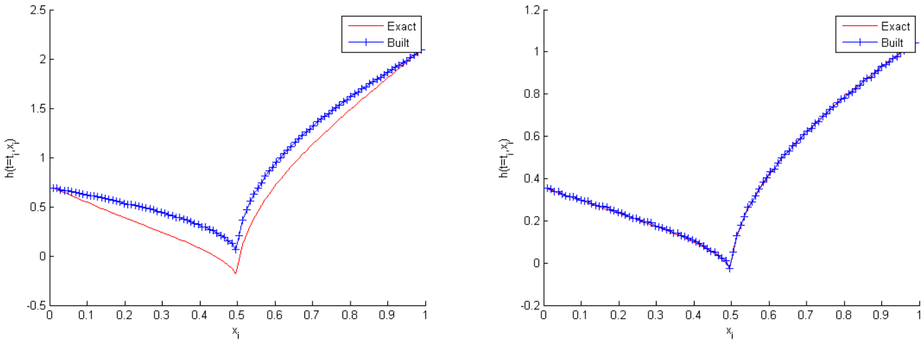


Figure 2.3: Graph of $h(t = t_0, .)$ (left). And $h(t = t_{100}, .)$ (right).

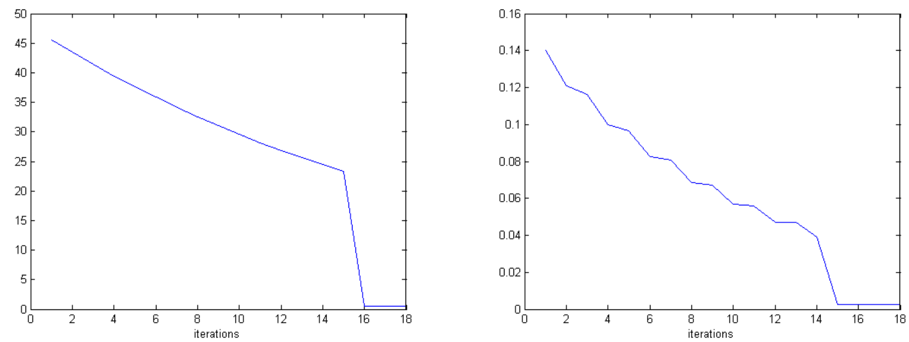


Figure 2.4: Graph of $\|\nabla J\|_2$ (left). And of $J$ (right).

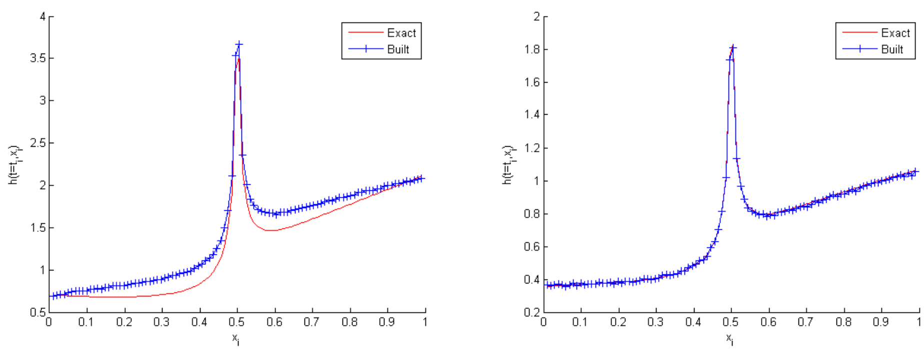


Figure 2.5: Graph of $h(t = t_0, .)$ (left). And $h(t = t_{100}, .)$ (right).

In all case (purely singular case $\alpha = 0$, purely degenerate case $\lambda = 0$, and degenerate and singular case), (Fig. 2.1,Fig. 2.3,Fig. 2.5) show that we can rebuild the source term

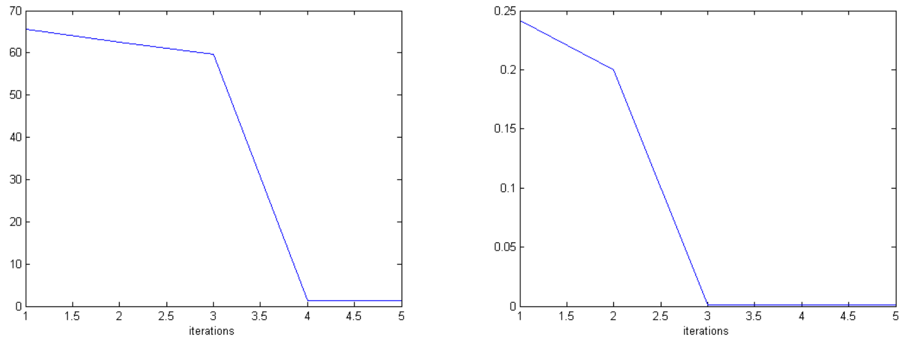


Figure 2.6: Graph of $\|\nabla J\|_2$ (left). And of $J$ (right).

$h$. And (Fig. 2.2,Fig. 2.4,Fig. 2.6) manifest that $J$ and $\|\nabla J\|_2$ converges both to 0. which shows numerically the convergence of the descent' method

# Chapter 3

# Inverse diffusion problem for degenerate/singular parabolic equations by Hybrid method

This chapter deals with the determination of a coefficient in the diffusion term of some degenerate /singular one-dimensional linear parabolic equation from final data observations. The mathematical model leads to a non convex minimization problem. To solve it, we propose a new approach based on a hybrid genetic algorithm (married genetic with descent method type gradient). Firstly, with the aim of showing that the minimization problem and the direct problem are well-posed, we prove that the solution's behavior changes continuously with respect to the initial conditions. Secondly, we chow that the minimization problem has at least one minimum. Finally, the gradient of the cost function is computed using the adjoint state method. Also we present some numerical experiments to show the performance of this approach.

## 3.1 Introduction

This chapter is devoted to the identification of a diffusion coefficient in degenerate/singular parabolic equation by the variational method, from final data observation. The problem we treat can be stated as follows:

Consider the following degenerate parabolic equation with singular potential

$$\begin{cases} \partial_t u + \mathcal{A}(u) = f \\ u(0,t) = u(1,t) = 0 \qquad \forall t \in (0,T) \\ u(x,0) = u_0(x) \quad \forall x \in \Omega \end{cases} \tag{3.1}$$

where, $\Omega = (0,1)$, $f \in L^2(\Omega \times (0,T))$ and $\mathcal{A}$ is the operator defined as

$$\mathcal{A}(u) = -\partial_x(a(x)\partial_x u(x)) - \frac{\lambda}{x^\beta}u, \quad a(x) = k(x)x^\alpha$$

with $\alpha \in (0,1)$, $\beta \in (0, 2-\alpha)$, and $\lambda \leq 0$, $0 < k(x) \leq c_1$

The formulation of the inverse problem is

$$\begin{cases} \text{find } k \in A_{ad} \text{ such that} \\ J(k) = \min\limits_{\kappa \in A_{ad}} J(\kappa), \end{cases} \tag{3.2}$$

where the cost function $J$ is defined as follows

$$J(k) = \frac{1}{2} \left\| u(t=T) - u_{obs} \right\|^2_{L^2(\Omega)}, \tag{3.3}$$

subject to $u$ is the weak solution of the parabolic problem (3.1). The space $A_{ad}$ is the admissible set of diffusion coefficients.
The functional $J$ is non convex, any descent algorithm will be stopped at the meeting of the first local minimum. To stabilize this problem, the classical method is to add to $J$ a regularizing term coming from the regularization of Tikhonov. So, we obtain the functional

$$J_T(\kappa) = \frac{1}{2} \left\| u(t=T) - u_{obs} \right\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \left\| \kappa - k^b \right\|^2_{L^2(\Omega)}. \tag{3.4}$$

But, in reality, $k^b$ is partially known, than the determination of the parameter $\varepsilon$, which presents an important difficulty. Until these lines are written, there is no effective method for determining this parameter. In the literature, we found two popular methods : cross-validation and Lcurve (see [31],[34],[32],[33]). For these two methods it is necessary to solve the problem with several different values of the regularization parameter.

To show the difficulty of determining the parameter $\varepsilon$ when we have a partial knowledge of $k^b$ (example 20%) in points of space, we did several test for different epsilon values, the results are as follows

| $\varepsilon$ | Minimum value of $J$ |
|---|---|
| 1 | $2.230238.10^{-02}$ |
| $10^{-01}$ | $1.277236.10^{-02}$ |
| $10^{-02}$ | $1.093206.10^{-02}$ |
| $10^{-03}$ | $2.093763.10^{-02}$ |
| $10^{-04}$ | $3.029143.10^{-02}$ |
| $10^{-05}$ | $2.92163.10^{-03}$ |
| $10^{-06}$ | $1.12187.10^{-02}$ |

Table 01. Results on the Tikhonov approach. Comparison between different values of regularizing coefficient $\varepsilon$. The smallest value of $J$ is reached when $\varepsilon = 10^{-05}$.

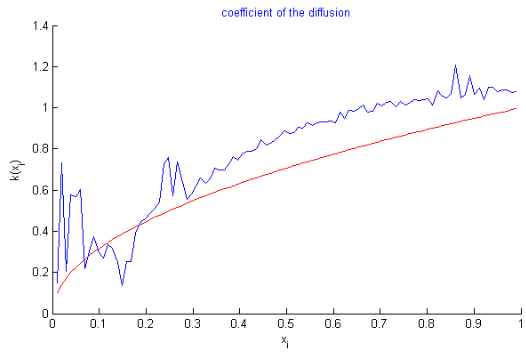


Figure 01. Graph of $k$ in case $\varepsilon = 10^{-05}$

The figure 01 shows that we can't rebuild the coefficient $k$ in case $\varepsilon = 10^{-05}$, and the method of choosing $\varepsilon$ by doing several tests for different epsilon values is not effective.

To overcome this problem, in case where $k^b$ is partially known, we propose in this work, a new approach based on Genetic Hybrid algorithms which consists to minimize the functional $J$ without any regularization. This work is the continuity of [1] where the authors identify the initial state of a degenerate parabolic problem.

Firstly, with the aim of showing that the minimization problem and the direct problem are well-posed, we prove that the solution's behavior changes continuously with respect to the initial conditions. Secondly, we show that the minimization problem has at least one minimum. Finally, The gradient of the cost function is computed using the adjoint state method. Also we present some numerical experiments to show the performance of this approach.

## 3.2 Well-posedness

Now we specify some notations we shall use. Let introduce the following functional spaces (see [8], [10], [18])

$$V = \left\{ u \in L^2(\Omega), u \text{ absolutely continuous on } [0,1] \right\}, \tag{3.5}$$

$$S = \left\{ u \in L^2(\Omega), \sqrt{a} u_x \in L^2(\Omega) \text{ and } u(0) = u(1) = 0 \right\}, \tag{3.6}$$

$$H_a^1(\Omega) = V \cap S, \tag{3.7}$$

$$H_a^2(\Omega) = \left\{ u \in H_a^1(\Omega),\ a u_x \in H^1(\Omega) \right\}, \tag{3.8}$$

$$H_{\alpha,0}^1 = \{ u \in H_\alpha^1 \mid u(0) = u(1) = 0 \},$$

$$H^1_\alpha = \{u \in L^2(\Omega) \cap H^1_{Loc}(]0,1]) \mid x^{\frac{\alpha}{2}} u_x \in L^2(\Omega)\}.$$

With

$$\| u \|^2_{H^1_a(\Omega)} = \| u \|^2_{L^2(\Omega)} + \| \sqrt{a} u_x \|^2_{L^2(\Omega)},$$

$$\| u \|^2_{H^2_a(\Omega)} = \| u \|^2_{H^1_a(\Omega)} + \| (a u_x)_x \|^2_{L^2(\Omega)},$$

$$< u, v >_{H^1_\alpha} = \int_\Omega (uv + k(x) x^\alpha u_x v_x)\, dx.$$

We recall that (see [18]) $H^1_a$ is an Hilbert space and it is the closure of $C^\infty_c(0,1)$ for the norm $\| \,.\, \|_{H^1_a}$ . If $\frac{1}{\sqrt{a}} \in L^1(\Omega)$ then the following injections

$$H^1_a(\Omega) \hookrightarrow L^2(\Omega),$$

$$H^2_a(\Omega) \hookrightarrow H^1_a(\Omega),$$

$$H^1(0,T; L^2(\Omega)) \cap L^2(0,T; D(A)) \hookrightarrow L^2(0,T; H^1_a) \cap C(0,T; L^2(\Omega))$$

are compacts.

Firstly, we prove that the problem (3.1) is well-posed, the functional $J$ is continuous and G-derivable in $A_{ad}$.

The weak formulation of the problem (3.1) is :

$$\int_\Omega \partial_t u v\, dx + \int_\Omega \left( a(x) \partial_x u \partial_x v - \frac{\lambda}{x^\beta} uv \right)\, dx = \int_\Omega f v\, dx, \quad \forall v \in H^1_0(\Omega). \tag{3.9}$$

Let the bilinear form

$$\mathcal{B}[u,v] = \int_\Omega \left( a(x) \partial_x u \partial_x v - \frac{\lambda}{x^\beta} uv \right)\, dx. \tag{3.10}$$

Since $a(x) = 0$ at $x = 0$ and $\lim_{x\to 0} \dfrac{\lambda}{x^\beta} = +\infty$, the bilinear form $\mathcal{B}$ is noncoercive and is non-continuous at $x = 0$.

Consider the not bounded operator $(\mathcal{O}, D(\mathcal{O}))$ where

$$\mathcal{O} g = (k(x) x^\alpha g_x)_x + \frac{\lambda}{x^\beta} g, \quad \forall g \in D(\mathcal{O}) \tag{3.11}$$

and

$$D(\mathcal{O}) = \{g \in H^1_{\alpha,0} \cap H^2_{Loc}(]0,1]) \mid (x^\alpha g_x)_x + \frac{\lambda}{x^\beta} g \in L^2(\Omega)\}.$$

Let

$$A_{ad} = \{g \in H^1(\Omega); \|g\|_{H^1(\Omega)} \le r\}, \text{ where } r \text{ is a real strictly positive constant.} \tag{3.12}$$

We recall the following theorem (see [10],[30]) If f = 0 then for all $u \in L^2(\Omega)$, the problem (3.1) has a unique weak solution

$$u \in C([0,T]; L^2(\Omega)) \cap C(]0,T]; D(\mathcal{O})) \cap C^1(]0,T]; L^2(\Omega)) \tag{3.13}$$

if more $u_0 \in D(\mathcal{O})$ then

$$u \in C([0,T]; D(\mathcal{O})) \cap C^1([0,T]; L^2(\Omega)) \tag{3.14}$$

if $f \in L^2(]0,1[\times(0,T))$ then for all $u_0 \in L^2(\Omega)$, the problem (3.1) has a unique solution

$$u \in C([0,T]; L^2(\Omega)).\square \tag{3.15}$$

We recall the following theorem (see [35]) For every $u_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$, where $Q_T = ((0,T) \times \Omega)$ there exists a unique solution of problem (3.1). In particular, the operator $\mathcal{O} : D(\mathcal{O}) \longmapsto L^2(\Omega)$ is non positive and self-adjoint in $L^2(\Omega)$ and it generates an analytic contraction semigroup of angle $\pi/2$. Moreover, let $u_0 \in D(\mathcal{O})$; then
$f \in W^{1,1}(0,T; L^2(\Omega) \Rightarrow u \in C^0(0,T; D(\mathcal{O})) \cap C^1([0,T]; L^2(\Omega))$,
$f \in L^2(L^2(Q_T)) \Rightarrow u \in H^1(0,T; L^2(\Omega)$.

Let $u$ the weak solution of (3.1), the function

$$\begin{aligned} \varphi : H^1(\Omega) &\longrightarrow C([0,T]; L^2(\Omega)) \\ k &\longmapsto u \end{aligned} \tag{3.16}$$

is continuous, and the functional $J$ is continuous in $A_{ad}$. Therefore, the problem 3.2 has at least one solution in $A_{ad}$ .$\square$

**Proof of Theorem 3.2.** Let $\delta k \in H^1(\Omega)$ a small variation such that $k + \delta k \in A_{ad}$ and $u_0 \in D(\mathcal{O})$. Consider $\delta u = u^\delta - u$, with $u$ is the weak solution of (3.1) with diffusion coefficient $k$, and $u^\delta$ is the weak solution of the following problem (3.17) with diffusion coefficient $k^\delta = k + \delta k$.

$$\begin{cases} \partial_t u^\delta - \partial_x((k+\delta k)x^\alpha \partial_x u^\delta) - \dfrac{\lambda}{x^\beta} u^\delta = f(x,t) \in \Omega \times (0,T) \\ u^\delta \mid_{x=0} = u^\delta \mid_{x=1} = 0, \\ u^\delta(x,0) = u_0(x) \end{cases} \tag{3.17}$$

(3.17)-(3.1) give

$$\begin{cases} \partial_t(\delta u) - ((k+\delta k)x^\alpha \partial_x \delta u) - (\delta k x^\alpha \partial_x u) - \dfrac{\lambda}{x^\beta}\delta u = 0 \\ \delta u(0,t) = \delta u(1,t) = 0 \quad \forall t \in (0,T) \\ \delta u(x,0) = 0 \quad \forall x \in \Omega. \end{cases} \tag{3.18}$$

The weak formulation for (3.18) is

$$\int_\Omega \partial_t(\delta u) v \, dx + \int_\Omega \left( (k+\delta k)x^\alpha \partial_x(\delta u)\partial_x(v) - \frac{\lambda}{x^\beta}(\delta u) v \right) dx - \int_\Omega (\delta k x^\alpha \partial_x u)_x \, v \, dx = 0, \forall v \in H_0^1(\Omega). \tag{3.19}$$

Take $v = \delta u$, then

$$\int_\Omega \partial_t(\delta u)\delta u \; dx + \int_\Omega \left( (k+\delta k)x^\alpha (\partial_x \delta u)^2 - \frac{\lambda}{x^\beta}(\delta u)^2 \right) dx - \int_\Omega \left(\delta k x^\alpha \partial_x u\right)_x \delta u \; dx = 0. \tag{3.20}$$

We have

$$\int_\Omega \left( (k+\delta k)x^\alpha (\partial_x \delta u)^2 - \frac{\lambda}{x^\beta}(\delta u)^2 \right) dx \geq 0,$$

this implies that

$$\int_\Omega \partial_t(\delta u)\delta u \; dx - \int_\Omega \left(\delta k x^\alpha \partial_x u\right)_x \delta u \; dx \leq 0 \tag{3.21}$$

and consequently

$$\left| \int_\Omega \partial_t(\delta u)\delta u \; dx \right| \leq \int_\Omega \left| \left(\delta k x^\alpha \partial_x u\right)_x \delta u \right| \; dx, \tag{3.22}$$

then

$$\int_\Omega \partial_t(\delta u)\delta u \; dx \leq \|\delta k\|_{L^\infty(\Omega)} \int_\Omega |\partial_x u \partial_x \delta u| \; dx. \tag{3.23}$$

By integrating between 0 and $t$ with $t \in [0,T]$ we obtain

$$\frac{1}{2}\|\delta u(t)\|^2_{L^2(\Omega)} \leq \|\delta k\|_{L^\infty(\Omega)} \int_0^T \int_\Omega |\partial_x u \partial_x \delta u| \; dxdt, \tag{3.24}$$

since $u, \delta u \in H^1(0,T;L^2(\Omega))$, we have $\displaystyle\int_0^T \int_\Omega |\partial_x u \partial_x \delta u| \; dxdt < \infty$, there is $C > 0$, such that

$$\sup_{t\in[0,T]} \|\delta u(t)\|^2_{L^2(\Omega)} \leq 2C\|\delta k\|_{L^\infty(\Omega)}, \tag{3.25}$$

which give

$$\|\delta u(t)\|^2_{C([0,T];L^2(\Omega))} \leq 2C\|\delta k\|_{H^1(\Omega)}. \tag{3.26}$$

Hence, the functional $J$ is continuous in

$$A_{ad} = \{u \in H^1(\Omega); \|u\|_{H^1(\Omega)} \leq r\}. \tag{3.27}$$

We have $H^1(\Omega) \underset{compact}{\hookrightarrow} L^2(\Omega)$. Since the set $A_{ad}$ is bounded in $H^1(\Omega)$, then $A_{ad}$ is a compact in $L^2(\Omega)$. Therefore, $J$ has at least one minimum in $A_{ad}$

■

Now we compute the gradient of $J$ with the adjoint state method.

## 3.3 Gradient of $J$

We define the Gâteaux derivative of $u$ at $k$ in the direction $h \in L^2(\Omega\times]0,T[)$, by

$$\hat{u} = \lim_{s\to 0} \frac{u(k+sh) - u(k)}{s}, \tag{3.28}$$

$u(k+sh)$ is the weak solution of (3.1) with diffusion coefficient $k+sh$, and $u(k)$ is the weak solution of (3.1) with diffusion coefficient $k$.

The Gâteaux (directional) derivative of (3.1) at $k$ in some direction $h \in L^2(\Omega)$ gives

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} - \dfrac{\partial}{\partial x}\left(kx^\alpha \dfrac{\partial \hat{u}}{\partial x}\right) - \dfrac{\partial}{\partial x}\left(hx^\alpha \dfrac{\partial u}{\partial x}\right) - \dfrac{\lambda}{x^\beta}\hat{u} = 0 \\ \hat{u}(x=0,t) = \hat{u}(x=1,t) = 0, \\ \hat{u}(x,t=0) = 0. \end{cases}$$

We introduce the adjoint variable $P$, and we integrate,

$$\int_0^T \int_0^1 \left( \frac{\partial \hat{u}}{\partial t} p - \frac{\partial}{\partial x}\left(kx^\alpha \frac{\partial \hat{u}}{\partial x}\right) p - \frac{\partial}{\partial x}\left(hx^\alpha \frac{\partial u}{\partial x}\right) p \right) dxdt = 0.$$

Calculate separately each term :

$$\int_0^T \int_0^1 \frac{\partial \hat{u}}{\partial t} p = \int_0^1 [\hat{u}p]_0^T \, dx - \int_0^1 \int_0^T \frac{\partial p}{\partial t} \hat{u} dt dx,$$

$$\begin{aligned} \int_0^T \int_0^1 \frac{\partial}{\partial x}\left(kx^\alpha \frac{\partial \hat{u}}{\partial x}\right) p dxdt &= \int_0^T \left[\left(kx^\alpha \frac{\partial \hat{u}}{\partial x}\right) p\right]_0^1 dt - \int_0^T \int_0^1 kx^\alpha \frac{\partial \hat{u}}{\partial x}\frac{\partial p}{\partial x} dxdt \\ &= \int_0^T \left[\left(kx^\alpha \frac{\partial \hat{u}}{\partial x}\right) p\right]_0^1 dt - \int_0^T \left[kx^\alpha \hat{u} \frac{\partial p}{\partial x}\right]_0^1 dt + \int_0^T \int_0^1 \frac{\partial}{\partial x}\left(kx^\alpha \frac{\partial p}{\partial x}\right) \hat{u} dxdt \\ &= \int_0^T \left[\left(kx^\alpha \frac{\partial \hat{u}}{\partial x}\right) p\right]_0^1 dt + \int_0^T \int_0^1 \frac{\partial}{\partial x}\left(kx^\alpha \frac{\partial p}{\partial x}\right) \hat{u} dxdt. \end{aligned}$$

$$\int_0^T \int_0^1 p \left( \frac{\partial}{\partial x}\left(hx^\alpha \frac{\partial u}{\partial x}\right)\right) dxdt = \int_0^T \left[hx^\alpha \frac{\partial u}{\partial x} p\right]_0^1 dt - \int_0^T \int_0^1 h \frac{\partial u}{\partial x}\frac{\partial p}{\partial x} dxdt.$$

We pose $\quad p(x=1,t) = 0, \quad p(x=0,t) = 0, \quad p(x,t=T) = 0,$

we obtain

$$\begin{gathered}\int_0^T \langle \hat{u}, \partial_t P - AP\rangle_{L^2(\Omega)} dt = \langle h, \int_0^T x^\alpha \frac{\partial p}{\partial x}\frac{\partial u}{\partial x} dt\rangle_{L^2(\Omega)} \\ P(x=0) = P(x=1) = 0, \quad P(T) = 0.\end{gathered} \tag{3.29}$$

The discretization in time of (3.29), using the Rectangular integration method, gives

$$\begin{gathered}\sum_{j=0}^{M+1} \langle \hat{u}(t_j), \partial_t P(t_j) - AP(t_j)\rangle_{L^2(\Omega)} \Delta t = \langle h, \int_0^T x^\alpha \frac{\partial p}{\partial x}\frac{\partial u}{\partial x} dt\rangle_{L^2(\Omega)} \\ P(x=0) = P(x=1) = 0, \quad P(T) = 0.\end{gathered} \tag{3.30}$$

With

$$t_j = j\Delta t, \quad j \in \{0, 1, 2, \ldots, M+1\},$$

where $\Delta t$ is the step in time and $T = (M+1)\Delta t$.

The Gâteaux derivative of $J$ at $k$ in the direction $h \in L^2(\Omega)$ is given by

$$\hat{J}(h) = \lim_{s\to 0} \frac{J(k+sh) - J(k)}{s}.$$

After some computations, we arrive at

$$\hat{J}(h) = \langle u(T) - u_{obs}, \hat{u}(T)\rangle_{L^2(\Omega)}. \tag{3.31}$$

The adjoint model is

$$\begin{gathered}\partial_t P(T) - AP(T) = \frac{1}{\Delta t}(u(T) - u_{obs}), \quad \partial_t P(t_j) - AP(t_j) = 0 \quad \forall t_j \neq T \\ P(x=0) = P(x=1) = 0 \quad \forall t_j \in ]0; T[ \\ P(T) = 0.\end{gathered} \tag{3.32}$$

From equations (3.30), (3.31) and (3.32), the gradient of $J$ is given by

$$\frac{\partial J}{\partial k} = \int_0^T x^\alpha \frac{\partial p}{\partial x}\frac{\partial u}{\partial x} dt. \tag{3.33}$$

Problem (3.32) is retrograde, we make the change of variable $t \longleftrightarrow T - t$.

## 3.4 Numerical scheme

**Step 1.** Full discretization

Discrete approximations of these problems need to be made for numerical implementation. To resolve the direct problem and adjoint problem, we use the Method $\theta$-schema in time. This method is unconditionally stable for $1 > \theta \geq \frac{1}{2}$.

Let $h$ the steps in space and $\Delta t$ the steps in time.
Let

$$x_i = ih, \quad i \in \{0, 1, 2...N+1\},$$

$$c(x_i) = a(x_i),$$

$$t_j = j\Delta t, \quad j \in \{0, 1, 2...M+1\},$$

$$f_i^j = f(x_i, t_j),$$

we put

$$u_i^j = u(x_i, t_j). \tag{3.34}$$

Let

$$da(x_i) = \frac{c(x_{i+1}) - c(x_i)}{h}, \tag{3.35}$$

and

$$b(x) = -\frac{\lambda}{x^\beta}. \tag{3.36}$$

Therefore

$$\partial_t u + \mathcal{A}u = f \tag{3.37}$$

is approximated by

$$-\frac{\theta\Delta t}{h^2}c(x_i)u_{i-1}^{j+1} + \left(1 + \frac{2\theta\Delta t}{h^2}c(x_i) + da(x_i)\frac{\theta\Delta t}{h} + b(x_i)\theta\Delta t\right)u_i^{j+1} - (\frac{\theta\Delta t}{h^2}c(x_i) + da(x_i)\frac{\theta\Delta t}{h})u_{i+1}^{j+1}$$

$$= \frac{(1-\theta)\,\Delta t}{h^2}c(x_i)u_{i-1}^j + \left(1 - \frac{(1-\theta)\,\Delta t}{h}da(x_i) - \frac{2\,(1-\theta)\,\Delta t}{h^2}c(x_i) - (1-\theta)\,b(x_i)\Delta t\right)u_i^j$$

$$+ \left(\frac{(1-\theta)\,\Delta t}{h}da(x_i) + \frac{(1-\theta)\,\Delta t}{h^2}c(x_i)\right)u_{i+1}^j + \Delta t.[(1-\theta)\,f_i^j + \theta f_i^{j+1}]. \tag{3.38}$$

Let us define

$$g_1(x_i) = -\frac{\theta\Delta t}{h^2}c(x_i), \tag{3.39}$$

$$g_2(x_i) = 1 + \frac{2\theta\Delta t}{h^2}c(x_i) + da(x_i)\frac{\theta\Delta t}{h} + b(x_i)\theta\Delta t, \tag{3.40}$$

$$g_3(x_i) = -\frac{\theta\Delta t}{h^2}c(x_i) - da(x_i)\frac{\theta\Delta t}{h}, \tag{3.41}$$

$$k_1(x_i) = \frac{(1-\theta)\,\Delta t}{h^2}c(x_i), \tag{3.42}$$

$$k_2(x_i) = 1 - \frac{(1-\theta)\,\Delta t}{h}da(x_i) - \frac{2\,(1-\theta)\,\Delta t}{h^2}c(x_i) - (1-\theta)\,b(x_i)\Delta t, \tag{3.43}$$

$$k_3(x_i) = \frac{(1-\theta)\,\Delta t}{h}da(x_i) + \frac{(1-\theta)\,\Delta t}{h^2}c(x_i). \tag{3.44}$$

Let $u^j = \left(u_i^j\right)_{i\in\{1,2,..N\}}$, finally we get

$$\begin{cases} \mathcal{D}u^{j+1} = \mathcal{B}u^j + \mathcal{V}^j \quad \text{wher} \quad j \in \{1,2,..M\} \\ u^0 = (u_0(ih))_{i\in\{1,2,..N\}}, \end{cases} \tag{3.45}$$

where

$$\mathcal{D} = \begin{bmatrix} g_2(x_1) & g_3(x_1) & 0 & & & & & 0 \\ g_1(x_2) & g_2(x_2) & g_3(x_2) & 0 & & & & \\ 0 & g_1(x_3) & g_2(x_3) & g_3(x_3) & 0 & & & \\ & 0 & g_1(x_4) & g_2(x_4) & g_3(x_4) & 0 & & \\ & & 0 & . & . & . & 0 & \\ & & & . & . & . & . & 0 \\ & & & & 0 & g_1(x_{N-1}) & g_2(x_{N-1}) & g_3(x_{N-1}) \\ 0 & & & & & 0 & g_1(x_N) & g_2(x_N) \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} k_2(x_1) & k_3(x_1) & 0 & & & & & 0 \\ k_1(x_2) & k_2(x_2) & k_3(x_2) & 0 & & & & \\ 0 & k_1(x_3) & k_2(x_3) & k_3(x_3) & 0 & & & \\ & 0 & k_1(x_4) & k_2(x_4) & k_3(x_4) & 0 & & \\ & & 0 & . & . & . & 0 & \\ & & & . & . & . & . & 0 \\ & & & & 0 & k_1(x_{N-1}) & k_2(x_{N-1}) & k_3(x_{N-1}) \\ 0 & & & & & 0 & k_1(x_N) & k_2(x_N) \end{bmatrix}$$

$$\mathcal{V}^j = \begin{bmatrix} \Delta t.[(1-\theta)\, f(x_1,t_j) + \theta f(x_1, t_j + \Delta t)] \\ \Delta t.[(1-\theta)\, f(x_2,t_j) + \theta f(x_2, t_j + \Delta t)] \\ . \\ . \\ . \\ . \\ \Delta t.[(1-\theta)\, f(x_{N-1},t_j) + \theta f(x_{N-1}, t_j + \Delta t)] \\ \Delta t.[(1-\theta)\, f(x_N,t_j) + \theta f(x_N, t_j + \Delta t)] \end{bmatrix}$$

**Step 2.** Discretization of the functional $J$

$$J(\kappa) = \frac{1}{2}\int_0^1 (u(x,t=T) - u_{obs}(x))^2 dx. \tag{3.46}$$

We recall that the method of Thomas Simpson to calculate an integral is

$$\int_a^b f(x)\; dx \simeq \frac{h}{2}\left[ f(x_0) + 2\sum_{i=1}^{\frac{N+1}{2}-1} f(x_{2i}) + 4\sum_{i=1}^{\frac{N+1}{2}} f(x_{2i+1}) + f(x_{N+1})\right]$$

with $x_0 = a$, $x_{N+1} = b$, $x_i = a + ih$, $i \in \{1..N+1\}$.

Let the function

$$\varphi(x) = (u(x,T) - u_{obs}(x))^2 \quad \forall x \in \Omega. \tag{3.47}$$

We have

$$\int_0^1 \varphi(x)\, dx \simeq \frac{h}{2}\left[\varphi(0) + 2\sum_{i=1}^{\frac{N+1}{2}-1} \varphi(x_{2i}) + 4\sum_{i=1}^{\frac{N+1}{2}} \varphi(x_{2i+1}) + \varphi(1)\right].$$

Therefore

$$J(\kappa) \simeq \frac{h}{4}\left[\varphi(0) + 2\sum_{i=1}^{\frac{N+1}{2}-1} \varphi(x_{2i}) + 4\sum_{i=1}^{\frac{N+1}{2}} \varphi(x_{2i+1}) + \varphi(1)\right].\Box \tag{3.48}$$

## 3.5 Genetic algorithme and Hybrid methode

The Genetic Algorithms (GA) are adaptive search and optimization methods that are based on the genetic processes of biological organisms. Their principles have been first laid down by Holland. The aim of GA is to optimize a problem-defined function, called the fitness function. To do this, GA maintain a population of individuals (suitably represented candidate solutions) and evolve this population over time. At each iteration, called generation, the new population is created by the process of selecting individuals according to their level of fitness in the problem domain and breeding them together using operators borrowed from natural genetics, as, for instance, crossover and mutation. As the population evolves, the individuals in general tend toward the optimal solution. The basic structure of a GA is the following:

*1. Initialize a population of individuals;*
*2. Evaluate each individual in the population;*
*3. while termination criterion not reached do*
{
*4. Select individuals for the next population;*
*5. Apply genetic operators (crossover, mutation) to produce new individuals;*
*6. Evaluate the new individuals;*
}
*7. return the best individual.*

The hybrid methods combine principles from genetic algorithms and other optimization methods. In this approach, we will combine the genetic algorithm with method descent (steepest descent algorithm (FP)) as following:

We assume that we have a partial knowledge of background state $k^b$ at certain points $(x_i)_{i\in I}$, $I \subset \{1, 2, .., N+1\}$.

We assume the individual is a vector $k$, the population is a set of individuals.

The initialization of individual is as following

$$\begin{array}{l} \textit{for } i = 1 \textit{ to } N+1 \\ \quad \textit{if } i \in I \\ \qquad k(x_i) \textit{ is chosen in the vicinity of } k^b(x_i) \\ \quad \textit{else} \\ \qquad k(x_i) \textit{ is chosen randomly} \\ \quad \textit{end if} \\ \textit{end for} \end{array} \tag{3.49}$$

Starting by initial population, we apply genetic operators (crossover, mutation) to produce a new population in which each individual is an initial point for the descent method (FP). When a specific number of generations is reached without improvement of the best individual, only the fittest individuals (e.g. the first 10% fittest individuals in the population) survive. The remaining die and their place is occupied by new individuals with new genetic ( 45% are chosen randomly, the other 45% are chosen as (3.49)). At each generation we keep the best. The algorithm ends when $\mid J(k) \mid < \mu$ or $generation \geqslant Maxgen$, where $\mu$ is a given precision (see Figure 02).

The main steps for descent method (FP) at each iteration are:

- *Calculate* $u^j$ *solution of (3.1) with coefficient* $k^j$
- *Calculate* $P^j$ *solution of adjoint problem*
- *Calculate the descent direction* $d_j = -\nabla J(k^j)$
- *Find* $t_j = \underset{t>0}{argmin}\ J(k^j + td_j)$
- *Update the variable* $k^{j+1} = k^j + t_j d_j$.

The algorithm ends when $\mid J(k^j) \mid < \mu$, where $\mu$ is a given small precision.

$t_j$ is chosen by the inaccurate linear search by Rule Armijo-Goldstein as following:

*let* $\alpha_i, \beta \in [0, 1[$ *and* $\alpha > 0$

*if* $J(k^j + \alpha_i d_j) \leqslant J(k^j) + \beta\alpha_i d_j^T d_j$

$t_j = \alpha_i$ *and stop*

*if not*

$\alpha_i = \alpha\alpha_i$.

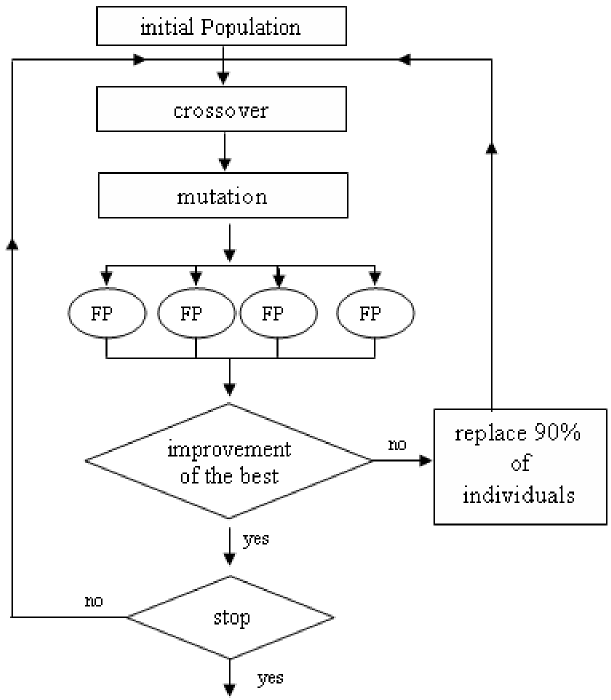


Figure 02. Hybrid Algorithm

## 3.6 Numerical experiments

In this section, we do three tests: In the first test, we recall the result obtained by the algorithm of simple descent with $\varepsilon = 10^{-5}$ (Fig 01), In the second test, we turn only the genetic algorithm (Fig 02). Finally, in the third test, we turn test with hybrid approach with parameters $\alpha = 1/3$, $\beta = 3/4$, $\lambda = -2/3$, $N = M = 99$, number of individuals $= 40$ , and number of generations $= 2000$.

In the figures below, the exact function is drawn in red and rebuild function in blue.

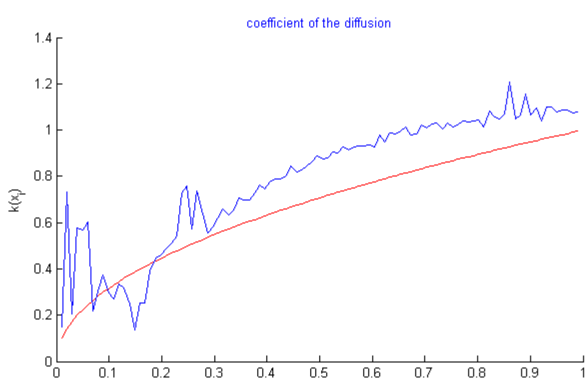


Figure 01. Test with simple descent and $\varepsilon = 10^{-5}$.

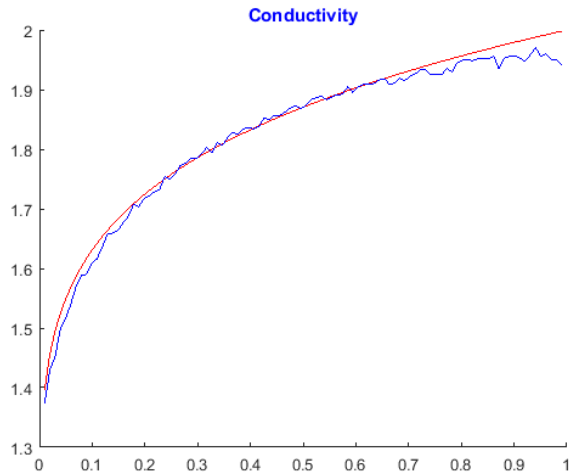


Figure 02. Test with genetic algorithm.

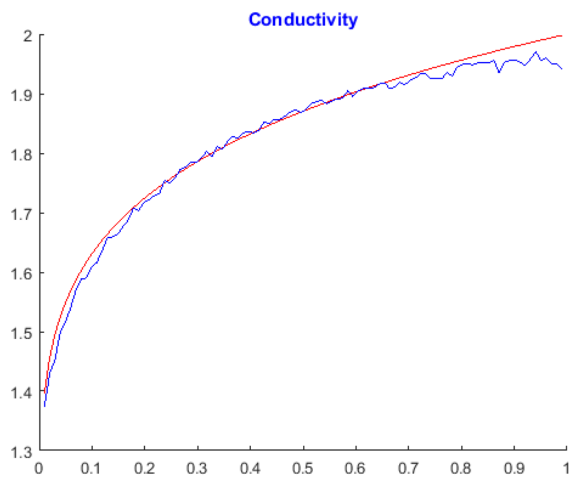


Figure 03. Test with hybrid algorithm.

These tests show that we can't rebuild the coefficient in the diffusion $k$ by the descent method and genetic approach (Fig. 1 and Fig. 2), and the hybrid approach proves effective to reconstruct this coefficient (Fig. 3).

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