ON THE EXISTENCE OF SOLUTION OF BOUNDARY VALUE PROBLEMS

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C O N T E N T S

Introduction

Theory of operator-differential equations in abstract spaces that takes its origin in the papers of K. Yosida, E. Hille and R. Fillips, T. Kato and others, appeared as application of the methods of functional analysis in the theory of partial differential equations.

Interest to this problem is stipulated by the fact that it allows to investigate both systems of ordinary differential equations, integro-differential equations, quasidifferential equations and others from a unique point of view. It should be noted that the questions on solvability of operator-differential equations and boundary value problems for them are of independent scientific interest. Some significant results of the theory of operator- differential equation are cited in the books of S.G. Krein, A.A. Dezin, J.L. Lions and E. Majenes, V.I. Gorbachuk and M.L. Gorbachuk, S.A. Yakubov and others.

Interest to the investigations of solvability of the Cauchy problem and boundary value problems for operator-differential equations and also the increased amount of papers devoted to this theme proceed from the fact that these questions are closely mixed up with the problems of spectral theory of not selfadjoint operators and operator pencils that at the present time are one of developing sections of functional analysis. The beginning of devolepment of these theories is the known paper of the academician M.V. Keldysh. In this paper M.V. Keldysh introduced the notion of multiple completeness of eigen and adjoint vectors for a wide class of operator bundless, and also showed how the notion of n-fold completeness of eigen and adjoint vectors of operator pencil is associated with appropriate Cauchy problem. After this, there appears a great deal of papers in which significant theorems on multiple completeness, on busicity of a system of eigen and adjoint vectors and on multiple expansion in this system were obtained for different classes of operator bundless. Many problems of mechanics and mathematical physics are related to investigations of completeness of some part of eigen and adjoint vectors of operator pencils. A great deal of papers was devoted to these problems. There are some methods for solving these problems and one of them is the consideration of appropriate boundary value problems on semi-axis, that arises while investigating completeness of eigen and adjoint vectors responding to eigen values from the left half-plane. This method was suggested by academican M.G. Gasymov. He showed relation of completeness of a part of eigen and adjoint vectors and solvability a boundary value problem on a semi-axis with some analytic properties of a resolvent of an operator pencil, that was developed in the S.S. Mirzoevs paper.

The suggested book is devoted to similar questions of solvability of operator-differential equations of higher order and boundary value problems for them, to investigation of spectral properties of appropriate operator pencils. A theorem of Phragmen-Lindeloff type in some vector is proved. The principal part of the investigated operator-differential equations have multiple characteristics.

Chapter I

Many problems of mechanics and mathematical physics are connected with the investigation of solvability of operator differential equations. As an example, we can show the following papers.

As is known, stress-strain state of a plate may be separated into internal and external layers [1-4]. Construction of a boundary layer is related with sequential solution of plane problems of elasticity theory in a semi-strip. In Papkovich's paper [5] and in others a boundary value problem of elasticity theory in a semi-strip $x > 0, |y| \leq 1$ is reduced to the definition of Airy biharmonic functions that is found in the form

$$u = \sum_{\mathrm{Im}\ \sigma_k > 0} C_k \varphi_k(y) e^{i\sigma_k x},$$

where φ_k are Papkovich functions [5-6], σ_k are corresponding values of a self-adjoint boundary value problem, C_k are unknown coefficients. In this connection in [6] there is a problem on representation of a pair of functions f_1 and f_2 in the form

$$\sum_{k=1}^{\infty} C_k P_k \varphi_k = f_1, \quad \sum_{k=1}^{\infty} C_k Q_k \varphi_k = f_2, \tag{1}$$

where P_k, Q_k are differential operators defined by boundary conditions for $x = 0$. In the paper [7,8] some sufficient conditions of convergence of expansion (1) is given for the cases when the coefficients C_k are obviously defined with the help of generalized orthogonality.

In [9] the coefficients C_k are uniquely defined by the boundary values of a biharmonic function and its derivatives. The trace problem for a two-dimensional domain with piecewise smooth boundary was studied in the paper [10]. The paper [11] deals with differential properties of solutions of general elliptic equations in the domains with canonical and corner points. Some new results for a biharmonic equation are in [12]. Investigation of behaviour of solution of problems of elasticity theory in the vicinity of singular points of the boundary is in the papers [13-14]. M.B. Orazov [15] and S.S. Mirzoyev [16] studied the problem when a principal part of the equation is of the form: $(-1)^m \frac{d^{2m}}{dt^{2m}} + A^{2m}$

where A is a self-adjoint operator pencil and it has a multiple characteristics, that differs it from above-mentioned papers.

§ 1.1. On generalized solution of a class of higher order operator–differential equations

In this section the sufficient conditions on the existence and uniqueness of generalized solution on the axis are obtained for higher order operator - differential equations the main part of which has the multi characteristic.

1.1. Some definition and auxiliary facts

Let H be a separable Hilbert space, A be a positive – definite self-adjoint operator in H with domain of definition $D(A)$. Denote by H_γ a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma), (\gamma \geq 0), (x,y)_\gamma = (A^\gamma x, A^\gamma y), x, y \in D(A^\gamma)$.

We denote by $L_2((a,b);H_\gamma)$ $(-\infty \leq a < b \leq +\infty)$ a Hilbert space of vector-functions $f(t)$ determined in (a,b) almost everywhere with values from H measurable, square integrable in the Bochner's sense

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f\|_\gamma^2 \, dt \right)^{1/2}.$$

Assume

$$L_2((-\infty,+\infty);H) \equiv L_2(R;H).$$

Further, we define a Hilbert space for natural $m \geq 1$ [17].

$$W_2^m((a;b);H) = \left\{ u \left| u^{(m)} \in L_2((a;b);H), \ A^m u \in L_2((a,b);H_m) \right. \right\}$$

with norm

$$\|u\|_{W_2^m((a,b);H)} = \left(\left\| u^{(m)} \right\|_{L_2((a,b);H)}^2 + \|A^m u\|_{L_2((a,b);H)}^2 \right)^{1/2}.$$

Here and in sequel the derivatives are understood in the sense of distributions theory [17]. Here we assume

$$W_2^m((-\infty,+\infty);H) \equiv W_2^m(R;H).$$

We denote by $D(R;H)$a set of infinitely-differentiable functions with values in H.

In the space H we consider the operator – differential equation

$$P\left(\frac{d}{dt}\right)u(t)\equiv\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t)+\sum_{j=0}^{2m}A_j u^{(2m-j)}(t)=f(t)\,,t\in R=(-\infty,+\infty)\,, \tag{2}$$

where $f(t)$ and $u(t)$ are vector-valued functions from H, and the coefficients A and A_j $\left(j=\overline{0,2m}\right)$ satisfy the following conditions:

1) A is a positive-definite self-adjoint operator in H ;

2) the operators A_j $\left(j=\overline{0,2m}\right)$ are linear in H.

In the paper we'll give definition of generalized solution of equation (1) and prove a theorem on the existence and uniqueness of generalized solution (1). Notice that another definition of generalized solution of operator - differential equations and their existence is given in the book [18], in the paper when $m=2$ on the semi-axis $R_+=(0,+\infty)$ was studied by the author [19].

First of all we consider some facts that we'll need in future. Denote

$$P_0\left(\frac{d}{dt}\right)u(t)=\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t)\,,\quad u(t)\in D(R;H)$$

and

$$P_1\left(\frac{d}{dt}\right)u(t)=\sum_{j=0}^{2m}A_j u^{(2m-j)}(t)\,,\qquad u(t)\in D(R;H)\,.$$

Now let's formulate a lemma that shows the conditions on operator coefficients (1) under which the solution of the equation from the class $W_2^m(R;H)$ has sense.

Lemma 1.1. *Let conditions 1) and 2) be fulfilled, moreover , $B_j=A_j\times$ $\times A^{-j}$ $\left(j=\overline{0,m}\right)$ and $D_j=A^{-m}A_jA^{m-j}$ $\left(j=\overline{m+1,2m}\right)$ be bounded in H. Then a bilinear functional $L(u,\psi)\equiv\left(P_1(d/dt)\,u,\psi\right)_{L_2(R;H)}$ determined for all vector-functions $u\in D(R;H)$ and $\psi\in D(R;H)$ continuous on the* *space $W_2^m(R;H)\oplus W_2^m(R;H)$ that acts in the following way*

$$L(u,\psi)=(P_1(d/dt)u,\psi)_{L_2(R;H)}=\sum_{j=0}^{2m}\left(A_ju^{(2m-j)},\psi\right)_{L_2(R;H)}=$$

$$=\sum_{j=0}^{2m}(-1)^m\left(A_ju^{(m-j)},\psi^m\right)_{L_2(R;H)}+\sum_{j=m+1}^{2m}\left(A_ju^{(2m-j)},\psi\right)_{L_2(R;H)}.$$

Proof. Let $u\in D(R;H)$, $\psi\in D(R;H)$. Then integrating by parts we get

$$L(u,\psi)=(P_1(d/dt)u,\psi)_{L_2(R;H)}=\sum_{j=0}^{2m}\left(A_ju^{(2m-j)},\psi\right)_{L_2(R;H)}=$$

$$=\sum_{j=0}^{m}(-1)^m\left(A_ju^{(m-j)},\psi^m\right)_{L_2(R;H)}+\sum_{j=m+1}^{2m}\left(A_ju^{(2m-j)},\psi^{(m)}\right)_{L_2(R;H)}. \qquad (3)$$

On the other hand , for $j=\overline{0,m}$ we apply the intermediate derivatives theorem [17] and get

$$\left|\left(A_ju^{(m-j)},\psi^{(m)}\right)_{L_2(R;H)}\right|=\left|\left(B_jA_ju^{(m-j)},\psi^m\right)_{L_2(R;H)}\right|\leq$$

$$\leq\|B_j\|\cdot\left\|A^ju^{(m-j)}\right\|_{L_2(R;H)}\cdot\left\|\psi^{(m)}\right\|_{L_2(R;H)}\leq C_{m-j}\|D_j\|\cdot\|u\|\cdot\|\psi\|_{W_2^m(R;H)}. \qquad (4)$$

And for $j=\overline{m+1,2m}$ we again use the theorem on intermediate derivatives [17] and get

$$\left|\left(A_ju^{(2m-j)},\psi^m\right)_{L_2(R;H)}\right|=\left|D_j\left(A^{m-j}u^{(2m-j)},A^m\psi\right)_{L_2(R;H)}\right|\leq$$

$$\leq\|D_j\|\left\|A^{m-j}u^{(2m-j)}\right\|_{L_2(R;H)}\cdot\|A^m\psi\|_{L_2(R;H)}\leq$$

$$\leq C_{2m-j}\|D_j\|\cdot\|u\|_{W_2^m(R;H)}\|\psi\|_{W_2^m(R;H)}. \qquad (5)$$

Since the set $D(R;H)$ in dense in the space $W_2^m(R;H)$, allowing for inequality (4) and (5) in (3) we get that the inequality

$$\|L(u,\psi)\|\leq const\,\|u\|_{W_2^m(R;H)}\cdot\|\psi\|_{W_2^m(R;H)}$$

is true for all $u,\varphi \in W_2^m(R;H)$, i.e. $L(u,\psi)$ continues by continuity up to a bilinear functional acting on the spaces $W_2^m(R;H) \oplus W_2^m(R;H)$. We denote this functional by $L(u,\psi)$ as well. The lemma is proved.

Definition 1.1. *The vector function $u(t) \in W_2^m(R;H)$ is said to be a generalized solution of (1) if for any vector-function $\psi(t) \in W_2^m(R;H)$ it holds the identity*

$$(u,\psi)_{W_2^m(R;H)} + \sum_{k=1}^{2m-1} C_{2m}^k \left(A^{m-k}u^{(k)}, A^{m-k}\psi^{(k)}\right)_{L_2(R;H)} = (f,\psi)_{L_2(R;H)}, \qquad (6)$$

where $C_{2m}^k = \frac{2m(2m-1)...(2m-k+1)}{k!}$.

To find the solvability conditions of equation (2) we prove the following Lemma by using the method of the paper [16].

Lemma 1.2. *For any $u(t) \in W_2^m(R;H)$ there hold the following estimates*

$$\left\| A^{m-j}u^{(j)} \right\|_{L_2(R;H)} \le d_{m,j}^{m/2} \, |||u|||_{W_2^m(R;H)}, \quad \left(j = \overline{0,m}\right), \qquad (7)$$

where

$$|||u|||_{W_2^m(R;H)} = \left(\|u\|^2_{W_2^m(R;H_m)} + \sum_{k=1}^{2m-1} c_{2m}^k \left|\left|\left| A^{m-k}u^{(k)} \right|\right|\right|^2_{L_2(R;H)} \right)^{1/2},$$

and the numbers from inequalities (7) are determined as follows

$$d_{m,j} = \begin{cases} \left(\frac{j}{m}\right)^{\frac{j}{m}} \cdot \left(\frac{m-j}{m}\right)^{\frac{m-j}{m}}, & j = \overline{1,m-1} \\ 1, & j = 0, m \end{cases}$$

Proof. Obviously, the norm $|||u|||_{W_2^m(R;H)}$ is equivalent to the norm $\|u\|_{W_2^m(R;H)}$. Then it follows from the intermediate derivatives theorem that the final numbers

$$b_j = \sup_{0 \ne u \in W_2^m(R;H)} \left\| A^{m-j}u^{(j)} \right\|_{L_2(R;H)} \cdot \|u\|^{-1}_{W_2^m(R;H)}, \quad j = \overline{0,m}.$$

Show that $b_j = d_{m,j}^{m/2}$, $j = \overline{0,m}$. Then $u(t) \in D(R;H)$.

For all $\beta \in [0, b_j^{-2})$, where

$$b_j = \sup_{\xi \in R} \left| \xi^j \left(\xi^2 + 1\right)^{-m/2} \right| = d_{m,j}^{m/2},$$

we use the Plancherel theorem and get

$$|||u|||_{W_2^m(R;H)} - \beta \left\| A^{m-j} u^{(j)} \right\|^2_{L_2(R;H)} = \sum_{k=0}^{2m} C_{2m}^k \left\| A^{m-k} \xi^k \hat{u}(\xi) \right\|^2_{L_2(R;H)} -$$

$$-\beta \left\| A^{m-j} \xi^j \hat{u}(\xi) \right\|^2_{L_2(R;H)} = \sum_{k=0}^{2m} C_{2m}^k \left(A^{m-k} \xi^k \hat{u}(\xi), A^{m-k} \xi^k \hat{u}(\xi) \right)_{L_2(R;H)} -$$

$$-\beta \left(A^{m-j} \xi^j \hat{u}(\xi), A^{m-j} \xi^j \hat{u}(\xi) \right)_{L_2(R;H)} =$$

$$= \int_{-\infty}^{+\infty} \left(\left[(\xi^2 E + A^2)^m - \beta \xi^{2j} A^{(2m-j)} \right] \hat{u}(\xi), \hat{u}(\xi) \right)_{L_2(R;H)} d\xi, \qquad (8)$$

where $\hat{u}(\xi)$ is a Fourier transformation of the vector-function $u(t)$. Since for $\beta \in [0, b_j^{-2})$ it follows from the spectral expansion of the operator A that

$$\left(\left(\left(\xi^2 E + A^2 \right)^m - \beta \xi^{2j} A^{2(m-j)} \right) x, x \right) = \int_{-\infty}^{+\infty} \left(\left(\xi^2 + \mu^2 \right)^m - \beta \xi^{2j} \mu^{2(m-j)} \right) (dE_\mu x, x) =$$

$$= \int_{\mu_0}^{\infty} \left(1 - \beta \frac{\xi^{2j} \mu^{2(m-j)}}{(\xi^2 + \mu^2)^\mu} \right) \left(\xi^2 + \mu^2 \right) (dE_\mu x, x) \geq \int_{\mu_0}^{\infty} \left(1 - \beta b_j^2 \right) \left(\xi^2 + \mu^2 \right) (dE_\mu x, x),$$

then equality (7) yields

$$|||u|||^2_{W_2^m(R;H)} \geq \beta \left\| A^{m-j} u^{(j)} \right\|^2_{L_2(R;H)}, \qquad (9)$$

for all $\beta \in [0, b_j^{-2}$ and $u(t) \in D(R;H)$. Passing to the limit as $\beta \to b_j^{-2}$ we get

$$|||u|||^2_{W_2^m(R;H)} \geq d_{m,j}^{-m/2} \left\| A^{m-j} u^{(j)} \right\|^2_{L_2(R;H)}.$$

Hence it follows

$$\left\| A^{m-j} u^{(j)} \right\|^2_{L_2(R;H)} \leq d_{m,j}^{m/2} |||u|||^2_{W_2^m(R;H)}, \quad \left(j = \overline{0, m} \right). \qquad (10)$$

Show that inequalities (10) are exact. To this end, for the given $\varepsilon > 0$ we show the existence of the vector-function $u_\varepsilon(t) \in W_2^m(R;H)$, such that

$$E(u_\varepsilon) = |||u|||^2_{W_2^m(R;H)} - \left(d^{-m}_{m,j} + \varepsilon\right) \left\|A^{m-j}u^{(j)}\right\|^2_{L_2(R;H)} < 0. \qquad (11)$$

We'll look for $u_\varepsilon(t)$ in the form $u_\varepsilon(t) = g_\varepsilon(t)\varphi_\varepsilon(t)$, where $g_\varepsilon(t)$ is a scalar function from the space $W_2^m(R)$ and $\varphi_\varepsilon \in H_{2m}$, where $\|\varphi_\varepsilon\| = 1$. Using the Plancherel theorem, we write $E(u_\varepsilon)$ in the equivalent form

$$E(u_\varepsilon) = \int\limits_{-\infty}^{+\infty} \left(\left(\left(\xi^2 E + A^2\right)^m - \left(d^{-m}_{m,j} + \varepsilon\right)\xi^{2j}A^{2m-2j}\right)\varphi_\varepsilon, \varphi_\varepsilon\right) |\hat{g}_\varepsilon(\xi)|^2 \, d\xi.$$

Note that $\hat{u}(\xi)$ and $\hat{g}_\varepsilon(\xi)$ are the Fourier transformations of the vector-functions $u(t)$ and $g_\varepsilon(t)$, respectively.

Notice that if A has even if one even value μ, then for φ_ε we can choose appropriate eigen-vector $\varphi_\varepsilon = \varphi\,(\|\varphi\| = 1)$. Indeed, then at the point $\xi_0 = (j/m)^{1/2} \cdot \mu$

$$\left(\left(\left(\xi_0^2 E + A^2\right)^m - \left(d^{-m}_{m,j} + \varepsilon\right)\xi_0^{2j}A^{2m-2j}\right)\varphi_\varepsilon, \varphi_\varepsilon\right) = \left(\xi_0^2 + \mu^2\right)^m -$$

$$- \left(d^{-m}_{m,j} + \varepsilon\right)\xi_0^{2j}\mu^{2m-2j} = \left(\xi_0^2 + \mu^2\right)^m \left[1 - \left(d^{-m}_{m,j} + \varepsilon\right)\frac{\xi_0^{2j}\mu^{2m-2j}}{\left(\xi^2 + \mu^2\right)^m}\right] < 0. \qquad (12)$$

If the operator A has no eigen-value, for $\mu \in \sigma(A)$ and for any $\delta > 0$ we can construct a vector φ_δ, $\|\varphi_\delta\| = 1$, such that

$$A^\delta \varphi_\delta = \mu^m \varphi_\delta + 0(1,\delta), \quad \delta \to 0, \quad m = 1, 2, ...$$

In this case, and for $\xi_0 = (j/m)^{1/2}\mu$

$$\left(\left(\left(\xi^2 E + A^2\right)^m - \left(d^{-m}_{m,j} + \varepsilon\right)\xi_0^{2j}A^{2m-2j}\right)\varphi_\delta, \varphi_\delta\right) =$$

$$= \left(\left(\xi_0^2 + \mu^2\right) - \left(d^{-m}_{m,j} + \varepsilon\right)\xi^{2j}\mu^{2m-2j}\right) + 0(1,\delta).$$

Thus, for small $\delta > 0$ it holds inequality (12). Consequently, for any $\varepsilon > 0$ there will be found a vector φ_ε $(\|\varphi_\varepsilon\| = 1)$, for which

$$\left(\left(\left(\xi^2 E + A^2\right)^m - \left(d^{-m}_{m,j} + \varepsilon\right)\xi_0^{2j}A^{2m-2j}\right)\varphi_\varepsilon, \varphi_\varepsilon\right) < 0 \qquad (13)$$

Now for $\xi = \xi_0$ we construct $g_\varepsilon(t)$. Since the left hand side of inequality (13) is a continuous function from the argument ξ, it is true at some vicinity of the point ξ_0. Assume that inequality (13) holds in the vicinity (η_1, η_2). Then we construct $\hat{g}(\xi)$-infinitely-differentiable function of argument ξ with support in (η_1, η_2) and denote it by

$$g_\varepsilon(t) = \frac{1}{\sqrt{2\pi}} \int_{\eta_1}^{\eta_2} \hat{g}(\xi)\, e^{i\xi t} d\xi.$$

It follows from the Paley-Wiener theorem that $g_\varepsilon(t) \in W_2^m(R)$ and obviously

$$E(u_\varepsilon) = E(g_e, \varphi_\varepsilon) =$$

$$= \int_{\eta_1}^{\eta_2} \left(\left(\left(\xi^2 E + A^2 \right)^m - \left(d_{m,j}^{-m} + \varepsilon \right) \xi^{2j} A^{2m-2j} \right) \varphi_\varepsilon, \varphi_\varepsilon \right) \left| \hat{g}_\varepsilon(\xi) \right|^2 d\xi < 0,$$

i.e. inequalities (10) are exact. The lemma is proved.

1.2. The basic result

Now let's prove the main theorem.

Theorem 2.1. *Let A be a positive-definite self-adjoint operator in H, the operators $B_j = A_j \cdot A^{-j}$ $\left(j = \overline{0, m}\right)$ and $D_j = A^{-m} A_j A^{m-j}$ $\left(j = \overline{m+1, 2m}\right)$ be bounded in H and it hold the inequality*

$$\gamma = \sum_{j=0}^{m} d_{m,j}^{m/2} \|B_j\| + \sum_{j=m+1}^{2m} d_{m,2m-j}^{m/2} \|D_j\| < 1, \tag{14}$$

where the numbers $d_{m,j}$ are determined from lemma 1.2.

Then equation (2) has a unique generalized solution and the inequality

$$\|u\|_{W_2^m(R;H)} \leq const \, \|f\|_{L_2(R;H)}$$

holds.

Proof. Show that for $\gamma < 1$ for all vector-functions $\psi \in W_2^m(R;H)$ it holds the inequality

$$\left(P(d/dt)\,\psi, \psi\right)_{L_2(R;H)} \equiv$$

$$\equiv \|\psi\|^2_{W_2^m(R;H)} + \sum_{k=1}^{2m} C_{2m}^k \left\|A^{m-k}\psi^{(k)}\right\|^2_{L_2(R;H)} + L(\psi,\psi) \geq C\|\psi\|^2_{W_2^m(R;H)},$$

where $C > 0$ is a constant number.

Obviously,

$$\left|(P(d/dt)\psi,\psi)_{L_2(R;H)}\right| \geq |||\psi|||^2_{W_2^m(R;H)} - |L(\psi,\psi)|. \tag{15}$$

Since

$$|L(\psi,\psi)| < \sum_{j=0}^{m} \left|\left(A_j\psi^{(m-j)},\psi^{(m)}\right)_{L_2(R;H)}\right| + \sum_{j=m+1}^{2m} \left|\left(A_j\psi^{(2m-j)},\psi\right)_{L_2(R;H)}\right|,$$

then we use lemma 1.2 and get

$$|L(\psi,\psi)| \leq \left(\sum_{j=0}^{m} \|B_j\|\, d_{m,j}^{m/2} + \sum_{j=m+1}^{2m} \|D_j\|\, d_{m,2m-j}^{m/2}\right) |||\psi|||^2_{W_2^m(R;H)} =$$

$$= \gamma\|\psi\|_{W_2^m(R;H)}. \tag{16}$$

Allowing for inequality (16) in (15) , we get

$$\left|(P(d/dt)\psi,\psi)_{L_2(R;H)}\right| \geq (1-\gamma)\,|||\psi|||^2_{W_2^m(R;H)}. \tag{17}$$

Further, we consider the problem

$$P_0(d/dt)\,u(t) = f(t),$$

where $f(t) \in L_2(R;H)$. It is easy to see that the vector - function

$$u_0(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\xi^2 + A^2\right)^m \int_{-\infty}^{+\infty} f(s)\, e^{-2(s-\xi)} ds d\xi \tag{18}$$

belongs to the space $W_2^m(R;H)$ and satisfies the condition

$$(u_0,\psi) = (f,\psi).$$

Now we'll look for the generalized solution of equation (1) in the form $u = u_0 + \xi_0$, where $\xi_0 \in W_2^m(R;H)$. Putting this expression into equality (5), we get

$$\left(P\left(d/dt\right)u, \psi\right)_{L_2(R;H)} = -L\left(u_0, \psi\right), \quad \psi \in W_2^m\left(R;H\right). \tag{19}$$

The right hand-side is a continuous functional in $W_2^m(R;H)$, the left hand-side satisfies Lax-Milgram [20] theorem's conditions by inequality (17). Therefore, there exists a unique vector - function $u(t) \in W_2^m(R;H)$ satisfying equality (19). On the other hand , for $\psi = u$ it follows from inequality (17) that

$$\left|\left(P\left(d/dt\right)u, u\right)_{L_2(R;H)}\right| = \left|\left(f, u\right)_{L_2(R;H)}\right| \geq C\,|||u|||^2_{W_2^m(R;H)} \geq C\,\|u\|^2_{W_2^m(R;H)},$$

then hence it follows

$$\|u\|_{W_2^m(R;H)} \leq const\,\|f\|_{L_2(R;H)}.$$

The theorem is proved.

§ 1.2. On the existence of solutions of boundary value problems for a class of higher order operator-differential equations

In this section we give the sufficient conditions on the existence and uniqueness of generalized solutions of boundary value problems of one class of higher order operator-differential equations at which the equation describes the process of corrosive fracture of metals in aggressive media and the principal part of the equation has a multiple characteristic.

2.1. Problem statement

Let H be a separable Hilbert space, A be a positive definite self-adjoint operator in H with domain of definition $D(A)$. By H_γ we denote a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(\gamma \geq 0)$, $(x, y)_\gamma = = (A^\gamma x, A^\gamma y)\,, x, y \in D(A^\gamma)$. By $L_2\left((a,b);H\right)$ $(-\infty \leq a < b \leq \infty)$ we denote a Hilbert space of vector-functions $f(t)$, determined in (a,b) almost everywhere with values in H, measurable, square integrable in Bochner sense

$$\|f\|_{L_2((a,b);H)} = \left(\int\limits_a^b \|f\|_\gamma^2 \, dt\right)^{1/2} .$$

Then we determine a Hilbert space for natural $m \geq 1$ [17]

$$W_2^m\left((a,b);H\right) = \left\{u/u^{(m)} \in L_2\left((a,b);H\right), A^m u \in L_2\left((a,b);H_m\right)\right\}$$

with norm

$$\|u\|_{W_2^m((a,b);H)} = \left(\left\|u^{(m)}\right\|^2_{L_2((a,b);H)} + \|A^m u\|^2_{L_2((a,b);H)}\right)^{1/2} .$$

Here and in sequel, derivatives are understood in the distributions theory sense [17]. Assume

$$L_2\left((0,\infty);H\right) \equiv L_2(R_+;H),\ L_2\left((-\infty,\infty);H\right) \equiv L_2(R;H),$$

$$W_2^m\left((0,\infty);H\right) \equiv W_2^m\left(R_+;H\right),\ W_2^m\left((-\infty,+\infty);H\right) \equiv W_2^m\left(R;H\right).$$

Then we determine the spaces

$$W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)=\left\{u|u\in W_2^m\left(R_+;H\right),u^{(\nu)}(0)=0,\ \nu=\overline{0,m-1}\right\}.$$

Obviously, by the trace theorem [17] the space $W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)$ is a closed subspace of the Hilbert space $W_2^m\left(R_+;H\right)$.

Let's define a space of $D\left([a,b];H_\gamma\right)$-times infinitely differentiable functions for $a\le t\le b$ with values in H_γ having a compact support in $[a,b]$. As is known a linear set $D\left([a,b];H_\gamma\right)$ is everywhere dense in the space $W_2^m\left((a,b);H\right)$, [17].

It follows from the trace theorem that the space

$$D\left(R_+;H_m;\{\nu\}_{\nu=0}^{m-1}\right)=\left\{u|u\in D\left(R_+;H_m\right),u^{(\nu)}(0)=0,\ \nu=\overline{0,m-1}\right\}$$

and also everywhere is dense in the space.

In the Hilbert space H we consider the boundary value problem

$$\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t)+\sum_{j=1}^{m}A_j u^{(m-j)}(t)=0,\quad t\in R_+=(0,+\infty),\tag{20}$$

$$u^{(\nu)}(0)=\varphi_\nu,\quad \nu=\overline{0,m-1},\ \varphi_\nu\in H_{m-\nu-1/2}.\tag{21}$$

Here we assume that the following conditions are fulfilled:

1) A is a positive-definite self-adjoint operator with completely continuous inverse $C=A^{-1}\in\sigma_\infty$;

2) The operators

$$B_j=A^{-j/2}A_jA^{-j/2}\ \left(j=2k,\ k=\overline{1,m}\right)$$

and

$$B_j=A^{-(j-1)/2}A_jA^{-(j-1)/2}\quad\left(j=2k-1,\ k=\overline{1,m-1}\right);$$

3) The operators $(B+E_m)$ are bounded in H.

Equation (20) descries a process of corrosion fracture in aggressive media that was studied in the paper [21].

2.2. Some definition and auxiliary facts

Denote

$$P_0\left(\frac{d}{dt}\right)u(t)\equiv\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t),\ \ u(t)\in D\left(R_+;H_m\right), \tag{22}$$

$$P_1\left(\frac{d}{dt}\right)u(t)\equiv\sum_{j=1}^{m-1}A_j u^{(m-j)}(t),\ \ u(t)\in D\left(R_+;H_m\right), \tag{23}$$

Lemma 2.1. *Let A be a positive-definite self-adjoint operator, the operators $B_j = A^{-j/2}A_jA^{-j/2}$ $\left(j=2k, k=\overline{1,m}\right)$ and $B_j = A^{-(j-1)/2}A_jA^{-(j-1)/2}$ $\left(j=2k-1, k=\overline{1,m-1}\right)$ be bounded in H. Then a bilinear functional*

$$P_1(u,\psi)\equiv\left(P_1(d/dt)u,\psi\right)_{L_2(R_+;H)}$$

determined for all vector-functions $u\in D\left(R_+;H_m\right)$ and $\psi\in D\left(R_+;H_m;\{\nu\}_{\nu=0}^{m-1}\right)$ continues by continuity on the space $W_2^m\left(R_+;H\right)\oplus W_2^m\left(R_+;H_m;\{\nu\}_{\nu=0}^{m-1}\right)$ up to bilinear functional $P_1(u,\psi)$ acting in the following way

$$P_1(u,\psi)=\sum_{(j=2k)}(-1)^{m-j/2}\left(A_ju^{(m-j/2)},\psi^{(m-j/2)}\right)_{L_2}+$$

$$+\sum_{j=(2k-1)}(-1)^{m-(j+1)/2}\left(A_ju^{(m-(j-1)/2)},\psi^{(m-(j-1)/2)}\right)_{L_2}. \tag{24}$$

Here in the first term, the summation is taken over even j, in the second term over odd j.

Proof. Let $u\in D(R_+;H_m),\psi\in D\left(R_+;H_m;\{\nu\}_{\nu=0}^{m-1}\right)$. After integration by parts we have

$$P_1(u,\psi)_{L_2}\equiv\left(P_1(d/dt)u,\psi\right)_{L_2}=\sum_{j=0}^{m}\left(A_ju^{(m-j)},\psi\right)_{L_2}=$$

$$=\sum_{(j=2k)}(-1)^{m-j/2}\left(A_ju^{(m-j/2)},\psi^{(m-j/2)}\right)_{L_2}+$$

$$+\sum_{j=(2k-1)}(-1)^{m-(j+1)/2}\left(A_ju^{(m-(j-1)/2)},\psi^{(m-(j-1)/2)}\right)_{L_2}.$$

Since

$$P_1(u,\varphi)=\sum_{(j=2k)}(-1)^{m-j/2}\left(B_jA^{j/2}u^{(m-j/2)},A^{j/2}\psi^{(m-j/2)}\right)_{L_2}+$$

$$+\sum_{j=(2k-1)}(-1)^{m-(j+1)/2}\left(B_jA_ju^{(m-(j-1)/2)},A^{(j-1)/2}\psi^{(m-(j-1)/2)}\right)_{L_2},$$

from belongless of $u\in D\left(R_+;H_m\right)$ and $\psi\in D\left(R_+;H_m;\{\nu\}_{\nu=0}^{m-1}\right)$ by intermediate derivatives theorem [17] it follows that

$$|P_1(u,\varphi)|\le\sum_{(j=2k)}\|B_j\|\left\|A^{j/2}u^{(m-j/2)}\right\|_{L_2}\left\|A^{j/2}\psi^{(m-j/2)}\right\|_{L_2}+$$

$$+\sum_{j=(2k-1)}\|B_j\|\left\|A^{(j+1)/2}u^{(m-(j-1)/2)}\right\|_{L_2}\left\|A^{(j-1)/2}\psi^{(m-(j-1)/2)}\right\|_{L_2}\le$$

$$\le const\,\|u\|_{W_2^m(R_+;H)}\,\|\psi\|_{W_2^m(R_+;H)},$$

i.e. $P_1(u,\varphi)$ is continuous in the space $D(R_+;H_m)\oplus D\left(R_+;H_m;\{\nu\}_{\nu=0}^{m-1}\right)$ therefore it continues by continuity on the space $W_2^m(R_+;H)\oplus W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)$. The lemma is proved.

Definition 2.1. *The vector-function $u(t)\in W_2^m(R_+;H)$ is said to be a generalized solution of (20), (21), if*

$$\lim_{t\to0}\left\|u^{(\nu)}(t)-\varphi_\nu\right\|_{H_{m-\nu-1/2}}=0,\quad\nu=\overline{0,m-1}$$

and for any $\psi(t)\in W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)$ it is fulfilled the identity

$$\langle u,\psi\rangle=(u,\psi)_{W_2^m(R_+;H)}+\sum_{p=1}^{m-1}C_m^p\left(A^pu^{(m-p)},A^p\psi^{(m-p)}\right)_{L_2(R_+;H)}+P_1(u,\psi)=0,$$

where

$$C_m^p=\frac{m(m-1)...(m-p+1)}{p!}=\begin{pmatrix}m\\p\end{pmatrix}.$$

First of all we consider the problem

$$P_0\left(\frac{d}{dt}\right)u(t)=\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t)=0,\ \ t\in R_+=(0,+\infty), \tag{25}$$

$$u^{(\nu)}(0)=\varphi_\nu,\quad \nu=\overline{0,m-1}. \tag{26}$$

It holds

Theorem 2.1. *For any collection $\varphi_\nu\in H_{m-\nu-1/2}$ $\left(\nu=\overline{0,m-1}\right)$ problem (25), (26) has a unique generalized solution.*

Proof. Let $c_0,c_1,...,c_{m-1}\in H_{m-\nu-1/2}$ $\left(\nu=\overline{0,m-1}\right)$, e^{-At} be a holomorphic semi-group of bounded operators generated by the operator $(-A)$. Then the vector-function

$$u_0(t)=e^{-tA}\left(c_0+\frac{t}{1!}Ac_1+...+\frac{t^{m-1}}{(m-1)!}A^{m-1}c_{m-1}\right)$$

belongs to the space $W_2^m(R_+;H)$. Really, using spectral expansion of the operator A we see that each term

$$\frac{t^{m-\nu}}{(m-\nu)!}A^{m-\nu}e^{-tA}\in W_2^m(R_+;H)\quad \text{for}\ \ c_\nu\in H_{m-1/2}\ \ \left(\nu=\overline{0,m-1}\right).$$

Then it is easily verified that $u_0(t)$ is a generalized solution of equation (25), i.e. it satisfies the relation

$$(u_0,\varphi)_{W_2^m}+\sum_{p=1}^{m-1}C_m^p\left(A^{m-p}u_0^{(p)},A^{m-p}\varphi^{(p)}\right)=0$$

for any $\varphi\in W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)$.

Show that $u^{(\nu)}(0)=\varphi_\nu,\ \nu=\overline{0,m-1}$.For this purpose we must determine the vectors c_ν $\left(\nu=\overline{0,m-1}\right)$ from condition (26). Obviously, in order to determine the vectors c_ν $\left(\nu=\overline{0,m-1}\right)$ from condition (26) we get a system of equations with respect to the vectors

$$\begin{pmatrix} E & 0 & \cdots & 0\\ -E & E & \cdots & 0\\ E & -E & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ (-1)^{m-1}\begin{pmatrix}1\\ m-1\end{pmatrix}E & (-1)^{m-2}\begin{pmatrix}2\\ m-2\end{pmatrix}E & \cdots & E\end{pmatrix}\times$$

$$\times \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{m-1} \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ A^{-1}\varphi_1 \\ A^{-2}\varphi_2 \\ \vdots \\ A^{-(m-1)}\varphi_{m-1} \end{pmatrix}, \tag{27}$$

where E is a unique operator in H and $\begin{pmatrix} p \\ m-s \end{pmatrix} = C^p_{m-s}$. Since the principal determinant of the operator is invertible, we can uniquely determine $c_\nu \quad \left(\nu = \overline{0, m-1}\right)$. Obviously, for any ν the vector $A^{-(m-\nu)}\varphi_\nu \in H_{m-1/2}$, since $\varphi_\nu \in H_{m-\nu-1/2}$. As the vector at the right hand side of the equation (27) belongs to the space

$$\underbrace{H_{m-1/2} \oplus ... \oplus H_{m-1/2}}_{m \text{ times}} = \left(H_{m-1/2}\right)^m,$$

then taking into account the fact that the principal operator matrix $\widetilde{E}$ as a product of the invertible scalar matrix by matrix where $\widetilde{E}$ is a unique matrix in $\left(H_{m-1/2}\right)^m$, then it is unique. Therefore, each vector c_ν $\left(\nu = \overline{0, m-1}\right)$ is a linear combination of elements $A^{-(m-\nu)}\varphi_\nu \in H_{m-1/2}$, that is why the vector c_ν $\left(\nu = \overline{0, m-1}\right)$ is determined uniquely and belongs to the space $H_{m-1/2}$. The theorem is proved.

In the space $W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$ we define a new norm

$$|||u|||_{W_2^m(R_+;H)} = \left(\|u\|^2_{W_2^m(R_+;H)} + \sum_{p=1}^{m-1} C^p_m \left\|A^{m-p}u^{(p)}\right\|^2_{L_2(R_+;H)} \right)^{1/2}.$$

By the intermediate derivatives theorem [17] the norms $|||u|||_{W_2^m(R_+;H)}$ and $\|u\|_{W_2^m(R_+;H)}$ are equivalent in the space $W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$. Therefore, the numbers

$$N_j\left(R_+; \{\nu\}_{\nu=0}^{m-1}\right) =$$
$$= \sup_{0 \neq u \in W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)} \left\|A^{m-j}u^{(j)}\right\|_{L_2(R_+;H)} |||u|||^{-1}_{W_2^m(R_+;H)}, \quad j = \overline{0, m}.$$

are finite.

The next lemma enables to find exact values of these numbers.

Lemma 2.2. *The numbers* $N_j\left(R_+;\{\nu\}_{\nu=0}^{m-1}\right)$ *are determined as follows:*

$$N_j\left(R_+;\{\nu\}_{\nu=0}^{m-1}\right)=d_{m,j}^{m/2},$$

where

$$d_{m,j}=\begin{cases}\left(\frac{j}{m}\right)^{\frac{j}{m}}\left(\frac{m-j}{m}\right)^{\frac{m-j}{m}}, & j=\overline{1,m-1}\\ 1, & j=0,m\end{cases}$$

Using the method of the papers [16, 22] the lemma is easily proved.

2.3. The basic results

Now, let's prove the principal theorems.

Theorem 2.2. *Let* A *be a positive-definite self-adjoint operator, the operators* $B_j=A^{-j/2}A_jA^{-j/2}$ $\left(j=2k,k=\overline{0,m}\right)$ *and* $B_j=A^{-(j-1)/2}A_jA^{-(j-1)/2}$ $\left(j=2k-1,k=\overline{1,m-1}\right)$ *be bounded in* H *and it hold the inequality*

$$\alpha=\sum_{j=1}^{m}C_j\left\|B_{m-j}\right\|<1,$$

where

$$C_j=\begin{cases}d_{m,j/2}^{m/2}, & j=2k,\ k=\overline{0,m}\\ \left(d_{m,(j-1)/2}d_{m,(j+1)/2}\right)^{m/2}, & j=2k-1,k=\overline{1,m-1}\end{cases}$$

and

$$d_{m,j}=\begin{cases}\left(\frac{j}{m}\right)^{\frac{j}{m}}\left(\frac{m-j}{m}\right)^{\frac{m-j}{m}}, & j=\overline{1,m-1}\\ 1, & j=0,m\end{cases}.$$

Then for any $\varphi_\nu\in D\left(A^{m-n-1/2}\right)$, $\left(\nu=\overline{0,m-1}\right)$ *problem (20), (21) has a unique generalized solution and it holds the inequality*

$$\|u\|_{W_2^m(R_+;H)}\le const\sum_{\nu=0}^{m-1}\|\varphi\|_{m-\nu-1/2}\,.$$

Proof. Let $\psi\in D\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)$. Then for any ψ

$$\operatorname{Re}P(\psi,\psi)=\operatorname{Re}P_0(\psi,\psi)+\operatorname{Re}P_1(\psi,\psi)=$$

$$= \left(\left(-\frac{d}{dt} + A \right)^m \psi, \left(-\frac{d}{dt} + A \right)^m \psi \right) +$$

$$+ \operatorname{Re} P_1(\psi, \psi) \geq \left\| \left(-\frac{d}{dt} + A \right)^m \psi \right\|^2 - |\operatorname{Re} P_1(\psi, \psi)| \geq$$

$$\geq \left\| \left(-\frac{d}{dt} + A \right)^m \psi \right\|^2_{L_2(R_+;H)} - |P_1(\psi, \psi)|_{L_2(R_+;H)} .$$

Since by lemma 2.2

$$\left\| A^k \psi^{(m-k)} \right\|_{L_2(R_+;H)} \leq d^{m/2}_{m,m-k} \|u\|_{W_2^m(R_+;H)} ,$$

then

$$|P_1(\psi, \psi)| \leq$$

$$\leq \left(\sum_{(j=2k)} \|B_{m-j}\| \, d^{m/2}_{m,m-k} + \sum_{(j=2k-1)} \|B_{m-j}\| \, d^{m/2}_{m,m-k+1} d^{m/2}_{m,m-k-1} \right) \|\psi\|^2_{W_2^m} .$$

Here $d_{0,0} = d_{m,m} = 1$ and

$$d_{m,k} = \left(\frac{k}{m} \right)^{\frac{k}{m}} \left(\frac{m-k}{m} \right)^{\frac{m-k}{m}} , \quad \left(k = \overline{1, m-1} \right),$$

thus

$$|P_1(\psi, \psi)| \leq \sum_{j=1}^{m} C_j \|B_{m-j}\| ,$$

where

$$C_j = \begin{cases} d^{m/2}_{m,j/2}, & j = 2k, k = \overline{0, m} \\ \left(d^{m/2}_{m,(j+1)/2} d^{m/2}_{m,(j-1)/2} \right)^{m/2}, & j = 2k-1, k = \overline{1, m-1}. \end{cases}$$

Consequently

$$|P_1(\psi, \psi)| \leq \alpha \|\psi\|^2_{W_2^m(R_+;H)} .$$

Then

$$\operatorname{Re} P(\psi, \psi)_{L_2(R_+;H)} \geq (1 - \alpha) P_0(\psi, \psi)_{L_2(R_+;H)}. \tag{28}$$

Now we look for a generalized solution of problem (20), (21) in the form

$$u(t) = u_0(t) + \theta(t),$$

where $u_0(t)$ is a generalized solution of problem (25), (26) and $\theta(t) \in$ $\in W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$. To define $\theta(t)$ we get relation

$$\langle\theta;\psi\rangle = (\theta,\psi)_{W_2^m(R_+;H)} + \sum_{p=1}^{m-1} C_m^p\left(A^{m-p}\theta, A^{m-p}\psi\right) + P_1(\theta,\psi) = P_1(u_0,\psi). \tag{29}$$

Since the right hand side of the equality is a continuous functional in $W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$, and the left hand side $\langle\theta;\psi\rangle$ is a bilinear functional in the space $W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right) \oplus W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$, then by inequality (28) it satisfies conditions of Lax-Milgram theorem [21]. Consequently, there exists a unique vector-function $\theta(t) \in W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$ that satisfies equality (29) and $u(t) = u_0(t) + \theta(t)$ is a generalized solution of problem (20), (21).

Further, by $J(R_+;H)$ we denote a set of generalized solutions of problem (20), (21) and define the operator $\Gamma : J(R_+;H) \to \widetilde{H} = \overset{m-1}{\underset{k=0}{\oplus}} H_{m-k-1/2}$ acting in the following way $\Gamma u = \left(u^{(k)}(0)\right)_{k=0}^{m-1}$. Obviously $J(R_+;H)$ is a closed set and by the trace theorem $\|\Gamma u\|_{\widetilde{H}} \leq C\|u\|_{W_2^m(R_+;H)}$. Then by the Banach theorem on the inverse operator there exists the inverse operator $\Gamma^{-1} : \widetilde{H} \to J(R_+;H)$. Consequently

$$\||u\||_{W_2^m(R_+;H)} \leq const\sum_{k=0}^{m-1}\|\varphi\|_{m-k-1/2}.$$

The theorem is proved.

Remark. From the proof we can show that for $m = 2$, $c_1 = c_3 = 1/2$, $c_2 = 1/4$, $c_4 = 1$.

§ 1.3. On completeness of elementary generalized solutions of a class of operator-differential equations of higher order

In this section we give definition of m- fold completeness and prove a theorem on completeness of elementary generalized solution of corresponding boundary value problems at which the equation describes the process of corrosive fracture of metals in aggressive media and the principal part of the equation has a multiple characteristic.

3.1. Problem statement

Let H be a separable Hilbert space, A be a positive definite self-adjoint operator in H with domain of definition $D(A)$. By H_γ we denote a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(\gamma \geq 0)$, $(x,y)_\gamma = = (A^\gamma x, A^\gamma y)\,, x, y \in D(A^\gamma)$. By $L_2\left((a,b);H\right)$ $(-\infty \leq a < b \leq \infty)$ we denote a Hilbert space of vector-functions $f(t)$, determined in (a,b) almost everywhere with values in H, measurable, square integrable in Bochner sense

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f\|_\gamma^2 \, dt\right)^{1/2}.$$

Then we determine a Hilbert space for natural $m \geq 1$ [17]

$$W_2^m\left((a,b);H\right) = \left\{u/u^{(m)} \in L_2\left((a,b);H\right), A^m u \in L_2\left((a,b);H_m\right)\right\}$$

with norm

$$\|u\|_{W_2^m((a,b);H)} = \left(\left\|u^{(m)}\right\|^2_{L_2((a,b);H)} + \|A^m u\|^2_{L_2((a,b);H)}\right)^{1/2}.$$

Here and in sequel, derivatives are understood in the distributions theory sense [17]. Assume

$$L_2\left((0,\infty);H\right) \equiv L_2(R_+;H),\ L_2\left((-\infty,\infty);H\right) \equiv L_2(R;H),$$

$$W_2^m\left((0,\infty);H\right)\equiv W_2^m\left(R_+;H\right),\ W_2^m\left((-\infty,+\infty);H\right)\equiv W_2^m\left(R;H\right).$$

Then we determine the spaces

$$W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)=\left\{u|u\in W_2^m\left(R_+;H\right),u^{(\nu)}(0)=0,\ \nu=\overline{0,m-1}\right\}.$$

Obviously, by the trace theorem [17] the space $W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)$ is a closed subspace of the Hilbert space $W_2^m\left(R_+;H\right)$.

Let's define a space of $D\left([a,b];H_\gamma\right)$-times infinitely differentiable functions for $a\le t\le b$ with values in H_γ having a compact support in $[a,b]$. As is known a linear set $D\left([a,b];H_\gamma\right)$ is everywhere dense in the space $W_2^m\left((a,b);H\right)$, [17].

It follows from the trace theorem that the space

$$D\left(R_+;H_m;\{\nu\}_{\nu=0}^{m-1}\right)=\left\{u|u\in D\left(R_+;H_m\right),u^{(\nu)}(0)=0,\ \nu=\overline{0,m-1}\right\}$$

and also everywhere is dense in the space.

Let's consider a polynomial operator pencil

$$P(\lambda)=\left(-\lambda^2E+A^2\right)^m+\sum_{j=1}^m A_j\lambda^{m-j}. \tag{30}$$

Bind the polynomial pencil (30) with the following boundary value problem

$$\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t)+\sum_{j=1}^m A_ju^{(m-j)}(t)=0,\ \ t\in R_+=(0,+\infty), \tag{31}$$

$$u^{(\nu)}(0)=\varphi_\nu,\ \ \nu=\overline{0,m-1},\ \varphi_\nu\in H_{m-\nu-1/2}. \tag{32}$$

Here we assume that the following conditions are fulfilled:

1) A is a positive-definite self-adjoint operator with completely continuous inverse $C=A^{-1}\in\sigma_\infty$;

2) The operators

$$B_j=A^{-j/2}A_jA^{-j/2}\ \left(j=2k,\ k=\overline{1,m}\right)$$

and

$$B_j=A^{-(j-1)/2}A_jA^{-(j-1)/2}\ \ \left(j=2k-1,\ k=\overline{1,m-1}\right);$$

3) The operators $(B+E_m)$ are bounded in H.

Equation (31) descries a process of corrosion fracture in aggressive media that was studied in the paper [21].

3.2. Some definition and auxiliary facts

Denote

$$P_0\left(\frac{d}{dt}\right)u(t) \equiv \left(-\frac{d^2}{dt^2}+A^2\right)^m u(t), \quad u(t)\in D\left(R_+;H_m\right), \tag{33}$$

$$P_1\left(\frac{d}{dt}\right)u(t) \equiv \sum_{j=1}^{m-1} A_j u^{(m-j)}(t), \quad u(t)\in D\left(R_+;H_m\right), \tag{34}$$

Definition 3.1. *The vector-function $u(t)\in W_2^m(R_+;H)$ is said to be a generalized solution of (31), (32), if*

$$\lim_{t\to 0}\left\|u^{(\nu)}(t)-\varphi_\nu\right\|_{H_{m-\nu-1/2}}=0, \quad \nu=\overline{0,m-1}$$

and for any $\psi(t)\in W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)$ it is fulfilled the identity

$$\langle u,\psi\rangle=(u,\psi)_{W_2^m(R_+;H)}+\sum_{p=1}^{m-1} C_m^p\left(A^p u^{(m-p)},A^p\psi^{(m-p)}\right)_{L_2(R_+;H)}+P_1(u,\psi)=0,$$

where

$$C_m^p=\frac{m(m-1)...(m-p+1)}{p!}=\left(\begin{array}{c} m\\ p\end{array}\right).$$

Definition 3.2. *If a non-zero vector $\varphi_0\neq 0$ is a solution of the equation $P(\lambda_0)\varphi_0=0$ then λ_0 is said to be an eigen-value of the pencil $P(\lambda)$ and φ_0 an eigen-vector responding to the number λ_0.*

Definition 3.3. *The system $\{\varphi_1,\varphi_2,...,\varphi_m\}\in H_m$ is said to be a chain of eigen and adjoint vectors φ_0 if it satisfies the following equations*

$$\sum_{i=0}^{q}\frac{1}{i}\frac{d^i}{d\lambda^i}P(\lambda)|_{\lambda=\lambda_0}\cdot\varphi_{q-i}=0, \quad q=\overline{1,m}.$$

Definition 3.4. *Let $\{\varphi_0,\varphi_1,...,\varphi_m\}$ be a chain of eigen and adjoint vectors responding to eigenvalues λ_0, then vector-functions*

$$\varphi_h(t)=e^{\lambda_0 t}\left(\frac{t^h}{h!}\varphi_0+\frac{t^{h-1}}{(h-1)!}\varphi_1+...+\varphi_n\right), \quad h=\overline{0,m}$$

satisfy equation (31) and are said to be its elementary solutions responding to the eigen value λ_0.

Obviously, elementary solutions $\varphi_h(t)$ have traces in the zero

$$\varphi_h^{(\nu)} = \frac{d^\nu}{dt^\nu}\varphi_h|_{t=0}, \quad \nu = \overline{0, m-1}.$$

By means of $\varphi_h^{(\nu)}$ we define the vectors

$$\left\{\widetilde{\varphi}_h = \left(\varphi_h^{(0)}, \varphi_h^{(1)}\right), \ \ h = \overline{0, m}\right\} \subset H^m = \underbrace{H \times ... \times H}_{m \ \text{ times}}.$$

Later by $K(\Pi_-)$ we denote all possible vectors $\widetilde{\varphi}_h$ responding to all eigen values from the left half-plane $(\Pi_- = \{\lambda / \mathrm{Re}\ \lambda < 0\})$.

Definition 3.5. *The system $K(\Pi_-)$ is said to be m-fold complete in the trace space, if the system $K(\Pi_-)$ is complete in the space* $\overset{m}{\underset{i=0}{\oplus}} H_{m-i-1/2}$.

It holds

Lemma 3.1. *Let conditions 1)-2) be fulfilled and*

$$\alpha = \sum_{j=1}^{m} C_j \left\| B_{m-j} \right\| < 1, \tag{35}$$

where

$$C_j = \begin{cases} d_{m,j/2}^{m/2}, & j = 2k, \ k = \overline{0, m} \\ \left(d_{m,(j-1)/2} d_{m,(j+1)/2}\right)^{m/2}, & j = 2k-1, k = \overline{1, m-1} \end{cases}$$

and

$$d_{m,j} = \begin{cases} \left(\frac{j}{m}\right)^{\frac{j}{m}} \left(\frac{m-j}{m}\right)^{\frac{m-j}{m}}, & j = \overline{1, m-1} \\ 1 \qquad\qquad\qquad , & j = 0, m \end{cases}$$

Then for any $\psi \in W_2^m\left(R_+; H_m; \{\nu\}_{\nu=0}^{m-1}\right)$ *it holds the inequality*

$$Re\ P(\psi, \psi) \geq (1-\alpha) P_0(\psi, \psi),$$

where

$$P_0\left(\psi, \psi\right) = \left(\left(-\frac{d}{dt} + A\right)^m \psi, \left(-\frac{d}{dt} + A\right)^m \psi\right)_{L_2}$$

and

$$P(u,\psi)=P_0(u,\psi)+P_1(u,\psi)$$

moreover $P_1(u,\psi)$ is determined from lemma 2.1.

Proof. Let $\psi\in D\left(R_+;H_m;\{\nu\}_{\nu=0}^{m-1}\right)$. Then for any ψ

$$\text{Re }P(\psi,\psi)=\text{ Re }P_0(\psi,\psi)+\text{ Re }P_1(\psi,\psi)=$$

$$=\left(\left(-\frac{d}{dt}+A\right)^m\psi,\left(-\frac{d}{dt}+A\right)^m\psi\right)_{L_2}+$$

$$+\text{ Re }P_1(\psi,\psi)\geq\left\|\left(-\frac{d}{dt}+A\right)^m\psi\right\|_{L_2}^2-|P_1(\psi,\psi)|.$$

Since

$$\left\|A^k\psi^{(m-k)}\right\|_{L_2}\leq d_{m,m-k}^{m/2}\|u\|_{W_2^m}$$

then

$$|P_1(\psi,\psi)|\leq$$

$$\leq\left(\sum_{(j=2k)}\|B_j\|\,d_{m,m-j/2}^{m/2}+\sum_{(j=2k-1)}\|B_j\|\,d_{m,m-(j-1)/2}^{m/2}d_{m,m-(j+1)/2}^{m/2}\right)\|\psi\|_{W_2^m}.$$

Here $d_{0,0}=d_{m,m}=1$, thus

$$|P_1(\psi,\psi)|\leq\sum_{j=0}^{m}\|B_{m-j}\|\,C_j,$$

where

$$C_j=\begin{cases} d_{m,j/2}^{m/2},\ j=2k,\quad k=\overline{0,m}\\ \left(d_{m,(j+1)/2}d_{m,(j-1)/2}\right)^{m/2},\ j=2k-1,\quad k=\overline{1,m-1}.\end{cases}$$

Thus

$$|P_1(\psi,\psi)_{L_2}|\leq\alpha\|\psi\|^2_{W_2^m(R_+;H)}.$$

Thus

$$P(\psi,\psi)_{L_2(R_+;H)}\geq(1-\alpha)P_0(\psi,\psi)_{L_2(R_+;H)}.$$

The lemma is proved.

Lemma 3.2. *Let the conditions of lemma 2 be fulfilled. Then for any $x \in H_m$ and $\xi \in R$ it holds the inequality*

$$\left(P(i\xi)X, X\right)_H > (1-\alpha)\left(P_0(i\xi)X, X\right)_H .$$

Proof. It follows from the conditions that for all $\psi(t) \in W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$ it holds the inequality

$$\left(P(\psi, \psi)\right)_{L_2(R_+;H)} \geq (1-\alpha)P_0(\psi, \psi)_{L_2(R_+;H)}. \tag{36}$$

Let $\psi(t) = g(t) \cdot X$, $X \in H_m$ and a scalar function $g(t) \in W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$. Then from (7)we get

$$\left(P(i\xi)g(t) \cdot X, g(t) \cdot X\right)_{L_2(R_+;H)} \geq (1-\alpha)\left(P_0(i\xi) \cdot X, X\right) \|g(t)\|^2_{L_2(R_+;H)} ,$$

then

$$\left(P(i\xi)X, X\right) \|g(t)\|^2_{L_2(R_+;H)} \geq (1-\alpha)\left(P_0(i\xi)X, X\right) \|g(t)\|^2_{L_2(R_+;H)} ,$$

i.e.

$$\left(P(i\xi)X, X\right) \geq (1-\alpha)\left(P_0(i\xi)X, X\right)$$

The lemma is proved.

Lemma 3.3. *Let conditions 1)-3) and solvability conditions be fulfilled, then estimation $\|A^m p^{-1}(i\xi) A^m\| \leq const$ is true.*

The proof of this lemma is easily obtained from Keldysh lemma [23,24] and lemma 3.1.

3.4. The basic result

Now, let's prove the principal theorems. It holds the following theorem.

Theorem 3.1. *Let conditions 1)-2) be fulfilled, solvability conditions and one of the following conditions hold*

a) $A^{-1} \in \sigma_p \ \ (0 \leq p < 1)$;

b) $A^{-1} \in \sigma_p \ \ (0 \le p < \infty)$ *and* $B_j \in \sigma_\infty$.

Then the system of eigen and adjoint vectors from $K(\Pi_-)$ *is complete in the trace space.*

Proof. Denote

$$L(\lambda) = A^{-m} p(\lambda) A^m,$$

where

$$L(\lambda) = \left(-\lambda^2 C^2 + E\right)^m + \sum_{j=1}^{m} \lambda^{m-j} T^j,$$

and

$$T_j = \begin{cases} C^{m-1/2} B_j C^{m-1/2} & \text{for} \quad j = 2k, k = \overline{1, m} \\ C^{m-(j-1)/2} B_j C^{m-(j-1)/2} & \text{for} \quad j = 2k-1, k = \overline{1, m-1}. \end{cases}$$

Obviously $T_j \in \sigma_{p/m-j}$. Then $L^{-1}(\lambda)$ is represented in the form of relation of two entire functions of order p and minimal order p. Then

$$A^{m-1/2} p^{-1}(\lambda) A^{m-1/2} = A^{-1/2} \left(A^m p^{-1}(\lambda) A^m\right) A^{-1/2}$$

is also represented in the relation of two entire functions of order p and of minimal type for order p. The proof of m-fold completeness of the system $K(\Pi_-)$ is equivalent to the proof of the fact that for any $\varphi_0, \varphi_1, ..., \varphi_{m-1}$ from holomorphic property of the vector-function

$$F(\lambda) = \left(L^*(\overline{\lambda})\right)^{-1} \left(f(\lambda), f(\overline{\lambda})\right),$$

where

$$f(\lambda) = \sum_{j=0}^{m-1} \lambda^j C^{j+1/2} \varphi_j.$$

For $\Pi_- = \{\lambda / \ \mathrm{Re}\ \lambda < 0\}$ it follows that $\varphi_j = 0$.

The theorem is proved.

Now we use theorem 2.2 and theorem 3.1 and prove the completeness of elementary solutions of problem (32), (33).

Theorem 3.2. *Let the conditions of theorem 2.2 be fulfilled. Then elementary solutions of problem (32), (33) is complete in the space of generalized solutions.*

Proof. It is easy to see that if there exists a generalized solution, then

$$\|u\|_{W_2^m(R_+;H)} \leq const \sum_{j=0}^{m-1} \|\varphi_j\|_{m-j-1/2}\,.$$

Then it follows from the trace theorem [17] and these inequalities that

$$C_k \sum_{\nu=0}^{m-1} \|\varphi_\nu\|_{m-\nu-1/2} \leq \|u\|_{W_2^m(R_+;H)} = C_k \sum_{\nu=0}^{m-1} \|\varphi_\nu\|_{m-\nu-1/2}\,. \tag{37}$$

Further, from the theorem on the completeness of the system $K(\Pi_-)$ it follows that for any collection $\{\varphi_\nu\}_{\nu=0}^{m-1}$ and $\varphi_\nu \in H_{m-\nu-1/2}$ there is such a number N and $C_k(\varepsilon, N)$ that

$$\left\|\varphi_\nu - \sum_{k=1}^{N} C_k \varphi_{i,j,h}^{(\nu)}\right\| < \varepsilon/m, \quad \nu = \overline{0, m-1}.$$

Then it follows from (37) that

$$\left\|u(t) - \sum_{k=1}^{N} C_k \varphi_{i,j,h}^{(\nu)}\right\| \leq \varepsilon.$$

The theorem is proved.

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Chapter II

In this chapter we'll give definition of regular holomorphic solutions of boundary value problems and prove theorems on existence and uniqueness of solutions in terms of coefficients of the studied higher order operator–differential equations. In addition we'll investigate ϕ-solvability of boundary value problems in a sector.

Further we prove theorem on m- fold completeness of a part of eigen and adjoint vectors for high order operator pencils responding to eigen- values from some angular sector, moreover a principal part of polynomial pencils has a multiple characteristics. Therewith we'll use main methods of M.G.Gasymov [1] and S.S.Mirzoyev [2,3] papers. We'll prove a Phragmen- Lindelof type theorem.

§2.1. On regular holomorphic solution of a boundary value problem for a class of operator- differential equations of higher order

In this section we give definition of regular holomorfic solutions of a boundary value problem and determine the conditions under which these solutions exists.

1.1. Introduction and problem statement.

Let H be a separable Hilbert space, A be a positive-definite self- adjoint operator in H, and H_γ be a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in D(A^\gamma)$.

In the space H we consider an operator–differential equation

$$P\left(\frac{d}{d\tau}\right) u(\tau) \equiv$$

$$\equiv \left(-\frac{d^2}{d\tau^2} + A^2\right)^m u(\tau) + \sum_{j=1}^{2m-1} A_{2m-j} u^{(j)}(\tau) = f(\tau),\ \tau \in S_\alpha \qquad (38)$$

with boundary conditions

$$u^{(\nu)}(0) = 0, \quad \nu = \overline{0, m-1}, \qquad (39)$$

where $u(\tau)$ and $f(\tau)$ are holomorphic in the angle

$$S_\alpha = \{\lambda / |\arg\lambda| < \alpha,\ 0 < \alpha < \pi/2\}$$

and are vector- functions with values in H, the derivatives are understood in the sense of complex analysis [4].

We denote

$$H_2(\alpha) = \left\{ f(\tau) / \sup_{|\varphi|<\alpha} \int_0^\infty \left\| f\left(te^{i\varphi}\right)\right\|^2 dt < \infty \right\}.$$

A linear set $H_2(\alpha)$ turns into Hilbert space if we determine the norm

$$\|f\|_\alpha = \frac{1}{\sqrt{2}} \left(\|f_\alpha(\tau)\|_{L^2_{2(R_+;H)}} + \|f_{-\alpha}(\tau)\|_{L^2_{2(R_+;H)}} \right)^{1/2}$$

where $f_\alpha(\tau) = f(te^{i\alpha})$, $f_{-\alpha}(\tau) = f(te^{-i\alpha})$are the boundary values of the vector- function $f(\tau)$ almost everywhere on the rays $\Gamma_{\pm\alpha} = \{\lambda / \arg\lambda = \pm\alpha\}$, and the space $L_2(R_+; H)$ is determined in [4]. Further we define the following Hilbert spaces

$$W_2^{2m}(\alpha) = \left\{u(\tau)/u^{(2m)}(\tau) \in H_2(\alpha), \quad A^{2m}u(\tau) \in H_2(\alpha)\right\}$$

$$\overset{\circ}{W}{}_2^{2m}(\alpha) = \left\{u(\tau) \in W_2^{2m}(\alpha), \quad u^{(\nu)}(0) = 0, \quad \nu = \overline{0, m-1}\right\}$$

with the norm

$$|||u|||_\alpha = \left(\left\|u^{(2m)}\right\|_\alpha^2 + \left\|A^{2m}u\right\|_\alpha^2\right)^{1/2}.$$

Definition 2.1. *If vector-function $u(\tau) \in W_2^{2m}(\alpha)$ satisfies the equation (38) in S_α identically, it said to be a regular holomorphic solution of equation (38).*
Definition 2.2. *If regular solution of equation (38) $u(\tau)$ satisfies the boundary values in the sense*

$$\lim_{\tau\to 0}\left\|u^{(\nu)}\right\|_{2m-\nu-1/2} = 0, \quad \nu = \overline{0, m-1}$$

and the inequality

$$|||u|||_\alpha \le const\|f\|_\alpha$$

is fulfilled, we'll say that the problem (38), (39) is regularly holomorpically solvable.

1.2. Some auxiliary facts

As first we prove some auxiliary statements.
Lemma 2.1. *An operator P_0 determined by the expression*

$$P_0\left(\frac{d}{d\tau}\right)u(\tau) = \left(-\frac{d^2}{dt^2} + A^2\right)^m u(\tau), \quad u(\tau) \in \overset{\circ}{W}{}_2^{2m}(\alpha)$$

realizes an isomorphism between the spaces $\overset{\circ}{W}{}_2^{2m}(\alpha)$ and $H_2(\alpha)$.

Proof. Let's consider the equation $P_0u(\tau) = 0$, $u(\tau) \in \overset{\circ}{W}{}_2^{2m}(\alpha)$. Obviously, the solution of equations

$$\left(-\frac{d^2}{dt^2}+A^2\right)^m u(\tau)=0$$

is of the form

$$u_0(\tau)=e^{-\tau A}\left(C_0+\frac{\tau}{1!}AC_1+...+\frac{\tau^{m-1}}{(m-1)!}A^{m-1}C_{m-1}\right),$$

where $C_0, C_1, ..., C_{m-1} \in H_{2m-1/2}$. Hence we determine $C_0, C_1, ..., C_{m-1}$ and boundary conditions (39) and get

$$\tilde{R}\tilde{C}=\begin{pmatrix} E & 0 & \cdots & 0 \\ -E & E & \cdots & 0 \\ E & -E & \cdots & 0 \\ \vdots & \vdots & \cdots & \\ (-1)^{m-1}EC^1_{m-1} & (-1)^{m-2}E & \cdots & E \end{pmatrix}\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_{m-1} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{40}$$

where $C^k_m=\frac{m(m-1)...)m-k}{k!}$. Since the operator $\tilde{R}$ is invertible, then all $C_i=0$, $i=\overline{0,m-1}$ i.e. $u_0(\tau)=0$. On the other hand, for any $f(\tau)$ vector-function

$$\upsilon(\tau)=\frac{1}{2\pi i}\int\limits_{\Gamma_{\frac{\pi}{2}+\alpha}}\left(-\lambda^2E+A^2\right)^{-m}\hat{f}(\lambda)\,e^{\lambda\tau}d\lambda-\frac{1}{2\pi i}\int\limits_{\Gamma_{\frac{3\pi}{2}-\alpha}}\left(-\lambda^2E+A^2\right)^{-m}\hat{f}(\lambda)\,e^{\lambda\tau}d\lambda$$

satisfies the equation

$$\left(-\frac{d^2}{d\tau^2}+A^2\right)^m u(\tau)=f(\tau)$$

in S_α and the estimations on the rays $\Gamma_{\frac{\pi}{2}+\alpha}=\left\{\lambda/\arg\lambda=\frac{\pi}{2}+\alpha\right\}$, $\Gamma_{\frac{3\pi}{2}+\alpha}=\left\{\lambda/\arg\lambda=\frac{3\pi}{2}-\alpha\right\}$

$$\left\|A^{2m}\left(-\lambda^2E+A^2\right)^{-m}\right\|+\left\|A\left(-\lambda^2E+A^2\right)^{-m}\right\|\le const$$

yield that $\upsilon(\tau)\in W_2^{2m}(\alpha)$. So, we look for the solution of the equation $P_0(d/d\tau)\,u(\tau)=f(\tau)$, $u(\tau)\in W_2^{2m}(\tau)$ in the form

$$u(\tau)=\upsilon(\tau)-e^{-\tau A}\left(C_0+\frac{\tau}{1!}AC_1+...+\frac{\tau^{m-1}}{(m-1)!}A^{m-1}C_{m-1}\right),$$

where the vectors $C_i \in H_{2m-1/2} \quad \left(i = \overline{0, m-1}\right)$ satisfy the equation $\tilde{R}\tilde{C} = \tilde{\varphi}$, where $\tilde{R}$ is determined from the left hand side of equations (40), and $\tilde{C} = (c_0, c_1, ..., c_{m-1})$, $\tilde{\varphi} = \left(\upsilon(0), A^{-1}\upsilon'(0), ..., A^{-(m-1)}\upsilon^{(m-1)}(0)\right)$. Since $\upsilon(\tau) \in W_2^{2m}(\alpha)$ it follows from the trace theorem [4] that $\upsilon(0),\ A^{-1}\upsilon_0'(t), ..., A^{-(m-1)}\upsilon^{(m-1)}(0) \in H_{2m-1/2}$, therefore the vectors $C_0, C_1, ..., C_{m-1} \in H_{2m-1/2}$, i.e. $u(\tau) \in W_2^{2m}(\alpha)$. On the other hand, it is easy to see that

$$\|P_0(d/d\tau)\,u(\tau)\|_\alpha \le const|||u(\tau)|||_\alpha.$$

Then the statement of the lemma follows from Banach theorem on the inverse operator.

It follows from this lemma and a theorem on intermediate derivatives that $|||u|||_\alpha$ and $\|P_0 u\|_\alpha$ are equivalent in the space $\overset{\circ}{W}{}_2^{2m}(\alpha)$ and the numbers

$$w_j = \sup_{0 \ne u \in \overset{\circ}{W}{}_2^{2m}(\alpha)} \left\|A^{2m-j}u^{(j)}\right\|_\alpha \|P_0 u\|_\alpha^{-1}, \quad j = \overline{1, 2m-1}$$

are finite. For estimating the numbers w_j we act in the following way. Since for $u(\tau) \in W_2^{2m}(\alpha)$

$$\frac{\partial^j}{\partial t^j} u\left(te^{i\varphi}\right) = \frac{d^j}{dt^j} u\left(te^{i\varphi}\right) e^{ij\varphi},$$

then

$$\|P_0 u\|_\alpha^2 = \left\|\left(-\frac{d^2}{dt^2} + A^2\right)^m u\right\|^2 = \frac{1}{2}\left\|\left(e^{-2i\alpha}\frac{d^2}{dt^2} + A^2\right)^m u_\alpha(t)\right\|_{L_2(R_+;H)}^2 +$$

$$+\frac{1}{2}\left\|\left(e^{-2i\alpha}\frac{d^2}{dt^2} + A^2\right)^m u_{-\alpha}(t)\right\|_{L_2(R_+;H)}^2.$$

On the other hand

$$\left\|\left(e^{-2i\alpha}\frac{d^2}{dt^2} + A^2\right)^m u_\alpha(t)\right\|_{L_2(R_+;H)}^2 =$$

$$= \left\|\sum_{l=0}^{m} (-1)^m \left(C_m^l\right) e^{-2i\alpha l} A^{2(m-l)} u_\alpha^{(2l)}\right\|_{L_2(R_+;H)}^2 =$$

$$= \left\| \sum_{l=0}^{m} C_l A^{2m-l} u_\alpha^{(l)} \right\|^2_{L_2(R_+;H)}, \tag{41}$$

where

$$C_l = \begin{cases} (-1)^m \left(e^m\right) e^{-2i\alpha l} & \text{for} \quad l = 0, 2, ..., 2m \\ 0 & \text{for} \quad l = 1, 2, ..., 2n-1. \end{cases}$$

Denote

$$Q_0\left(\lambda, A\right) = \sum_{l=0}^{m} C_l \lambda^l A^{2m-l}$$

and construct polynomial operator pencils ([3])

$$P_j\left(\lambda; \beta; A\right) = Q_0\left(\lambda; A\right) Q_0^*\left(-\lambda; A\right) - \beta \left(i\lambda\right)^{2j} A^{2(2m-j)}$$

where $\beta \in 0, b_j^{-2}$ and

$$b_j = \sup_{\xi \in R} \left| \frac{\xi^j}{Q_0\left(-i\xi; 1\right)} \right| \equiv \sup_{\xi \in R} \left| \frac{\xi^j}{\left(\xi^2 e^{-2i\alpha} + 1\right)^m} \right| =$$

$$= \sup_{\xi \in R} \left| \frac{\xi^j}{\left(\xi^4 + 1 + 2\xi^2 6s^2\alpha\right)^{\frac{m}{2}}} \right|, \tag{42}$$

that may be represented in the form

$$P_j\left(\lambda; \beta; A\right) = \phi_j\left(\lambda; \beta; A\right) \phi_j^*\left(-\lambda; \beta; A\right)$$

moreover

$$\phi_j\left(\lambda; \beta; A\right) = \sum_{l=0}^{m} \alpha_{j,l}\left(\beta\right) \lambda^l A^{m-l} = \prod_{l=1}^{m} \left(\lambda E - \alpha_{j,l}\left(\beta\right) A\right)$$

where $Re\alpha_{j,l}\left(\beta\right) < 0$

It holds

Lemma 2.2.[3] *For any* $\upsilon \in W_2^{2m}\left(R_+; H\right)$ *and* $\beta \in \left[0, b_j^{-2}\right]$

$$\left\| Q_0\left(d/dt\right) \upsilon\left(t\right) \right\|^2_{L_2(R_+;H)} - \beta \left\| A^{m-j} \upsilon^{(j)}\left(t\right) \right\|^2_{L_2(R_+;H)} =$$

$$= \|\phi_j (d/dt; \beta; A) \upsilon (t)\|^2_{L_2(R_+;H)} + \left(s_j (\beta) \tilde{\psi}, \tilde{\psi} \right),$$

where

$$\tilde{\psi} = \left(A^{m-j} \upsilon^{(j)} (t) \right)_{j=0}^{m-1} \in H_m, \quad s_j (\beta) = R_j (\beta) - T, \tag{43}$$

moreover $R_j (\beta) = (r_{pq,j} (\beta))$, $T = (t_{pq})$ *and for*

$$r_{pq,j} (\beta) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \overline{\alpha_{p+\nu,j} (\beta)} \alpha_{q-\nu-1} (\beta) \quad (\alpha_\nu = 0,\ \nu < 0,\ \nu > m),$$

$$t_{pq,j} (\beta) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \overline{C_{p+\nu} (\beta)} C_{q-\nu-1} (\beta) \quad (C_\nu = 0,\ \nu < 0,\ \nu > m)$$

for $p = q$

$$r_{pq,j} (\beta) = \sum_{\nu=0}^{\infty} (-1)^{\nu}\ Re \overline{\alpha_{p+\nu,j} (\beta)} \alpha_{q-\nu-1} (\beta)$$

$$t_{pq,j} (\beta) = \sum_{\nu=0}^{\infty} (-1)^{\nu} Re \overline{C_{p+\nu} (\beta)} C_{q-\nu-1} (\beta),$$

and for $p < q$ *we assume*

$$r_{pq,j} (\beta) = \overline{r_{qp,j} (\beta)}, \quad t_{pq} = \overline{t_{qp}}.$$

The next lemma follows from 2.2.

Lemma 2.3. *Let* $\beta \in \left[0, b_j^{-2}\right)$. *Then for any* $u(\tau) \in W_2^{2m} (\alpha)$ *it holds the equality*

$$\|P_0 (d/d\tau) u (\tau)\|^2_\alpha - \beta \left\| A^{2m-j} u^{(j)} (\tau) \right\|^2_\alpha = \frac{1}{2} \|\phi_j (d/dt; u_\alpha (t))\|^2_{L_2(R_+;H)} +$$

$$+ \frac{1}{2} \|\phi_j (d/dt; \beta; A) u_{-\alpha} (t)\|^2_{L_2(R_+;H)} + (M_j (\beta) \tilde{\varphi}, \tilde{\varphi})$$

where $\tilde{\varphi} = \left(A^{2m-j-1/2} u^{(j)} (0) \right)_{j=0}^{m-1}$, $\quad M_j (\beta) = \frac{1}{2} [\tilde{u}_\alpha^{-1} s_j (\beta) \tilde{u}_\alpha + \tilde{u}_\alpha s_j (\beta) \tilde{u}_\alpha^{-1}]$ *and* $s_j (\beta)$ *is determined from (43)*

$$\tilde{u}_\alpha \equiv diag \left(1, e^{i\alpha}, e^{2i\alpha}, ..., e^{i(m-1)\alpha} \right).$$

The proof of lemma 2.3 follows from lemma 2.2. and the fact that

$$\left(A^{2m-j-1/2}u_{-\alpha}^{j}(0)\right)=u_{\alpha}\left(A^{2m-j-1/2}u^{(j)}(0)\right)=u_{\alpha}\tilde{\varphi}$$

and

$$\left(A^{2m-j-1/2}u_{-\alpha}^{j}(0)\right)=u_{-\alpha}\left(A^{2m-j-1/2}u^{(j)}(0)\right)=u_{-\alpha}\tilde{\varphi}.$$

1.3. The basic result

By $\mu_{j,m}(\beta)$ we denote a matrix obtained by rejecting the first m rows and columns from $M_j(\beta)$. It holds

Theorem 2.1. *Let A be a self- adjoint positive –definite operator, the operators $B_j=A_jA^{-j}$ $\left(j=\overline{1,2m-1}\right)$ be bounded in H and the inequality*

$$\sigma=\sum_{j=0}^{2m-1}\chi_j\|B_{2m-j}\|<1, \tag{44}$$

is full filled, where the numbers χ_j are determined as follows

$$\chi_j\geq\begin{cases} b_j & \text{if } \det\mu_{j,m}(\beta)\neq 0\\ \mu_{\alpha}^{-1/2} & \text{otherwise}\end{cases}$$

here $\mu_j^{-1/2}$ is the least root of the equation $\det\mu_{j,m}(\beta)=0$. Then the problem (38), (39) is regularly holomorphically solvable.

Proof. It follows from lemma 2.3 that for $u(\tau)\in\overset{\circ}{W}{}_2^{2m}(\alpha)$and $\beta\in\left[0,b_j^{-2}\right)$ it holds the inequality

$$\|P_0(d/d\tau)\|_{\alpha}^2-\beta\left\|A^{2m-j}u^{(j)}\right\|_{\alpha}^2=\frac{1}{2}\|\phi_j(d/dt;\beta;A)\,u_{\alpha}(t)\|_{L_2(R_+;H)}^2+$$

$$+\frac{1}{2}\|\phi_j(d/dt;\beta;A)\,u_{-\alpha}(t)\|_{L_2(R_+;H)}^2+\left(\mu_{j,m}(\beta)\,\tilde{\varphi},\tilde{\varphi}\right)_{H^m}, \tag{45}$$

where, $\tilde{\varphi}=\left(A^{2m-j-1/2}u^{(j)}(0)\right)_{j=0}^{m-1}$. We can easily verify that $\mu_{j,m}(0)$ is a positive operator. Then $\lambda_1(0)$ is east is the least eigen value of the matrix $\mu_{j,m}(0)$ is positive as well. There fore, for each $\beta>0$ $\lambda_1(\beta)>0$. For estimating χ_j we consider two cases.

1) If $\chi_j > b_j$, then $\chi_j^{-2} \in \left[0, b_j^{-2}\right)$. Then from definition of χ_j it follows that for all $\left(\chi_j^{-2}; b_j^{-2}\right)$ there exists a vector-function $u_\beta\left(\tau\right) \in \overset{\circ}{W}{}_2^{2m}\left(\alpha\right)$ such that

$$\left\|P_0\left(d/d\tau\right)u\left(\tau\right)\right\|_\alpha^2 < \beta\left\|A^{2m-j}u^{(j)}\right\|_\alpha^2.$$

Then it follows from the equality (45) that

$$\frac{1}{2}\left\|\phi_j\left(d/dt;\beta;A\right)u_\alpha\right\|_{L_2(R_+;H)}^2 + \frac{1}{2}\left\|\phi_j\left(d/dt;\beta;A\right)u_{-\alpha}\right\|_{L_2(R_+;H)}^2 + \left(\mu_{j,m}\left(\beta\right)\tilde{\varphi}_\beta, \tilde{\varphi}_\beta\right) < 0,$$

i.e.$\left(\mu_{j,m}\left(\beta\right)\tilde{\varphi}, \tilde{\varphi}\right) < 0$. Thus, for $\beta \in \left(\chi_j^{-2}, b_j^{-2}\right)$ the last eigen value $\lambda_1\left(\beta\right) < 0$. It follows from the continuity of $\lambda_1\left(\beta\right)$ that the function $\lambda_1\left(\beta\right)$ vanisnes of some points $\left[0, b_j^{-2}\right)$, i.e. at the points $\det\mu_{j,m}\left(\beta\right) = 0$. Thus, in this case the equation $\det\mu_{j,m}\left(\beta\right) = 0$ has a solution from the interval $\left[0, b_j^{-2}\right)$. Since the last of them is the number μ_α, then $\chi_j^{-2} \geq \mu_\alpha$. Thus, for $u\left(\tau\right) \in \overset{\circ}{W}{}_2^{2m}\left(\alpha\right)$

$$\left\|A^{2m-j}u^{(j)}\left(\tau\right)\right\|_\alpha \leq \mu_\alpha^{-1/2}\left\|P_0u\left(\tau\right)\right\|_\alpha$$

2) Let $\chi_j < b_j$. Then of the equation $\det\mu_{j,m}\left(\beta\right) = 0$ has a solution from the interval $\left[0, b_j^{-2}\right)$, then $\chi_j \leq b_j < \mu_j^{-1/2}$. But if $\det\mu_{j,m}\left(\beta\right) \neq 0$ for $\beta \in \left[0, b_j^{-2}\right)$, then $\mu_{j,m}\left(\beta\right)$ is positive for all $\beta \in \left[0, b_j^{-2}\right)$. Therefore it follows from the equality (45) in this case that for all $u\left(\tau\right) \in \overset{\circ}{W}{}_2^{2m}\left(\alpha\right)$ and $\beta \in \left[0, b_j^{-2}\right)$

$$\left\|P_0\left(d/d\tau\right)u\left(\tau\right)\right\|_\alpha^2 - \beta\left\|A^{2m-j}u^{(j)}\left(\tau\right)\right\|_\alpha^2 > 0.$$

Passing to limit as $\beta \to b_j^{-2}$ we have

$$\left\|A^{2m-j}u^{(j)}\left(\tau\right)\right\|_\alpha \leq \beta\left\|P_0u\left(\tau\right)\right\|_\alpha,$$

i.e. $\chi_j \leq b_j$. Thus, for all $u\left(\tau\right) \in \overset{\circ}{W}{}_2^{2m}\left(\alpha\right)$ it holds the inequality

$$\left\|A^{2m-j}u^{(j)}\left(\tau\right)\right\|_\alpha \leq \chi_j\left\|P_0u\left(\tau\right)\right\|_\alpha, \quad j = \overline{1, 2m-1},$$

where χ_j is determined from the condition of the theorem. Now we look for the solution of the boundary value problem (38), (39) in the form of the equation

$$P_0u\left(\tau\right) + P_1u\left(\tau\right) = f\left(\tau\right), \quad u\left(\tau\right) \in \overset{\circ}{W}{}_2^{2m}\left(\alpha\right), \quad f\left(\tau\right) \in H_2\left(\alpha\right)$$

where

$$P_0 u(\tau) \equiv \left(-\frac{d^2}{d\tau^2} + A^2\right)^m u(\tau), \qquad P_1 u(\tau) \equiv \sum_{j=1}^{2m-1} A_{2m-j} u^{(j)}(\tau).$$

By lemma 2.1 the operator P_0^{-1} exists and is bounded. Then after substitution $P_0 u(\tau) = \upsilon(\tau)$ we obtain a new equation in $H_2(\alpha)$

$$\upsilon(\tau) + P_1 P_0^{-1} \upsilon(\tau) = f(\tau)$$

or

$$\left(E + P_1 P_0^{-1}\right) \upsilon(\tau) = f(\tau).$$

Since

$$\left\| P_1 P_0^{-1} \upsilon \right\|_\alpha = \|P_1 u\|_\alpha = \left\| \sum_{j=1}^{2m-1} A_{2m-j} u^{(j)} \right\|_\alpha \le \sum_{j=1}^{2m-1} \|B_{2m-j}\| \times$$

$$\left\| A_{2m-j} u^{(j)} \right\|_\alpha \le \sum_{j=1}^{2m-1} \|B_{2m-j}\| \chi_j \|P_0 u\|_\alpha = \sigma \|P_0 u\|_\alpha = \sigma \|\upsilon\|_\alpha,$$

and by the condition of the theorem $\sigma < 1$, the operator $\left(E + P_1 P_0^{-1}\right)$ is invertible in $H_2(\alpha)$. Hence we find

$$u(\tau) = P_0^{-1} \left(E + P_1 P_0^{-1}\right) f(\tau)$$

where it follows that

$$\||u|\|_\alpha \le const \|f\|_\alpha.$$

The theorem is proved.

§2.2. On m-fold completeness of eigen and adjoint vectors of a class of polynomial operator bundles of higher order

2.1 Introduction and problem statement

Let H be a separable Hilbert space, A be a self-adjoint positive definite operator in H with completely continuous inverse A^{-1}. Let's denote by H_γ a Hilbert scale generated by the operator A. Let $S_\alpha = \{\lambda / |\arg \lambda| < \alpha\}$, $0 < \alpha < \pi/2$ be some sector from a complex plane, and $\tilde{S}_\alpha = \left\{\lambda / |\arg \lambda - \pi| < \frac{\pi}{2} - \alpha\right\}$. In the given paper shall search m-fold completeness of eigen and adjoint vectors of the bundle

$$P(\lambda) = \left(-\lambda^2 E + A^2\right)^m + \sum_{j=0}^{2m-1} \lambda^j A_{2m-j} \qquad (46)$$

corresponding to eigen values from the sector $\tilde{S}_\alpha$. To this end we introduce some notation and denotation. In the sequel, we shall assume the fulfillment of the following conditions: 1) A is a self-adjoint positive definite operator; 2) A^{-1} is a completely continuous operator; 3) The operators $B_j = A_j A^{-j}$, $j = \overline{1, 2m}$ are bounded in H.

Denote by $L_2(R_+; H)$ a Hilbert space whose elements $u(t)$ are measurable and integrable in the sense of Bochner, i.e.

$$L_2(R_+; H) = \left\{ u(t) / \|u(t)\|_{L_2(R_+:H)} = \left(\int_0^\infty \|u(t)\|_H^2 dt \right)^{1/2} < \infty \right\}.$$

Let $H_2(\alpha; H)$ be a liner set of holomorphic in $S_\alpha = \{\lambda / |\arg \lambda| < \lambda\}$ vector functions $u(z)$ for which

$$\sup_{|\varphi| < \alpha} \int_0^\infty \left\| u\left(te^{i\varphi}\right) \right\|^2 dt < \infty.$$

The elements of this set have boundary values in the sense of $L_2(R_+; H)$ and equal $u_\alpha(t) = u\left(te^{i\alpha}\right)$ and $u_{-\alpha}(t) = u\left(te^{-i\alpha}\right)$. This linear set turns into Hilbert space with respect to the norm

$$\|u(t)\|_\alpha = \frac{1}{\sqrt{2}} \left(\|u_\alpha(t)\|_{L_2(R_+:H)}^2 + \|u_{-\alpha}(t)\|_{L_2(R_+:H)}^2 \right)^{1/2}.$$

Denote by $W_2^{2m}(\alpha; H)$ a Hilbert space

$$W_2^{2m}(\alpha; H) = \left\{ u(z) / u^{(2m)}(z) \in H_2(\alpha; H), \quad A^{2m} u(z) \in H_2(\alpha; H) \right\}$$

with norm

$$|||u|||_\alpha = \left(\left\| u^{(2m)} \right\|_\alpha^2 + \left\| A^{2m} u \right\|_\alpha^2 \right)^{1/2}.$$

Bind a bundle $P(\lambda)$ from the equality (46) with the following initial value problem:

$$P(d/dz)\, u(z) = 0, \tag{47}$$

$$u^{(j)}(0) = \psi_j, \quad j = \overline{0, m-1}, \tag{48}$$

where we'll understand (48) in the sense

$$\lim_{\substack{z \to 0 \\ |\arg \lambda| < \alpha}} \left\| u^{(j)}(z) \right\|_{2m-j-1/2} = 0.$$

2.2. Some auxiliary facts

Definition 2.1. *If for any $\psi_j \in H_{2m-j-1/2}$ $(j = \overline{0, m-1})$ there exists a vector-function $u(z) \in W_2^{2m}(\alpha; H)$ satisfying equation (47) in S_α identically and inequality*

$$|||u|||_\alpha \leq const \sum_{j=0}^{m-1} \|\psi_j\|_{2m-j-1/2}$$

they say that problem (47), (48) is regularly solvable and $u(z)$ will be called a regular solution of problem (47), (48).

Let

$$\varphi_0(\lambda) = \left(-\lambda^2 e^{2i\alpha} + 1\right)^4 = \sum_{k=1}^{2m} c_k \lambda^k$$

$$\psi(\lambda,\beta)=\varphi_0(\lambda)\,\varphi_0(-\lambda)-\beta\,(i\lambda)^{2j}\,,\quad \beta\in\left[0,b_j^{-2}\right).$$

then

$$\psi(\lambda,\beta)=F\left(\overline{\lambda};\beta\right)F(-\lambda;\beta)$$

moreover, $F(\lambda;\beta)$has roots in the left half-plane and is of the form

$$F(\lambda;\beta)=\sum_{k=1}^{2m}\alpha_k(\beta)\,\lambda^k.$$

Denote by $M_{j,m}(\beta)$ a matrix obtained by $M_j(\beta)$ by rejecting m first rows and m first columns, where

$$M_j(\beta)=\frac{1}{2}\left[\tilde{u}_\alpha^{-1}S_j(\beta)\,\tilde{u}_\alpha+\tilde{u}_\alpha S_j(\beta)\,\tilde{u}_\alpha^{-1}\right],$$

$$\tilde{u}_\alpha\equiv diag\left(1,e^{i\alpha},e^{2i\alpha},...,e^{i(m-1)\alpha}\right),\quad \beta\in\left[0,b_j^{-2}\right],$$

$$b_j=\sup_{\xi\in R}\left|\frac{\xi^j}{\left(\xi^4+1+2\xi^2\cos 2\alpha\right)^{m/2}}\right|$$

and

$$S_j(\beta)=R_j(\beta)-T,$$

moreover $R_j(\beta)=(r_{pq,j}(\beta))$, $T=(t_{pq})$. For $p>q$

$$r_{pq,j}(\beta)=\sum_{\nu=0}^{\infty}(-1)^\nu\,\overline{\alpha_{p+\upsilon,j}(\beta)}\alpha_{q-\nu-1}(\beta)\quad(\alpha_\upsilon=0,\ \ \upsilon<0,\ \ \upsilon>2m)$$

$$t_{pq,j}(\beta)=\sum_{\nu=0}^{\infty}(-1)^\nu\,\overline{c_{p+\upsilon,j}(\beta)}c_{q-\nu-1}(\beta)\quad(c_\upsilon=0,\ \ \upsilon<0,\ \ \upsilon>2m)$$

for $p=q$

$$r_{pq,j}(\beta)=\sum_{\nu=0}^{\infty}(-1)^\nu\,Re\overline{\alpha_{p+\upsilon,j}(\beta)}\alpha_{q-\nu-1}(\beta)$$

$$t_{pq,j}\left(\beta\right)=\sum_{\nu=0}^{\infty}\left(-1\right)^{\nu}Re\overline{c_{p+\upsilon,j}\left(\beta\right)}c_{q-\nu-1}\left(\beta\right)$$

and for $p<q$

$$r_{pq,j}\left(\beta\right)=\overline{r_{pq,j}\left(\beta\right)},\quad t_{pq,j}\left(\beta\right)=\overline{t_{pq,j}\left(\beta\right)}$$

moreover $\alpha_k\left(\beta\right)$ and c_k are the coefficients of the polynomial $\varphi_0\left(\lambda\right)$ and $F\left(\lambda;\beta\right)$. The following theorem is obtained from the results of the papers [2] and [5].

Theorem 2.1. *Let A be a self-adjoint, positive definite operator, the operators $B_j=A_jA^{-j}\quad\left(j=\overline{1,2m}\right)$ be bounded in H and the inequality*

$$\sum_{j=1}^{2m}\chi_j\|B_j\|<1,$$

be fulfilled, where the numbers χ are determined as

$$\chi_j\geq\begin{cases}b_j, & \text{if }\det M_{j,m}\left(\beta\right)\neq 0\\ \mu_j^{-1/2}, & \text{in otherwise}\end{cases}\tag{49}$$

Here $\mu_j^{-1/2}$ is the least root of the equation$M_{j,m}\left(\beta\right)=0$. Then problem (47)-(48) is regularly solvable.

In order to study m- fold completeness of a system of eigen and self-adjoint vectors corresponding to eigen-values from the sector $\tilde{S}_\alpha$, we have to investigate some analytic properties of the resolvent.

Definition 2.2. *Let $K\left(\tilde{S}_\alpha\right)$ be a system of eigen and adjoint vectors corresponding to eigen values from the sector $\tilde{S}_\alpha$. If for any collection of m vectors from the holomorphy of the vector-function*

$$R\left(\lambda\right)=\sum_{j=0}^{2m-1}\left(A^{2m-j-1/2}p^{-1}\left(\lambda\right)\right)^{*}A^{2m-j-1/2}\lambda^j\psi^j$$

in the sector $\tilde{S}_\alpha$ it yields that $K\left(\tilde{S}_\alpha\right)$ in strongly m-fold complete in H.

Note that this definition is a prime generalization of m-hold completeness of the system in H in the sense of M.V. Keldysh and in fact it means that the derivatives of the chain in the sense of M.V. Keldysh are complete in the space of traces $\tilde{H} = \mathop{\oplus}\limits_{j=0}^{m-1} H_{m-j-1/2}$.

It holds the following theorem on an analytic property of the resolvent.

Theorem 2.2. *Let the conditions 1)-3) be fulfilled and it holds the inequality*

$$\sum_{j=0}^{2m-1} b_j \|B_{2m-j}\| < 1,$$

where

$$b_j = \sup_{\xi \in R} \left(\frac{\xi^{2j/m}}{1 + \xi^4 + 2\xi^2 \cos 2\alpha} \right)^{m/2}, \quad j = \overline{0, 2m}$$

Besides, let one of the following conditions be fulfilled:

a)$A^{-1} \in \sigma_p \quad (0 < p < \pi / (\pi - 2\alpha))$;

b) $A^{-1} \in \sigma_p \quad (0 < p < \infty)$, $\quad B_j \quad \left(j = \overline{0, 2m}\right)$ *are completely continuous operators in* H

Then the resolvent of the operator pencil $p(\lambda)$ *posseses the following properties:*

1) $A^{2m} p^{-1}(\lambda)$ *is represented in the form of relation of two entire functions of order not higher than* p *and has a minimal type order* p*;*

2) there exists a system $\{\Omega\}$ *of rays from the sector* $\tilde{S}_\alpha$ *where the rays*

$$\Gamma_{\frac{\pi}{2}+\alpha} = \left\{ \lambda / \arg \lambda = \frac{\pi}{2} + \alpha \right\}, \quad \Gamma_{\frac{3\pi}{2}-\alpha} = \left\{ \lambda / \arg \lambda = \frac{3\pi}{2} - \alpha \right\},$$

are also contained, and the angle between the neighboring rays is no greater than π/p *and the estimation*

$$\left\| p^{-1}(\lambda) \right\| \leq const |\lambda|^{-2m}$$

$$\left\| A^{2m} p^{-1}(\lambda) \right\| \leq const$$

holds on these rays.

Proof. Since

$$P(\lambda) A^{-2m} = (E + B_{2m}) (E + T(\lambda)),$$

where

$$T(\lambda)=\sum_{j=1}^{2m} c_j A^{-j},$$

$$c_j=\begin{cases}(E+B_{2m})^{-1} B_j A^{j-2m}, & j=1,3,..,2m-1,2m\\ (E+B_{2m})^{-1}\left(B_j+(-1)^j\begin{pmatrix}2m\\ j\end{pmatrix}E\right)A^{-2j}, & j=2,4,..,2m-2.\end{cases}$$

Then applying Keldysh lemma [6], we get that

$$A^{2m}p^{-1}(\lambda)=(E+T(\lambda))^{-1}(E+B_{2m})^{-1}$$

is represented in the form of relation of two entire functions of order not higher than p and of minimal type at order p.

On the other hand

$$p(\lambda)=p_0(\lambda)+p_1(\lambda),$$

therefore

$$p^{-1}(\lambda)=p_0^{-1}(\lambda)\left(E+p_1(\lambda)\,p_0^{-1}(\lambda)\right)^{-1}$$

$$A^{2m}p^{-1}(\lambda)=A^{2m}p_0^{-1}(\lambda)\left(E+p_1(\lambda)\,p_0^{-1}(\lambda)\right)^{-1}.$$

Since by fulfilling the condition a) of the theorem we get on the rays $\Gamma_{\frac{\pi}{2}+\alpha}$ and $\Gamma_{\frac{3\pi}{2}-\alpha}$ (i.e. for $\lambda=re^{i\left(\frac{\pi}{2}+\alpha\right)}$, $\lambda=re^{i\left(\frac{3\pi}{2}-\alpha\right)}$)

$$\left\|p_1(\lambda)\,p_0^{-1}(\lambda)\right\|\le\sum_{j=0}^{2m-1}\|B_{2m-j}\|\left\|\lambda_j A^{2m-j}p_0^{-1}(\lambda)\right\|_{H\to H}, \tag{50}$$

therefore we should first estimate the norm

$$\left\|\lambda_j A^{2m-j}p_0^{-1}(\lambda)\right\|_{H\to H}$$

on the rays $\Gamma_{\frac{\pi}{2}+\alpha}$ and $\Gamma_{\frac{3\pi}{2}-\alpha}$. Let $\lambda=re^{i\left(\frac{\pi}{2}+\alpha\right)}\in\Gamma_{\frac{\pi}{2}+\alpha}$.

Then it follows from the spectral expansion of the operator A

$$\left\|\lambda_j A^{2m-j} p_0^{-1}(\lambda)\right\|_{H\to H} = \sup_{\mu\in\tau(A)}\left|\lambda_j \mu^{2m-j}\left(-\lambda^2+\mu^2\right)^{-m}\right| =$$

$$= \sup_{\mu\in\tau(A)}\left|r^j \mu^{2m-j}\left(r^2 e^{2i\alpha}+\mu^2\right)^{-m}\right| = \sup_{\mu\in\tau(A)}\left|r^j \mu^{2m-j}\left(r^4+\mu^4+2r^2\mu^2\cos 2\alpha\right)^{-\frac{m}{2}}\right| =$$

$$= \sup_{\mu\in\tau(A)}\left(\frac{r^{\frac{2j}{m}}\mu^{\frac{2(2m-j)}{m}}}{1+\left(\frac{r}{\mu}\right)^4+2\left(\frac{r}{\mu}\right)^2\cos 2\alpha}\right)^{\frac{m}{2}} = b_j.$$

So, we get from inequality (50)

$$\left\|p_1(\lambda)\,p_0^{-1}(\lambda)\right\| \le \sum_{j=0}^{2m-1}\|B_{2m-j}\| b_j < \gamma < 1.$$

Therefore on this ray

$$\left\|p^{-1}(\lambda)\right\| \le \left\|p_0^{-1}(\lambda)\right\| \left\|\left(E+p_1(\lambda)p_0^{-1}(\lambda)\right)^{-1}\right\| \le \left\|p_0^{-1}(\lambda)\right\|\frac{1}{\gamma}.$$

On the other hand, on the ray $\Gamma_{\frac{\pi}{2}+\alpha}$ it holds the estimation

$$\left\|p_0^{-1}(\lambda)\right\| = \left\|\left(-\lambda^2 E + A^2\right)^{-m}\right\| = \sup_{\mu\in\sigma(A)}\left|\frac{1}{\left(\lambda^4+\mu^4+2\lambda^2\mu^2\cos 2\alpha\right)^{\frac{m}{2}}}\right|.$$

If $\cos 2\alpha \ge 0 \quad \left(0\le\alpha\le\frac{\pi}{4}\right)$, then

$$\sup_{\mu\in\sigma(A)}\left|\frac{1}{\left(\lambda^4+\mu^4+2\lambda^2\mu^2\cos 2\alpha\right)^{\frac{m}{2}}}\right| \le$$

$$\sup_{\mu\in\sigma(A)}\left|\frac{1}{\left(\lambda^4+\mu^4\right)^{\frac{m}{2}}}\right| \le const|\lambda|^{-2m}.$$

It is analogously proved that on these rays

$$\left\|A^{2m}p^{-1}(\lambda)\right\| = const.$$

The theorem is proved.

2.3. The basic theorem

Theorem 2.3. *Let conditions 1)-3) be fulfilled and it holds*

$$\sum_{j=0}^{2m-1} \chi_j \|B_{2m-j}\| < 1,$$

where the numbers χ_j are determined from formula (49).

Besides, one of the conditions a) or b) of theorem 2.2 is fulfilled.

Then the system $K\left(\tilde{S}_\alpha\right)$ is strongly m-fold complete in H.

Proof. Prove the theorem by contradiction. Then there exist vectors $\psi_k \quad \left(k=\overline{0,m-1}\right) H_{2m-k-1/2}$ for which even if one of them differs from zero and

$$R(\lambda) = \sum_{j=0}^{m-1} \left(A^{2m-k-1/2} p^{-1}(\lambda)\right)^* \lambda^k A^{2m-j-1/2} \psi_k$$

is a holomorphic vector-function in the sector $\tilde{S}_\alpha$. By theorem 2.2 and Phragmen-Lindelof theorem in the sector $R(\lambda)$ the $\tilde{S}_\alpha$ has the estimation

$$\|R(\lambda)\| \leq const|\lambda|^{-1/2}.$$

On the other hand, by theorem 2.1. problem (38)-(39) has a unique regular solution $u(z)$ for any $\psi_k \in H_{2m-k-1/2}$. Denote by $\hat{u}(\lambda)$ its Laplace transformation. Then $u(z)$ is represented as

$$u(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{\pi}{2}+1}} \hat{u}(\lambda) e^{\lambda_z} d\lambda - \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{3\pi}{2}-\alpha}} \hat{u}(\lambda) e^{\lambda_z} d\lambda,$$

where $\hat{u}(\lambda) = p^{-1}(\lambda) g(\lambda)$ and $g(\lambda) = \sum\limits_{j=0}^{m-1} \lambda^{m-j} Q_j u^{(j)}(0)$, Q_j are some operators, obviously, for $t>0$

$$\sum_{k=0}^{m-1} \left(u^{(k)}(t), \psi_k\right)_{H_{2m-k-1/2}} = \frac{1}{2\pi i} \int\limits_{\Gamma_\alpha} \left(g(\lambda), R\left(\bar{\lambda}\right)\right) e^{\lambda t} dt.$$

Since the functions $\upsilon(\lambda) = \left(g(\lambda), R(\bar{\lambda})\right)_H$ is an entire function and on Γ_α it holds the estimation $\|\upsilon(\lambda)\| \leq c|\lambda|^{2m-1}$, then

$$\upsilon(\lambda) = \sum_{j=0}^{m-1} a_j \lambda^j.$$

Since

$$\int_{\Gamma_2} \upsilon(\lambda) e^{\lambda t} d\lambda = 0,$$

then for $t > 0$

$$\sum_{j=0}^{m-1} \left(u^{(k)}(t), \psi_k\right)_{H_{2m-k-1/2}} = 0.$$

Passing to the limit as $t \to 0$, we get that

$$\sum_{k=0}^{m-1} \|\psi_k\|_{H_{2m-k-1/2}} = 0,$$

i.e. all $\psi_k = 0$, $k = \overline{0, m-1}$. The theorem is proved.

§ 2.3. On the existence of ϕ-solvability of boundary value problems

In this section sufficient conditions are found for ϕ-solvability of boundary value problems for a class of higher order differential equations whose main part contains a multi charachteristic.

3.1. Introduction and problem statement

In the present paper, using the method in the paper [1] we study the existence of holomorphic solution of the operator-differential equations

$$P\left(\frac{d}{dz}\right)u(z) \equiv \left(-\frac{d^2}{dz^2}+A^2\right)^m u(z) + \sum_{j=1}^{2m-1} A_j u^{(2m-j)}(z), \quad z \in S_{(\alpha,\beta)}, \quad (51)$$

with initial-boundary conditions

$$u^{(S\nu)}(0)=0, \qquad \nu=\overline{0,m-1}, \quad (52)$$

where A is a positive-definite selfadjoint operator, A_j $(j=\overline{0,m-1})$ are linear operators in an abstract separable space H, $u(z)$ and $f(z)$ are $H-$valued holomorphic functions in the domain

$$S_{(\alpha,\beta)} = \{z/ \ -\beta < \arg z < \alpha\}, \quad 0 \le \alpha < \frac{\pi}{2}, \quad 0 \le \beta < \frac{\pi}{2}$$

and the integers s_ν $(\nu=\overline{0,m-1})$ satisfy the conditions

$$0 < s_0 < s_1 < \ldots < s_{m-1} \le m-1.$$

Let H be α separable Hilbert space, A be a positive definite selfadjoint operator in H, and H_γ be a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $\|x\|_\gamma = \|A^\gamma x\|$, $x \in D(A^\gamma)$, $\gamma \ge 0$. Denote by $L_2(R_+ : H)$ Hilbert space of vector-functions $f(t)$ with values from H, defined in $R_+ = (0,+\infty)$, measurable, and for which

$$\|f\|_{L_2(R_+:H)} = \left(\int_0^\infty \|f(t)\|^2 \, dt\right)^{\frac{1}{2}} < \infty.$$

Then, denote by $H_2(\alpha,\beta:H)$ a set of vector-functions $f(z)$ with values from H, that are holomorphic in the sector $S_{(\alpha,\beta)}=\{z/\ -\beta<\arg z<\alpha\}$ and for any $\varphi\in[-\beta,\alpha]$ of the function $f(\xi e^{i\varphi})\in L_2(R_+:H)$. Note that for the vector-function $f(z)$ there exist boundary values $f_{-\beta}(\xi)=f(\xi e^{-i\beta})$ and $f_\alpha(\xi)=f(\xi e^{i\alpha})$ from the space $L_2(R_+:H)$ and we can reestablish the vector-function $f(z)$ with their help by Cauchy formula

$$f(z)=\frac{1}{2\pi i}\int\limits_0^\infty\frac{f_{-\beta}(\xi)}{\xi e^{-i\beta}-z}e^{-i\beta}d\xi-\frac{1}{2\pi i}\int\limits_0^\infty\frac{f_\alpha(\xi)}{\xi e^{i\alpha}-z}e^{i\alpha}d\xi.$$

The linear set $H_2(\alpha,\beta:H)$ becomes a Hilbert space with respect to the norm [4]

$$\|f\|_{(\alpha,\beta)}=\frac{1}{\sqrt{2}}\left(\|f_{-\beta}\|^2_{L_2(R_+:H)}+\|f_\alpha\|^2_{L_2(R_+:H)}\right)^{\frac{1}{2}}.$$

Now, define the space $W_2^{2m}(\alpha,\beta:H)$

$$W_2^{2m}(\alpha,\beta:H)=\left\{u/\ A^{2m}u\in H_2(\alpha,\beta:H),\quad u^{(2m)}\in H_2(\alpha,\beta:H),\right\}$$

with norm

$$\||u|\|_{(\alpha,\beta)}=\left(\left\|\ A^{2m}u\right\|^2_{(\alpha,\beta)}+\left\|u^{(2m)}\right\|^2_{(\alpha,\beta)}\right)^{\frac{1}{2}}.$$

Here and in the sequel, the derivatives are understood in the sense of complex analisys in abstract spaces ([7]).

Definition 3.1. *The vector-function $u(z)\in W_2^{2m}(\alpha,\beta:H)$ is said to be a regular solution of problem (51),(52), if $u(z)$ satisfies equation (51) in $S_{(\alpha,\beta)}$ identically and boundary conditions are fulfilled in the sense*

$$\lim_{\substack{z\to0\\-\beta<\arg z<\alpha}}\left\|u^{(s_j)}(z)\right\|_{2m-s_j-\frac{1}{2}}=0,\qquad j=\overline{0,m-1}.$$

Definition 3.2. *Problem (51),(52) is said to be ϕ-solvable, if for any $f(z)\in H_1\subset H_2(\alpha,\beta:H)$ there exists $u(z)\in W_1\subset W_2^{2m}(\alpha,\beta:H)$, which is a regular solution of boundary value problem (51),(52) and satisfies the inequality*

$$\||u|\|_{(\alpha,\beta)}\leq const\,\|f\|_{(\alpha,\beta)},$$

moreover, the spaces H_1 and W_1 have finite-dimensional orthogonal complements in the spaces $H_2(\alpha,\beta:H)$ and $W_2^{2m}(\alpha,\beta:H)$, respectively.

In the present paper we study the ϕ-solvability of problem (51),(52). The similar problem was investigated in general form in [1], when the principal part doesn't contain a multiple characteric. In the author's paper [5] for $\alpha=\beta=\pi/4$ the one valued and correct solvability conditions of problem (51),(52) are found in the case when the principal part of equation (51) is biharmonic. For simplicity we consider equation (51) with boundary conditions

$$u^{(j)}(0)=0, \qquad j=\overline{0,m-1}. \tag{53}$$

The general case is considered similarly.

3.2. Some auxiliary facts

At first, let's prove some lemmas.

Lemma 3.1. *The boundary-value problem*

$$P_0\left(\frac{d}{dz}\right)u(z)\equiv\left(-\frac{d^2}{dz^2}+A^2\right)^m u(z)=v(z), \qquad z\in S_{(\alpha,\beta)} \tag{54}$$

$$u^{(j)}(0)=0, \qquad j=\overline{0,m-1} \tag{55}$$

is regularly solvable.

Proof. It is easily seen that ([1]) the vector function

$$u_0(z)=\frac{1}{2\pi i}\int\limits_{\Gamma}P_0^{-1}(\lambda)\widehat{v}(\lambda)e^{\lambda z}d\lambda \tag{56}$$

satisfies equation (54) identically in $S_{(\alpha,\beta)}$ where $\widehat{v}(\lambda)$ is a Laplace transform of the vector-function $v(z)$:

$$\widehat{v}(z)=\int\limits_0^\infty v(t)e^{-\lambda t}dt,$$

that is an analytic vector-function in the domain

$$\widetilde{S}_{(\alpha,\beta)}=\left\{\lambda/\ -\frac{\pi}{2}-\alpha<\arg\lambda<\frac{\pi}{2}+\beta\right\}$$

and for $\lambda \in \widetilde{S}_{(\alpha,\beta)}$

$$\|\widehat{v}(\lambda)\| \to 0, \qquad |\lambda| \to \infty, \quad ([8])$$

in formula (56) the integration contour $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \left\{\lambda / \ \arg\lambda = \frac{\pi}{2} + \beta\right\}$, $\Gamma_2 = \left\{\lambda / \ \arg\lambda = -\frac{\pi}{2} - \alpha\right\}$. Thus,

$$u_0(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_1} P_0^{-1}(\lambda)\widehat{v}(\lambda)e^{\lambda z}d\lambda - \frac{1}{2\pi i}\int\limits_{\Gamma_2} P_0^{-1}(\lambda)\widehat{v}(\lambda)e^{\lambda z}d\lambda, \qquad z \in S_{(\alpha,\beta)}.$$

On the other hand, it is easy to check that on the rays Γ_1 and Γ_2 it holds the estimate

$$\left\|\lambda^{2m}P_0^{-1}(\lambda)\right\| + \left\|A^{2m}P_0^{-1}(\lambda)\right\| \le const.$$

Then using the analogies of Plancherel formula for a Laplace transform we get $u_0(z) \in W_2^{2m}(\alpha,\beta : H)$. Further, we seek a general regular solution of the equation in the form

$$u(z) = u_0(z) + \sum_{p=0}^{m-1} (zA)^p \, e^{-zA} C_p, \tag{57}$$

where $C_p \in H_{2m-\frac{1}{2}}$, and e^{-zA} is a holomorphic in $S_{(\alpha,\beta)}$ group of bounded operators generated by the operator $(-A)$. Now, let's define the vectors C_p $(p = \overline{0, m-1})$ from condition (55). Then, obviously, for the vectors C_p $(p = \overline{0, m-1})$ we get the following system of equations:

$$\begin{cases} c_0 = -u_0(0), \\ -c_0 + c_1 = -A^{-1}u'(0), \\ c_0 - 2c_1 + 2c_2 = -A^{-2}u''(0), \\ \dots\dots\dots\dots\dots\dots\dots \\ (-1)^{m-1}c_0 + (-1)^{m-2}\begin{pmatrix} 1 \\ m-1 \end{pmatrix} c_1 + \dots + c_{m-1} = -A^{-m+1}u^{(m-1)}(0). \end{cases}$$

It is evident that the main matrix differs from zero, since it is triangle. Therefore, we can define all the vectors C_p $(p = \overline{0, m-1})$ in a unique way. On the other hand, $u_0^{(j)}(z) \in H_{2m-j-\frac{1}{2}}$, since $u_0^{(j)}(z) \in W_2^{2m}(\alpha, \beta : H)$, therefore the vectors $C_p \in H_{2m-j-\frac{1}{2}}$.

Thus,

$$u(z) = u_0(z) + \sum_{p=0}^{m-1} (zA)^p e^{-zA} \sum_{q=0}^{p} \alpha_{pq} A^{-q} u_0^{(q)}(0). \tag{58}$$

The form $u(z)$ and the trace theorem implies that the inequality

$$\|u\|_{W_2^{2m}(\alpha,\beta:H)} \leq const\, \|v\|_{H_2(\alpha,\beta:H)}$$

holds. The lemma is proved.

For the further study we transform the form of the vector function $u_0(z)$. From formula (56) after simple transformations we get

$$u_0(z) = \int_0^{\infty} \left(\frac{1}{2\pi i} \int_0^{i\infty} P_0^{-1}(\lambda e^{i\beta}) e^{\lambda(ze^{i\beta} - \xi)} d\lambda \right) v_{-\beta}(\xi) d\xi -$$

$$- \int_0^{\infty} \left(\frac{1}{2\pi i} \int_0^{-i\infty} P_0^{-1}(\lambda e^{-i\alpha}) e^{\lambda(ze^{-i\alpha} - \xi)} d\lambda \right) v_{\alpha}(\xi) d\xi =$$

$$= \int_0^{\infty} G_1(ze^{i\beta} - \xi) v_{-\beta}(\xi) d\xi - \int_0^{\infty} G_2(ze^{-i\alpha} - \xi) v_{\alpha}(\xi) d\xi, \tag{59}$$

where

$$v_{\alpha}(t) = v(te^{i\alpha}), \qquad v_{\beta}(t) = v(te^{-i\beta})$$

and

$$\left. \begin{aligned} G_1(s) &= \tfrac{1}{2\pi i} \int_0^{i\infty} P_0^{-1}(\lambda e^{i\beta}) e^{\lambda s} d\lambda \\ G_1(s) &= \tfrac{1}{2\pi i} \int_0^{-i\infty} P_0^{-1}(\lambda e^{-i\alpha}) e^{\lambda s} d\lambda \end{aligned} \right\} . \tag{60}$$

Now, let's prove the main result of the paper.

3.3. The basic results

Theorem 3.1. *Let A be a positive self-adjoint operator with completely continuous inverse A^{-1}. The resolvent $P^{-1}(\lambda)$ exist on the rays $\Gamma_1 = \left\{\lambda / \ \arg\lambda = \frac{\pi}{2} + \beta\right\}$, $\Gamma_2 = \left\{\lambda / \ \arg\lambda = -\frac{\pi}{2} - \alpha\right\}$ and be iniformly bounded, the operators $B_j = A_j \times A^{-j}$ $(j = \overline{1, 2m-1})$ be completely continuous in H. Then, problem (51),(53) is ϕ-solvable.*

Proof. Write $P(d/dz)$ in the form

$$P(d/dz)u(z) = P_0(d/dz)u(z) + P_1(d/dz)u(z),$$

where

$$P_0(d/dz)u(z) = \left(-\frac{d^2}{dz^2} + A^2\right)^m u(z),$$

$$P_1(d/dz)u(z) = \sum_{j=1}^{2m-1} A_j u^{(2m-j)}(z).$$

Having applied the operator $P(d/dz)$ to both sides of equality (58) we get

$$v(z) = P_0(d/dz)u_0(z) + P_1(d/dz)\sum_{p=0}^{m-1}(zA)^p e^{-zA}\sum_{q=0}^{p}\alpha_{pq}A^{-q}u_0^{(q)}(0). \qquad (61)$$

Passing in equality (61) to the limit as $z \to te^{i\alpha}$ and $z \to te^{-i\beta}$ $(t \in R_+ = = (0,\infty))$ and using for $u_0^{(q)}(0)$ the expressions found from equality (59) allowing for (60) we get the following system of integral equations in the space $L_2(R_+ : H)$

$$\left.\begin{array}{l} v_\alpha(t) + \int\limits_0^\infty \left(K_2(t-\xi) + K_4(te^{i\alpha},\xi)\right) v_\alpha(\xi)d\xi + \\ + \int\limits_0^\infty \left(K_1(te^{i(\alpha+\beta)} - \xi) + K_3(te^{i\alpha},\xi)\right) v_{-\beta}(\xi)d\xi = f_\alpha(t) \\ v_{-\beta}(t) + \int\limits_0^\infty \left(K_1(t-\xi) + K_3(te^{-i\beta},\xi)\right) v_{-\beta}(\xi)d\xi + \\ + \int\limits_0^\infty \left(K_2(te^{-i(\alpha+\beta)} - \xi) + K_4(te^{-i\beta} - \xi)\right) v_\alpha(\xi)d\xi = f_{-\beta}(t) \end{array}\right\} \qquad (62)$$

where

$$K_1(te^{i\beta} - \xi) = P_1(e^{i\beta}d/dt)G_1(te^{i\beta} - \xi);$$

$$K_2(te^{-i\alpha}-\xi)=P_1(e^{-i\alpha}\frac{d}{dt})G_2(te^{-i\alpha}-\xi);$$

$$K_3(t,\xi)=-P_1(e^{i\beta}d/dt)\sum_{p=0}^{m-1}(te^{-i\beta}A)^p e^{-te^{-i\beta}A}\sum_{q=0}^{p}\alpha_{pq}A^{-q}G_1^{(q)}(-\xi),$$

$$K_4(t,\xi)=P_1(e^{-i\alpha}d/dt)\sum_{p=0}^{m-1}(te^{-i\beta}A)^p e^{-te^{i\alpha}A}\sum_{q=0}^{p}\alpha_{pq}A^{-q}G_2^{(q)}(-\xi).$$

Since the operator $P_0(d/dz)$ maps isomorphically the domain $\overset{0}{W}{}_2^{2m}(\alpha,\beta:H)$ onto $H_2(\alpha,\beta:H)$ where $\overset{0}{W}{}_2^{2m}(\alpha,\beta:H)=\{u(z)/\ u(z)\in W_2^{2m}(\alpha,\beta:H),$ $u^{(j)}(0)=0,\ j=\overline{0,m-1}\}$, then the ϕ-solvability of problem (51),(53) is equivalent to the ϕ-solvability of a system of integral equations (61) in $L_2(R_+:H)$. Therefore, we study the ϕ-solvability of the system of integral equations (61) in $L_2(R_+:H)$. Since $P^{-1}(\lambda)$ exists on the rays Γ_1 and Γ_2, then each equation

$$\widetilde{v}(t)+\int_{-\infty}^{+\infty}K_j(t-\xi)\widetilde{v}(\xi)d\xi=\widetilde{f}(\xi),\qquad j=1,2$$

is correctly and uniquely solvable in the space

$$L_2(R:H)=L_2(R:H)\oplus L_2(R:H)$$

where $\widetilde{f}(t)\in L_2(R:H)$, $\widetilde{v}(t)\in L_2(R:H)$. Therefore, for ϕ-solvability of the system of integral equations,

$$\left.\begin{array}{r}v_\alpha(t)+\int_0^{+\infty}K_2(t-\xi)v_\alpha(\xi)d\xi=f_\alpha(\xi)\\ v_{-\beta}(t)+\int_0^{+\infty}K_1(t-\xi)v_{-\beta}(\xi)d\xi=f_{-\beta}(\xi)\end{array}\right\}$$

in the space $L_2(R_+:H)$ it suffices to prove that the kernels $K_1(t+\xi)$ and $K_2(t+\xi)$ generate completely continuous operators in $L_2(R:H)$. Then, to prove the ϕ-solvability of the system of integral equations (61) in the space $L_2(R_+:H)$, we have to prove that the kernels $K_1(te^{i(\alpha+\beta)}-\xi)$, $K_2(te^{-i(\alpha+\beta)}-\xi)$, $K_3(te^{i\alpha},\xi)$, $K_4(te^{i\alpha},\xi)$, $K_3(te^{-i\beta},\xi)$, $K_4(te^{-i\beta},\xi)$ also generate completely continuous operators in $L_2(R_+:H)$. The proof of complete continuity of these operators is similar. Therefore, following [1] we shall prove the complete continuity of the operator generated by the kernel $K_1(t+\xi)$. Since

$$K_1(t+\xi) = \sum_{j=1}^{2m-1} A_{2m-j} e^{ij\beta} \frac{d^j}{dt^j} \left(\frac{1}{2\pi i} \int_0^{i\infty} (-\lambda^2 e^{2i\beta} E + A^2)^{-m} e^{\lambda(t+\xi)} d\lambda \right),$$

and taking into account that for $\lambda \in (0, i\infty)$ and for sufficiently small sector adherent to the axis $i\infty$ it holds the estimation

$$\left\| (-\lambda^2 e^{2i\beta} E + A^2)^{-m} \right\| \leq const(1+|\lambda|)^{-2m},$$

we can represent $K_1(t+\xi)$ in the form

$$K_1(t+\xi) = \sum_{j=1}^{2m-1} A_{2m-j} e^{ij\beta} \frac{d^j}{dt^j} \left(\frac{1}{2\pi i} \int_0^{(i-\xi)\infty} (-\lambda^2 e^{2i\beta} E + A^2)^{-m} e^{\lambda(t+\xi)} \right) d\lambda =$$

$$= \sum_{j=1}^{2m-1} \frac{B_{2m-j}}{2\pi i} \int_0^{(i-\xi)\infty} \lambda^{2m-j} A^j (-\lambda^2 e^{2i\beta} E + A^2)^{-m} e^{\lambda(t+\xi)} d\lambda$$

$$\equiv \frac{1}{2\pi i} \sum_{j=1}^{2m-1} B_{2m-j} K_{1,j}(t+\xi),$$

where $\varepsilon > 0$ is a sufficiently small number, and

$$K_{1,j}(t+\xi) = \int_{t,\varepsilon>0}^{(i-\xi)\infty} \lambda^{2m-j} A^j (-\lambda^2 e^{2i\beta} E + A^2)^{-m} e^{\lambda(t+\xi)} d\lambda.$$

Then

$$\|K_{1,j}(t+\xi)\|_{H\to H} = \left\| \int_0^{(i-\xi)\infty} \lambda^{2m-j} A^j (-\lambda^2 e^{2i\beta} E + A^2)^{-m} e^{\lambda(t+\xi)} d\lambda \right\| =$$

$$= \left\| \int_0^{\infty} (i-\varepsilon)^{2m+1-j} \lambda^{2m-j} A^j (-\lambda^2 (i-\varepsilon)^2 e^{2i\beta} E + A^2)^{-m} e^{-\varepsilon\lambda(t+\xi)} e^{i\lambda(t+\xi)} d\lambda \right\| \leq$$

$$\leq |(i-\varepsilon)| \int\limits_0^\infty \left\| ((i-\varepsilon)\lambda)^{2m-j} A^j (-\lambda^2 (i-\varepsilon)^2 e^{2i\beta} E + A^2)^{-m} e^{-\varepsilon\lambda(t+\xi)} \right\| d(\lambda\varepsilon) \leq$$

$$\leq C_\varepsilon \int\limits_0^\infty e^{-\varepsilon\lambda(t+\xi)} d(\lambda\varepsilon) \leq \frac{C_\varepsilon}{t+\varepsilon}.$$

Using Hilbert's inequality [9] we get from the last inequality that $K_{1,j}(t+\xi)$ generates a continuous operator in $L_2(R_+ : H)$. To prove the complete continuity of the operator generated by the operator $B_{2m-j}K_{1,j}(t+\xi)$ we act as follows. Let $\{e_n\}$ be an orthonormal system of eigen vector of the operator A responding to $\{\mu_n\} : Ae_n = \mu_n e_n,\ 0 < \mu_1 < \ldots < \mu_n < \ldots$ and let $L_m = \sum\limits_{i=1}^{m} (\cdot, e_i) e_i$ be an orthogonal projector on a sub-space generated by the first m vectors. Since B_{2m-j} is a completely continuous operator, then as $m \to \infty$

$$\|Q_{m,j}\|_{H \to H} = \|B_j - B_j L_m\|_{H \to H} \to 0.$$

On the other hand

$$\|B_j L_m K_{1,j}(t+\xi)\| =$$

$$= \left\| \sum_{n=1}^{m} \int\limits_0^{(i-\xi)\lambda} \lambda^{2m-j} (i-\varepsilon)^{2m+1-j} \mu_n^j (-\lambda^2 (i-\varepsilon)^2 e^{2i\beta} + \mu_n^2)^{-m} (\cdot, e_n) B_j e_n e^{i\lambda(t+\xi)} e^{-\lambda\varepsilon(t+\xi)} d\lambda \right\| \leq$$

$$\leq C_\varepsilon \sum_{n=1}^{m} \int\limits_0^\infty \frac{|(\varepsilon\lambda)^{2n-j}| \mu_n^j}{|-\lambda^2 (i-\varepsilon)^2 e^{2i\beta} + \mu_n^2|} e^{-\varepsilon\lambda(t+\xi)} d(\lambda\varepsilon) \leq C_\varepsilon(m) \int\limits_0^\infty \frac{\lambda^{2m-j}}{1+\lambda^{2m}} e^{-\lambda(t+\xi)} d\lambda.$$

Hence, it follows that the kernel $B_{2m-j} L_m K_{1,j}(t+\xi)$ generates a Hilbert-Schmidt operator, since for $j = \overline{1, 2m-1}$ the following inequality holds

$$\int\limits_0^\infty \int\limits_0^\infty \|B_j L_m K_{1,j}(t+\xi)\|^2 d\xi dt \leq$$

$$\leq \int\limits_0^\infty \int\limits_0^\infty \left(\int\limits_0^\infty \frac{\lambda^{2m-j}}{1+\lambda^{2m}} e^{-\lambda(t+\xi)} \int\limits_0^\infty \frac{s^{2m-j}}{1+s^{2ml}} e^{-s(t+\xi)} ds \right) dt d\xi =$$

$$= 2\int\limits_0^\infty\int\limits_0^\infty \frac{\lambda^{2m-j}s^{2m-j}}{(1+\lambda^{2m})(1+s^{2m})(1+s)^2}d\lambda ds \le 2\int\limits_0^\infty\int\limits_0^\infty \frac{\lambda^{2m-1-j}s^{2m-1-j}}{(1+\lambda^{2m})(1+s^{2m})}d\lambda ds =$$

$$= 2\int\limits_0^\infty \frac{\lambda^{2m-1-j}}{1+\lambda^{2m}}d\lambda \int\limits_0^\infty \frac{s^{2m-1-j}}{1+s^{2m}}ds < 0 \qquad (j = \overline{1, 2m-1}).$$

On the other hand

$$B_{2m-j}K_{1,j}(t+\xi) = Q_{m,j}K_{1,j}(t+\xi) + B_{2m-j}L_mK_{1,j}(t+\xi)$$

then the boundedness of the operator $\widetilde{K_{1,j}}$ generated by the kernel implies that the operator $\widetilde{T_{1,j}}$ generated by the kernel $B_{2n-j}K_{1,j}(t+\xi)$ is the limit of completely continuous operators $T_{1,j,m}$ generated by the kernels $B_jL_mK_{1,j}(t+\xi)$. In fact the difference operators

$$\left\|\widetilde{T_{1,j}} - T_{1,j,m}\right\|_{L_2(R_+:H)\to L_2(R_+:H)} \le \|Q_{m,j}\| \left\|\widetilde{K_{1,j}}\right\|_{L_2(R_+:H)\to L_2(R_+:H)} \to 0 \ \ (m\to\infty)$$

Thus, $B_{2m-j}K_{1,j}(t+\xi)$ generates a completely continuous operator in $L_2(R_+ : H)$. Since $K_1(t+\xi)$ generates a completely continuous operator in $L_2(R_+ : H)$. The theorem is proved.

§ 2.4. ON SOME PROPERTIES OF REQULAR HOLOMORPHIC SOLUTIONS OF CLASS OF HIGHER ORDER OPERATOR-DIFFERENTIAL EQUATIONS

In this section we give definition of regular holomorphic solutions of a class of higher order operator-differential equations and Phragmen-Lindelof type theorem is proved for these solutions.

4.1. Introduction and problem statement

In the paper [7] P.D.Lax gives definition of intrinsic compactness for some spaces of solutions in infinite interval and indicates its close connection with Phragmen-Lindelof principles for the solutions of elliptic equations. Such theorems for abstract equations were obtained in the papers [10-11]. In our case the main difference from the above-indicated papers is that a principal part of the equation has a complicated- miltiple character and therefore our conditions essentially differ from the ones of the indicated papers. Notice that the found conditions are expressed by the operator coefficients of the equation.

On a separable Hilbert space H consider an operator differential equation

$$P\left(\frac{d}{d\tau}\right)u(\tau) \equiv \left(\frac{d^2}{d\tau^2} - A^2\right)^m u(\tau) + \sum_{i=0}^{2m} A_{2m-j} u^{(j)}(\tau) = 0, \quad \tau \in S_{\pi/2}, \tag{63}$$

where $S_{\pi/2}$ is a corner vector

$$S_{\pi/2} = \{\tau / |\arg \tau| < \pi/2\},$$

and $u(t)$ is a vector –valued holomorphic function determined in $S_{\pi/2}$ with values from H, the operator A is positive- definite self-adjoint, $A_j \left(j = \overline{1,2m}\right)$ are linear operators in H, $A_j A^{-j} \left(j = \overline{1,2m}\right)$, are bounded in H. All the derivatives are understood in the sense of complex variable theory [4]. By $H_{2,\alpha}$ we denote a space of vector-functions $f(\tau)$ with values in H that are holomorphic in the sector $S_{\pi/2}$, moreover

$$\sup_{\varphi:|\varphi|<\frac{\pi}{2}} \int_0^\infty \left\| f\left(te^{i\varphi}\right)\right\|_H^2 dt < \infty, \quad \left(\tau = te^{i\varphi}\right).$$

Lets introduce the space $W_{2,\alpha}^{2m}(H)$ as a class of vector- functions $u(\tau)$ with values in H that are holomorphic in the sector $S_{\pi/2}$ and for which

$$\sup_{\varphi:|\varphi|<\frac{\pi}{2}} \int_0^\infty \left(\left\| \frac{d^{2m}}{dt^{2m}} u\left(te^{i\varphi}\right)\right\|_H^2 + \left\| A^{2m} u\left(te^{i\varphi}\right)\right\|_H^2 \right)^{1/2} dt < \infty, \quad \left(\tau = te^{i\varphi}\right).$$

Definition 4.1. *If the vector-function $u(\tau) \in W_{2,\alpha}^{2m}(H)$ satisfies the equation (63) in $S_{\pi/2}$ identically, it is said to be a regular holomorphic solution of the equation (63).*

By $U_\beta^{(\alpha)}$ we denote a set of regular holomorphic solutions of the equation (63) for which $e^{\beta\tau} u(\tau) \in H_{2,\alpha}$.
Obviously,

$$U_0^{(\alpha)} = Ker P(d/d\tau) = \{u / P(d/d\tau) u(\tau) = 0\},$$

and for $\tau \geq 0$

$$U_\tau^{(\alpha)} = \left\{ u / u \in U_0^{(\alpha)}, e^{\tau\alpha} u(\tau) \in H_{2,\alpha} \right\}.$$

4.2. Some auxiliary facts

Lets consider some facts that well need in future. It holds

Lemma 4.1. *The set $U_0^{(\alpha)}$ is close in the norm $\|u\|_{w_{2,\alpha}^{2m}}$.*

Proof. Let $\{u_n(\tau)\}_{n=1}^\infty \subset U_0^{(\alpha)}$ and let $\|u_n(\tau) - u(\tau)\|_{W_{2,\alpha}^{2m}} \to 0$. Then, obviously $u(\tau) \in W_{2,\alpha}^{2m}$. Show that $u(\tau) \in U_0^{(\alpha)}$, i.e. $P(d/d\tau) u(\tau) = 0$. Since $\|u_n(\tau) - u(\tau)\|_{W_{2,\alpha}^{2m}} \to 0$, by the theorem on intermediate derivatives [4] for each j $(0 \leq j \leq 2m)$ a sequence $\left\{ A_{2m-j} u_n^{(j)} (\tau) \right\}_{n=1}^{\infty}$ converges on the space $H_{2,\alpha}$. Indeed, as $n \to \infty$

$$\left\| A_{2m-j} u_n^{(j)} (\tau) - A_{2m-j} u^{(j)} (\tau) \right\|_{H_{2,\alpha}} \leq const \left\| u (\tau) - u_n (\tau) \right\|_{W_{2,\alpha}^{2m}} \to 0.$$

Show that the sequence $\left\{ A_{2m-j} u_n^{(j)} (\tau) \right\}_{n=1}^{\infty}$ uniformly converges in any compact $S \subset S_\alpha$. Really, since $A_{2m-j} u_n^{(j)} (\tau)$ and $A_{2m-j} u^{(j)} (\tau) \in H_{2,\alpha}$, there exists their boundary value in the sense $L_2 (R_+; H)$, respectively $Z_{n,j}^{\pm} (t)$ and $Z_j^{\pm} (t) \in L_2 (R_+; H)$. Since

$$A_{2m-j} u^{(j)} (\tau) = \frac{1}{2\pi i} \int_0^{\infty} \frac{z_{n,j}^{-} (\xi)}{\xi e^{-i\alpha} - \tau} e^{-i\alpha} d\xi - \frac{1}{2\pi i} \int_0^{\infty} \frac{z_{n,j}^{+} (\xi)}{\xi e^{i\alpha} - \tau} e^{i\alpha} d\xi,$$

$$A_{2m-j} u^{(j)} (\tau) = \frac{1}{2\pi i} \int_0^{\infty} \frac{z_{j}^{-} (\xi)}{\xi e^{-i\alpha} - \tau} e^{-i\alpha} d\xi - \frac{1}{2\pi i} \int_0^{\infty} \frac{z_{j}^{+} (\xi)}{\xi e^{i\alpha} - \tau} e^{i\alpha} d\xi$$

and

$$\left\| A_{2m-j} u_n^{(j)} (\tau) - A_{2m-j} u^{(j)} (\tau) \right\|_{H_{2,\alpha}} =$$

$$= \frac{1}{\sqrt{2}} \left(\left\| z_{n,j}^{-} (\xi) - z_j^{-} (\xi) \right\|_{L_2(R_+;H)}^2 + \left\| z_{n,j}^{+} (\xi) - z_j^{+} (\xi) \right\|_{L_2(R_+;H)}^2 \right)^{1/2} \to 0,$$

then

$$\left\| z_{n,j}^{-} (\xi) - z_j^{-} (\xi) \right\|_{L_2(R_+;H)} \to 0, \quad \left\| z_{n,j}^{+} (\xi) - z_j^{+} (\xi) \right\|_{L_2(R_+;H)}^2 \to 0.$$

Obviously

$$\sup_{\tau \in s} \left\| A_{2m-j} u_n^{(j)} (\tau) - A_{2m-j} u^{(j)} (\tau) \right\| \leq \sup_{\tau \in s} \frac{1}{2\pi} \int_0^{\infty} \frac{\left\| z_{n,j}^{-} (\xi) - z_j^{-} (\xi) \right\|}{\left| \xi e^{-i\alpha} - \tau \right|} d\xi +$$

$$+ \sup_{\tau \in s} \frac{1}{2\pi} \int_0^{\infty} \frac{\left\| z_{n,j}^{+} (\xi) - z_n^{+} (\xi) \right\|}{\left| \xi e^{-i\alpha} - \tau \right|} d\xi \leq$$

$$\leq \frac{1}{2\pi}\left(\left\| z_{n,j}^{-}\left(\xi\right) - z_{j}^{-}\left(\xi\right)\right\|^{2} d\xi\right)^{1/2}\left(\int_{0}^{\infty}\frac{1}{\left|\xi e^{-i\alpha}-\tau\right|}d\xi\right)^{1/2}+$$

$$+\frac{1}{2\pi}\left(\left\| z_{n,j}^{+}\left(\xi\right) - z_{j}^{+}\left(\xi\right)\right\|^{2} d\xi\right)^{1/2}\left(\int_{0}^{\infty}\frac{1}{\left|\xi e^{-i\alpha}-\tau\right|}d\xi\right)^{1/2}. \qquad (64)$$

Since $\tau \in S \subset S_\alpha$ then

$$\sup_{\tau\in s}\int_{0}^{\infty}\left|\xi e^{-i\alpha}-\tau\right|^{-2}d\xi \leq \int_{0}^{\infty}\sup_{\tau\in s}\left|\xi e^{-i\alpha}-\tau\right|^{-2}d\xi \leq const.$$

Therefore, as $n\to\infty$ it follows from (64) that

$$\sup_{\tau\in s}\left\|A_{2m-j}u_n^{(j)}\left(\tau\right)-A_{2m-j}u^{(j)}\left(\tau\right)\right\|\to 0,$$

i.e. $\left\{A_{2m-j}u_n^{(j)}\right\}_{n=1}^{\infty}$ uniformly converges in the compact to the vector-function $A_{2m-j}u^{(j)}\left(\tau\right)$. On the other hand

$$\sup_{\tau\in s}\left\|P\left(d/d\tau\right)u_n\left(\tau\right)-P\left(d/d\tau\right)u\left(\tau\right)\right\|=$$

$$=\sup_{\tau\in s}\left\|\sum_{j=0}^{2m}A_{2m-j}u_n^{(j)}\left(\tau\right)-\sum_{j=0}^{2m}A_{2m-j}u^{(j)}\left(\tau\right)\right\|\leq$$

$$\leq\sum_{j=0}^{2m}\left\|A_{2m-j}A^{-(2m-j)}\right\|\sup_{\tau\in s}\left\|A^{(2m-j)}u_n^{(j)}\left(\tau\right)-A^{(2m-j)}u^{(j)}\left(\tau\right)\right\|\leq$$

$$\leq const\sum_{j=0}^{2m}\sup_{\tau\in s}\left\|A^{(2m-j)}u_n^{(j)}\left(\tau\right)-A^{(2m-j)}u^{(j)}\left(\tau\right)\right\|.$$

Then from $u_n\left(\tau\right)\in U_0^{(\alpha)}\left(P\left(d/d\tau\right)u\left(\tau\right)=0\right)$ we get

$$\sup_{\tau\in s}\left\|P\left(d/d\tau\right)u\left(\tau\right)\right\|\leq const\sum_{j=0}^{2m}\sup_{\tau\in s}\left\|A^{(2m-j)}u_n^{(j)}\left(\tau\right)-A^{(2m-j)}u^{(j)}\left(\tau\right)\right\|,$$

it follow from the convergence in S

$$A^{(2m-j)}u_n^{(j)}\left(\tau\right)\to A^{(2m-j)}u^{(j)}\left(\tau\right)$$

that $P(d/d\tau)\,u(\tau)=0$ i.e. $u(\tau)\in U_0^{(\alpha)}$. The lemma is proved.

Lemma 4.2. *Let A be a positive self-adjoint operator and one of the following conditions be fulfilled:*

1) $A^{-1}\in\sigma_p\,(0<p<\infty)$, $A_jA^{-j}\left(j=\overline{1,2m}\right)$ are bounded H and solvability condition holds;

2) $A^{-1}\in\sigma_p\,(0<p<\infty)$. Then if $u(\tau)\in U_0^{(\alpha)}$, then for its Laplace transformation $\hat{u}(\lambda)$ estimation

$$\left\|\hat{u}(\lambda)\right\|\le const\,(|\lambda+1|)^{-1},\ \lambda\in\left\{\lambda/\,|\arg\lambda|<\frac{\pi}{2}+\alpha\right\}$$

is true.

Proof. It is easily seen that

$$\left\|\hat{u}(\lambda)\right\|=\frac{1}{\sqrt{2}}P^{-1}(\lambda)\sum_{j=0}^{2m-1}Q_q(\lambda)\,u^{(q)}(0)\,,$$

where

$$Q_q(\lambda)=\sum_{j=0}^{2m-q-1}\lambda^{2m-q-j}\tilde{A}_j$$

and

$$\tilde{A}_j=\begin{cases}A_j, & j=1,3,..,2k-1,k=\overline{1,m}\\ A_j+(-1)^{j/2}\,C_m^{j/2}A^j, & j=2,4,..,2m-2,2m.\end{cases}$$

Obviously, an operator pencil $p(\lambda)$ is represented in the form $P(\lambda)=P_0(\lambda)+P_1(\lambda)$, where

$$P_0(\lambda)=\left(-\lambda^2E+A^2\right)^m,\quad P_1(\lambda)=\sum_{j=0}^{2m}\lambda^jA_{2m-j},$$

and on the rays $\Gamma_{\pm\left(\frac{\pi}{2}+\alpha\right)}=\left\{\lambda/\arg\lambda=\pm\frac{\pi}{2}+\alpha\right\}$ in case 1) from the solvability condition, in case 2) from Keldysh lemma [6] it follows that

$$\sup_{\tau\in\Gamma_{\pm\left(\frac{\pi}{2}+\alpha\right)}}\left\|P(\lambda)\,P_0^{-1}(\lambda)\right\|\le const. \tag{65}$$

Therefore, from these rays

$$\left\|\hat{u}(\lambda)\right\| = \left\|P^{-1}(\lambda)\left(\sum_{q=0}^{2m-1} Q_q(\lambda)\,u^{(q)}(0)\right\| =$$

$$= \left\|(P_0(\lambda) + P_1(\lambda))^{-1}\sum_{q=0}^{2m-1} Q_q u^{(q)}(\lambda)\,u^q(0)\right\| =$$

$$= \left\|\left(P(\lambda)\,P_0^{-1}(\lambda)\,P_0(\lambda)\right)^{-1}\sum_{q=0}^{2m-1} Q_q(\lambda)\,u^{(q)}(0)\right\| =$$

$$= \left\|\left(P_0^{-1}(\lambda)\,P(\lambda)\,P_0^{-1}(\lambda)\right)^{-1}\sum_{q=0}^{2m-1} Q_q(\lambda)\,u^{(q)}(0)\right\|. \tag{66}$$

But in these identities it holds (65) and

$$\left\|P_0^{-1}(\lambda)\right\| \leq const\left(|\lambda|^2 + 1\right)^{-1} \tag{67}$$

$$\left\|\sum_{q=0}^{2m-1} Q_q(\lambda)\,u^{(q)}(0)\right\| \leq const\left|\lambda^{2m-1}\right|. \tag{68}$$

Thus, from (66) allowing for (65), (67) and (68) on these rays we got the estimation

$$\left\|\hat{u}(\lambda)\right\| \leq const\,(|\lambda| + 1)^{-1}.$$

Notice that the angle between the rays $\Gamma_{\pm\left(\frac{\pi}{2}+\alpha\right)}$ equals $(\pi + 2\alpha)$. It follows from, conditions 1) and 2) that $P^{-1}(\lambda)$ is a meramorphic operator-function of order ρ and minimal type of order ρ, i.e. it is represented in the form of relations of two entire functions of order $\rho$and minimal type for order ρ. Since $\hat{u}(\lambda)$ is a holomorphic vector-function in the domain (see [12])

$$S_{\frac{\pi}{2}-\alpha} = \left\{\lambda / |\arg\lambda| < \frac{\pi}{2} + \alpha\right\}$$

and the angle between the rays equals $\pi+2\alpha$, for $0 \leq \rho \leq \pi/(\rho + 2\alpha)$ it follows from the Phragmen-Lindelof theorem that in the sector $S_{\frac{\pi}{2}+\alpha}$ the estimation

$$\left\|\hat{u}(\lambda)\right\| \leq const\,(|\lambda| + 1)^{-1}$$

holds.

In case 2) M.V. Keldysh theorem [6] yields that there exist the rays between which the angles are less than π/ρ and the estimation (66), (67), (68) and the estimation

$$\left\|\hat{u}(\lambda)\right\| \leq const\,(|\lambda|+1)^{-1}$$

hold. The lemma is proved.

4.3. The basic result

Now, lets prove a theorem on Phragmen-Lindelof principle.

Theorem 4.1. *Let A be a positive-definite self-adjoint operator, $A_j A^{-j}\left(j=\overline{1,2m}\right)$ be bounded in H and one of the two conditions hold:*

1) $A^{-1}\in\sigma_p\ (0<p<\pi/(\pi+2\alpha))$ and solvability condition holds;

2) $A^{-1}\in\sigma_p\ (0<p<\infty)$. Then, if a regular holomorphic solution $u(\tau)\in\bigcap\limits_{\tau\geq 0} u_\tau^{(\alpha)}$, then $u(\tau)=0$.

Proof. For $u(\tau)\in U_\tau^{(\alpha)}$ it follows that the Laplace transformation $\hat{u}(\lambda)$ admits holomorphic continuation to the domain $\left\{\lambda/|\arg(\lambda+\tau)|<\frac{\pi}{2}+\alpha\right\}$. The inclusion $u(\tau)\in\bigcap\limits_{\tau\geq 0}\left(U_\tau^{(\alpha)}\right)$ implies that $\hat{u}(\lambda)$is an entire function, i.e. it is an entire function and $\hat{u}(\lambda)=P^{-1}(\lambda)\sum\limits_{q=0}^{2m-1}Q_q(\lambda)\,u^{(q)}(0)$, moreover $\hat{u}(\lambda)$ is an entire function of order ρ and of minimal type for order ρ. Further, in the second case, we can use the Keldysh lemma and obtain that $\hat{u}(\lambda)$ is an entire function of order ρ and of minimal type for order ρ on all the complex half-plane

$$\left\|\hat{u}(\lambda)\right\| \leq const\,(|\lambda|+1)^{-1}.$$

In the first case, since the angle between $\Gamma_{\pm\left(\frac{\pi}{2}+\alpha\right)}$ in the left half-plane $\pi-2\alpha$ and $\pi-2\alpha<\pi+2\alpha$, then again from the Phragmen-Lindelof theorem it follows, that $\hat{u}(\lambda)$ is an entire function of order ρ and of minimal type for order ρ and on the complex plane

$$\left\|\hat{u}(\lambda)\right\| \leq const\,(|\lambda|+1)^{-1}.$$

Thus, as $n\to\infty$, $\hat{u}(\lambda)=0$. Hence, it follows that $u(\tau)=0$. The theorem is proved.

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Chapter III

In this chapter we give definition of smooth solutions. For a boundary value problem we prove theorems on the existence and uniqueness of these solutions in terms of the coefficients of the studied operator- differential equations of higher order, moreover the principal part of these equations has a multiple characteristic. Here we mainly use S.S.Mirzoyev's [1] metod.

§3.1. On the conditions of existence of smooth solutions for a class of operator- differential equations on the axis

1.1. Introduction and problem statement

Let H be a separable Hilbert space, A be a positive- definite self- adjoint operator in H, and H_γ be a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(x,y)_\gamma = (A^\gamma x,\ A^\gamma y)$, $(x,y) \in D(A^\gamma)$. Denote by $L_2(R;H)$ $\quad (R = (-\infty, +\infty))$ a Hilbert space of measurable vector- functions quadratically integrable on Bochner in R and define the norm in this space by the following way

$$\|f\|_{L_2(R;H)} = \left(\int_{-\infty}^{\infty} \|f\|^2 dt \right)^{1/2} < \infty.$$

Further, at natural $m \geq 1 \quad (m \in N)$ we define the following Hilbert space [2]

$$W_2^m(R;H) = \left\{ u/u^{(m)} \in L_2(R;H),\ \ A^m u \in L_2(R;H) \right\}$$

with the norm

$$\|u\|_{W_2^m(R;H)} = \left(\left\| u^{(m)} \right\|^2_{L_2(R;H)} + \|A^m u\|^2_{L_2(R;H)} \right)^{1/2}.$$

Here and further, the derivatives are understood in the sense of the theory of distributions [2].

In the given paper we consider the equation

$$P\left(\frac{d}{dt}\right) u(t) = \left(-\frac{d^2}{dt^2} + A^2 \right)^n u(t) + \sum_{j=1}^{2n-1} A_{2n-j} u^{(j)}(t) = f(t),\ t \in R, \quad (69)$$

where $A = A^* > cE \quad (c > 0)$, $A_j \quad \left(j = \overline{1, 2n-1}\right)$ are linear, generally speaking, unbounded operators, $f(t) \in W_2^s(R;H)$, $u(t) \in W_2^{2n+s}(R;H)$. Here S is a fixed positive integer.

Definition 1.1. *If the vector- function $u(t) \in W_2^{2n+s}(R;H)$ satisfies equation (38) at all $t \in (-\infty, +\infty)$, then we'll call $u(t)$ the smooth solution of equation (38).*

1.2. Some auxiliary facts

We prove first the following lemma, that we'll need later on.

Denote by

$$P_0\left(\frac{d}{dt}\right)u\left(t\right)=\left(-\frac{d^2}{dt^2}+A^2\right)^n u\left(t\right),\quad u\left(t\right)\in W_2^{2n+s}\left(R;H\right),\tag{70}$$

$$P_1\left(\frac{d}{dt}\right)u\left(t\right)=\sum_{j=1}^{2n-1}A_{2n-j}u^{(j)}\left(t\right),\quad u\left(t\right)\in W_2^{2n+s}\left(R;H\right).\tag{71}$$

Lemma 1.1. *The operator $P_0\left(d/dt\right)$ defined by equality (70), is an isomorphism from the space $W_2^{2n+s}\left(R;H\right)$ to $W_2^{s}\left(R;H\right)$.*

Proof. By virtue $W_2^{2n+s}\left(R;H\right)$

$$\left\|P_0\left(d/dt\right)u\right\|^2_{W_2^s}=\left\|A^s\left(-\frac{d^2}{dt^2}+A^2\right)^n u\right\|^2_{l_2}+\left\|\frac{d^s}{dt^s}\left(-\frac{d^2}{dt^2}+A^2\right)^n u\right\|^2_{l_2}=$$

$$=\left\|\sum_{q=0}^{n}C_n^qA^{2(n-q)+s}u^{(2q)}\right\|^2_{L_2}+\left\|\sum_{q=0}^{n}C_n^qA^{2n}u^{(2n-q+s)}\right\|^2_{L_2}\leq$$

$$\leq const\left(\sum_{q=0}^{n}\left\|A^{2(n-q)+s}u^{(q)}\right\|^2_{L_2}+\sum_{q=0}^{n}\left\|A^{2n+s}u^{(2n-q+s)}\right\|^2_{L_2}\right)$$

where $C_n^q=\frac{n(n-1)...(n-q+1)}{q!}$. Applying the theorem on intermediate derivatives [2] we get that

$$\left\|P_0\left(d/dt\right)u\right\|^2_{W_2^s}\leq const\left\|u\right\|^2_{W_2^{2n+s}},$$

i.e. the operator $P_0\left(d/dt\right)$:$W_2^{2n+s}\left(R;H\right)\rightarrow W_2^{2s}\left(R;H\right)$ is continuous. Let $u\left(t\right)\in W_2^{2n+s}\left(R;H\right)$. Then we denote by $P_0\left(d/dt\right)u\left(t\right)=g\left(t\right)$. Evidently after Fourier transformation we have

$$P_0\left(-i\xi\right)\hat{u}\left(\xi\right)=\hat{g}\left(\xi\right)\ \ or\ \hat{u}\left(\xi\right)=P_0^{-1}\left(-i\xi\right)\hat{g}\left(\xi\right).\tag{72}$$

By Plancherel theorem the inequality

$$\left\|g\left(t\right)\right\|^2_{W_2^s(R;H)}=\left\|\xi^s\hat{g}\left(\xi\right)\right\|^2_{L_2(R;H)}+\left\|A^s\hat{g}\left(\xi\right)\right\|^2_{L_2(R;H)}\leq const,\tag{73}$$

should be fulfilled. On the other hand, from (72) we have

$$\|u(t)\|^2_{W_2^{2n+s}} = \left\|\xi^{2n+s} u(\xi)\right\|^2_{L_2} + \left\|A^{2n+s}\hat{u}(\xi)\right\|^2_{L_2} =$$
$$= \left\|\xi^{2n+s} P_0^{-1}(-i\xi)\,\hat{g}(\xi)\right\|^2_{L_2} + \left\|A^{2n+s} P_0^{-1}(-i\xi)\,\hat{g}(\xi)\right\|^2_{L_2} \leq \tag{74}$$
$$\leq \sup_{\xi\in R}\left\|\xi^{2n} P_0^{-1}(-i\xi)\right\|^2 \left\|\xi^s \hat{g}(\xi)\right\|^2_{L_2} + \sup_{\xi\in R}\left\|A^{2n} P_0^{-1}(-i\xi)\right\|^2 \left\|A^s \hat{g}(\xi)\right\|^2_{L_2}.$$

In turn, at $\xi \in R$

$$\left\|A^{2n} P_0^{-1}(-i\xi)\right\| \leq \sup_{\sigma\in\sigma(A)}\left|\sigma^{2n}(-i\xi-\sigma)^{-n}(-i\xi+\sigma)^{-n}\right| =$$
$$= \sup_{\sigma\in\sigma(A)}\left|\sigma^{2n}\left(\xi^2+\sigma^2\right)^{-n}\right| \leq \sup_{\sigma>0}\left|\left(\tfrac{\sigma^2}{\xi^2+\sigma^2}\right)^n\right| < 1 \tag{75}$$

and

$$\left\|\xi^{2n} p_0^{-1}(-i\xi)\right\| \leq \sup_{\sigma\in\sigma(A)}\left|\xi^{2n}(-i\xi-\sigma)^{-n}(-i\xi+\sigma)^{-n}\right| =$$
$$= \sup_{\sigma\in\sigma(A)}\left|\xi^{2n}\left(\xi^2+\sigma^2\right)^{-n}\right| \leq \sup_{\sigma>0}\left|\left(\tfrac{\xi^2}{\xi^2+\sigma^2}\right)^n\right| \leq 1. \tag{76}$$

Allowing for inequalities (75) and (76) in (74) we get

$$\|u(t)\|^2_{W_2^{2n+s}} \leq \left\|\xi^s\hat{g}(\xi)\right\|^2_{L_2} + \left\|A^s\hat{g}(\xi)\right\|^2_{L_2} = \|g(t)\|^2_{W_2^s}, \tag{77}$$

i.e. $u(t) \in W_2^{2n+s}(R;H)$. Evidently

$$u(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} P_0^{-1}(-i\xi)\,\hat{g}(\xi)\,d\xi$$

satisfies the equation $P_0(d/dt)\,u(t) = g(t)$ almost everywhere in R. Further it follows from Banach theorem that this mapping, i.e. the mapping $P_0(d/dt)\,u(t) = g(t)$ is an isomorphism.

The lemma is proved.

Lemma 1.2. *Let the operators $B_j = A_j A^{-j}$, $D_j = A^s A_j A^{-(j+s)}$ $\left(j=\overline{1,2n-1}\right)$ be bounded in H, i.e. $A_j \in L(H_j H) \cap (H_{j+s}, H_s)$ $\left(j=\overline{1,2n-1}\right)$. Then the operator $P_1(d/dt)$ defined by equality (71) is bounded from $W_2^{2n+s}(R;H)$ to $W_2^s(R;H)$.*

Proof. Since $u(t) \in W_2^{2n+s}(R;H)$, then the inequality

$$\|P_1(d/dt)\,u(t)\|_{W_2^s}^2 = \left\|\sum_{j=1}^{2n-1} A_{2n-j}u^{(j+s)}\right\|_{L_2(R;H)}^2 + \left\|\sum_{j=1}^{2n-1} A^s A_{2n-j}u^j\right\|_{L_2(R;H)}^2 \leq$$

$$\leq 2n\left(\sum_{j=1}^{2n-1}\left\|A_{2n-j}A^{-(2n-j)}A^{2n-j}u^{(j+s)}\right\|_{L_2}^2\right)+$$

$$+2n\left(\sum_{j=1}^{2n-1}\left\|A^s A_{2n-j}A^{-(2n-j)}A^{2n-j}u^{(j+s)}\right\|_{L_2}^2\right)\leq$$

$$\leq 2n\sum_{j=1}^{2n-1}\max\left(\left\|A_{2n-j}A^{-(2n-j)}\right\|^2\left\|A^s A_{2n-j}A^{-(2n+s-j)}\right\|^2\right)\times$$

$$\times\left(\left\|A^{2n+s-j}u^{(j)}\right\|_{L_2}^2+\left\|A^{2n-j}u^{(j+s)}\right\|_{L_2}^2\right), \tag{78}$$

holds.

Then it follows from the theorem on intermediate derivatives that

$$\left\|A^{2n+s-j}u^{(j)}\right\|_{L_2}^2 \leq K_j\|u\|_{W_2^{2n+s}}^2, \quad \left\|A^{2n-j}u^{(j+s)}\right\|_{L_2}^2 \leq K_j\|u\|_{W_2^{2n+s}}^2,$$

where the number $K_j > 0$, $K_{j+s} > 0$. Allowing for these inequalities in (78), we complete the proof the lemma. From lemmas 1.1 and 1.2 we get

Corollary. *The operator $P(d/dt)$ defined by equation (38), by fulfilling conditions of lemma 1.2 is bounded from the space $W_2^{2n+s}(R;H)$ to $W_2^s(R;H)$.*

Lemma 1.3. *The operator $A^{2n-j}\frac{d_j}{dt^j}$ is a bounded operator from the space $W_2^{2n+s}(R;H)$ to $W_2^s(R;H)$ and it holds the exact inequality*

$$\left\|A^{2n-j}u^{(j)}(t)\right\|_{W_2^s(R;H)} \leq d_{2n,j}^n\|p_0(d/dt)\,u(t)\|_{W_2^s(R;H)},$$

where $d_{2n,j} = \left(\frac{j}{2n}\right)^{\frac{i}{2n}}\left(\frac{2n-j}{2n}\right)^{\frac{2n-j}{2n}}$, $j=\overline{1,2n-1}$.

Proof. Consider in $W_2^{2n+s}(R;H)$ the functional

$$E_j(u,\beta) = \|P_0u\|_{W_2^s}^2 - \beta\left\|A^{2n-j}u^{(j)}(t)\right\|_{W_2^s}, \quad j=\overline{1,2n-1}$$

where $\beta \in \left[0; d_{2n,j}^{-n}\right)$. Then

$$\begin{array}{l} E_j(u;\beta) = \left\| \frac{d^s}{dt^s} P_0(d/dt)\, u \right\|_{L_2}^2 + \left\| A^s P_0(d/dt)\, u \right\|_{L_2}^2 - \beta \left\| A^{2n+s-j} u^{(j)} \right\|_{L_2}^2 - \\ -\beta \left\| A^{2n-j} u^{(j+s)} \right\|_{L_2}^2 = \left\| (-i\xi)^s P_0(-i\xi) \hat{u}(\xi) \right\|_{L_2}^2 + \left\| A^s P_0(-i\xi) \hat{u}(\xi) \right\|_{L_2}^2 - \\ -\beta \left\| A^{2n+s-j} \xi^j \hat{u}(\xi) \right\|_{L_2}^2 + \left\| \xi^{j+s} A^{2n-j} \hat{u}(\xi) \right\|_{L_2}^2 = \\ = \int\limits_{-\infty}^{\infty} \left((\xi^{2s} E + A^{2s}) \left((\xi^2 E + A^2)^{2n} - \beta \xi^{2j} A^{4-2j} \right) \hat{u}(\xi), \hat{u}(\xi) \right) d\xi \equiv \\ \equiv \int\limits_{-\infty j}^{\infty} \left(P_j(\beta;\xi;A)\, \hat{u}(\xi), \hat{u}(\xi) \right) d\xi \end{array}, \tag{79}$$

where

$$\begin{array}{l} P_j(\beta;\xi;A) = (\xi^{2s} + \mu^{2s}) \left[(\xi^2 + \mu^2)^{2n} - \beta \xi^{2j} \mu^{4n-2j} \right] = \\ = (\xi^2 + \mu^2)^{2n+s} \left[1 - \beta \sup\limits_{t>0} \frac{t^{2j}}{(t^2+1)^{2n}} \right] > (\xi^2 + \mu^2) \left(1 - \beta d_{2n,j}^{-n} \right) > 0 \end{array}. \tag{80}$$

It follows from the spectral theory of self- adjoint operators that at $\xi \in R$ and $\beta \in \left[0; d_{2n,j}^{-n}\right)$

$$P_j(\beta; -i\xi; A) > 0$$

Thus, it follows from equation (79) that at $\xi \in R$ and $\beta \in \left[0; d_{2n,j}^{-n}\right)$ the equality

$$E_j(u;\beta) \equiv \left\| P_0(d/dt)\, u \right\|_{W_2^s(R;H)}^2 - \beta \left\| A^{2n-j} u^{(j+s)} \right\|_{W_2^s(R;H)}^2 > 0, \quad j = \overline{1, 2n-1}$$

holds. Going over to the limit at $\beta \to d_{2n,j}^{-2n}$ we get

$$\left\| A^{2n-j} u^{(j)} \right\|_{W_2^s} \leq d_{2n,j}^{n} \left\| P_0(d/dt)\, u \right\|_{W_2^s}, \quad j = \overline{1, 2n-1}. \tag{81}$$

The exactness of inequality (81) is proved similar to the paper [1]. Thus, lemma 1.3 is proved.

1.3. The basic theorem

Now prove the basic theorem.

Theorem 1.1. *Let A be a positive definite self-adjoint operator, the operators $B_j = A_j A^{-j}$, $D_j = A^s A_j A^{-(j+s)}$ $\left(j = \overline{1, 2n-1}\right)$ be bounded in H, i.e. $A_j \in L\left(H_j H\right) \cap \left(H_{j+s}, H_s\right)$ $\left(j = \overline{1, 2n-1}\right)$ and the inequality*

$$\alpha = \sum_{j=1}^{2n-1} \max\{\|B_{2n-j}\|, \|D_{2n-j}\|\} d_{2n,j}^{-n} < 1, \tag{82}$$

hold. Then equation (69) have a unique regular solution $u\left(t\right) \in W_2^{2n+s}\left(R; H\right)$ at any $f\left(t\right) \in W_2^{2s}\left(R; H\right)$ and the inequality

$$\|u\|_{W_2^{2n+s}(R;H)} \leq const\|f\|_{W_2^s(R;H)} \tag{83}$$

hold.

Proof. It follows from lemma 1.1 that the operator is an isomorphism $P_0\left(d/dt\right)$:$W_2^{2n+s}\left(R; H\right) \to W_2^s\left(R; H\right)$. Write equation (69) in the form

$$P_0\left(d/dt\right) u\left(t\right) + P_1\left(d/dt\right) u\left(t\right) = f\left(t\right) \tag{84}$$

Denote by $P_0\left(d/dt\right) u\left(t\right) = \upsilon\left(t\right)$. Then equation (84) has the form

$$\upsilon + P_1 P_0^{-1} \upsilon = f$$

Show that $\left\|P_1 P_0^{-1}\right\|_{W_2^s(R;H) \to W_2^s(R;H)} < 1$. Evidently

$$\left\|P_1 P_0^{-1} \upsilon\right\|_{W_2^s} = \|P_1 u\|_{W_2^s} < \sum_{j=1}^{2n-1} \left\|A_{2n-j} u^{(j)}\right\|_{W_2^s} \leq$$

$$\leq \sum_{j=1}^{2n-1} \left(\left\|A_{2n-j} u^{(j+s)}\right\|_{L_2}^2 + \left\|A^s A_{2n-j} u^{(j)}\right\|_{L_2}^2\right)^{1/2} \leq$$

$$\leq \sum_{j=1}^{2n-1} \max\left\{\|A_{2n-j}\|_{H_{2n-j} \to H} \|A_{2n-j}\|_{H_{2n+s-j} \to H}\right\} \times$$

$$\times \left(\left\|A^{2n+s-j} u^{(j)}\right\|_{L_2}^2 + \left\|A^{2n-j} u^{(j+s)}\right\|_{L_2}^2\right)^{1/2} =$$

$$= \sum_{j=1}^{2n-1} \max\{\|B_{2n-j}\|, \|D_{2n-j}\|\} \left\|A^{2n-j}u^{(j)}\right\|_{W_2^s}.$$

But from lemma 1.3 and the last inequality it follows that

$$\left\|P_1P_0^{-1}\upsilon\right\|_{W_2^s} \leq \sum_{j=1}^{2n-1} \{\|B_{2n-j}\|, \|D_{2n-j}\|\} d_{2n,j}^{-n} < 1.$$

Thus, the operator $\left(E + P_1P_0^{-1}\right)$ is invertible in the spaceW_2^s. Then

$$\upsilon = \left(E + P_1P_0^{-1}\right)^{-1} f,$$

and

$$u = P_0^{-1}\left(E + P_1P_0^{-1}\right)^{-1} f.$$

On the other hand

$$\|\upsilon\|_{W_2^{2n+s}(R;H)} \leq \left\|P_0^{-1}\right\|_{W_2^s \to W_2^{2n+s}} \left\|\left(E + P_1P_0^{-1}\right)^{-1}\right\|_{W_2^s \to W_2^s} \|f\|_{W_2^s} \leq const\|f\|_{W_2^s}$$

The theorem is proved.

§3.2. On smooth solution of boundary value problem for a class of operator- differential equations of high order

2.1. Introduction and problem statement.

"'Let H be a separable Hilbert space, A be a positive- definite self- adjoint operator in H. By H_γ denote a scale of Hilbert spaces generated by the operators A, i.e. $H_\gamma = D(A^\gamma)$, $(x,y)_\gamma = (A^\gamma x,\ A^\gamma y)$, $(x,y) \in D(A^\gamma)$. By $L_2(R_+;H)$ we denote a Hilbert space of vector- functions determined $R_+ = (0,+\infty)$ strongly measurable and quadratically integrable by Bochner with square, moreover

$$\|f\|_{L_2(R;H)} = \left(\int_{-0}^{\infty} \|f\|_H^2 dt\right)^{1/2} < \infty.$$

Following [2] we determine a Hilbert space W_2^l

$$W_2^l(R_+;H) = \left\{u/u^{(l)} \in L_2(R_+;H),\ \ A^l u \in L_2(R_+;H)\right\}$$

with norm

$$\|u\|_{W_2^l(R_+;H)} = \left(\left\|A^l u\right\|_{L_2(R_+;H)}^2 + \left\|u^{(l)}\right\|_{L_2(R_+;H)}^2\right)^{1/2}, \quad l = 1,2,...$$

Let's consider the following boundary value problem

$$P\left(\frac{d}{dt}\right)u(t) = \left(-\frac{d^2}{dt^2} + A^2\right)^n u(t) + \sum_{j=1}^{2n-1} A_{2n-j}u^{(j)}(t) = f(t),\ t \in R_+, \quad (85)$$

$$u^{(\nu)}(0) = 0, \quad \nu = \overline{0, n-1}, \quad (86)$$

moreover $f(t) \in W_2^s(R_+;H)$, $u(t) \in W_2^{n+s}(R_+;H)$, where $s \geq 1$ is a natural number, $A_j \quad \left(j = \overline{1, 2m-1}\right)$ are linear in the space H.

2.2. Some auxiliary facts.

Denote by

$$d_{2n,j} = \left(\frac{j}{2n}\right)^{\frac{i}{2n}} \left(\frac{2n-j}{2n}\right)^{\frac{2n-j}{2n}}, \quad j = \overline{1, 2n-1}, \quad (87)$$

and consider for $\beta \in \left[0; d_{2n,j}^{-2n}\right)$ the following operator pencils

$$P_j\left(\lambda;\beta;A\right) = \left(\left(-\lambda^2 E + A^2\right)^{2n} - \beta\left(i\lambda\right)^{2n} A^{4-2j}\right)\left(\left(i\lambda\right)^{2s} E + A^{2s}\right). \qquad (88)$$

As is known, the pencil $P_j\left(\lambda;\beta;A\right)$ has no on operaturm an imaginary axis and we can represent it in the form

$$P_j\left(\lambda;\beta;A\right) = F_j\left(\lambda;\beta;A\right) F_j\left(-\lambda;\beta;A\right)$$

where

$$F_j\left(\lambda;\beta;A\right) = \left(\lambda E - w_1 A\right)...\left(\lambda E - w_{n+s} A\right),$$

$Rew_j < 0, \quad j = \overline{1, n+s}$. Indeed, for $\xi \in R = (-\infty, +\infty)$ and $\mu \in \sigma\left(\lambda\right)$ the characteristic polynomial $P_j\left(i\xi;\beta;\mu\right)$ is of the form

$$P_j\left(i\xi;\beta;A\right) - \left(\left(\xi^2+\mu^2\right)^{2n} - \beta\xi^{2j}\mu^{4n-2j}\right)\left(\xi^{2s}+\mu^{2s}\right) =$$

$$= \left(\xi^2+\mu^2\right)\left(\xi^{2s}+\mu^{2s}\right)\left[1 - \beta\tfrac{\xi^{2j}\mu^{4-2j}}{\left(\xi^2+\mu^2\right)^{2n}}\right] > \left(\xi^2+\mu^2\right)\left(\xi^{2s}+\mu^{2s}\right) \times$$
$$\times\left[1 - \beta\sup_{\eta>0}\tfrac{\eta^{2j}}{\left(1+\eta^2\right)^{2n}}\right].$$

Let's consider the function

$$f\left(\eta\right) = \frac{\eta^{2j}}{\left(1+\eta^2\right)^{2n}}, \quad \eta > 0.$$

Obviously, it follows from the equation $f'\left(\eta\right) = 0$ that $2j\eta^{2j-1}\left(\eta^2+1\right)^{2n} - 2\eta 2\eta\left(1+\eta\right)^{2n-1}\eta^{2j} = 0$, i.e. $j\left(1+\eta^2\right) - 2n\eta^2 = 0$ or $\eta^2 = j\left(2n-j\right)$. Then the function $f\left(\eta\right)$ takes its maximal value at the point $\eta = \left(\frac{j}{2n-j}\right)^{1/2}$. Thus

$$f_{\max}(\eta) = \frac{\left(\frac{j}{2n-j}\right)^j}{\left(1+\frac{j}{2n-j}\right)^{2n}} - \frac{\left(\frac{j}{2n}\right)^j}{\left(\frac{2n}{2n-j}\right)^{2n}} =$$
$$= \left(\left(\frac{j}{2n}\right)^{\frac{j}{2n}}\right)^{2n}\left(\left(\frac{2n-j}{2n}\right)^{\frac{2n-j}{2n}}\right)^{2n} = d_{2n,j}^{2n}.$$

Therefore for $\beta \in \left[0; d_{2n,j}^{-2n}\right)$

$$P_j\left(i\xi;\beta;A\right) = \left(\xi^2+\mu^2\right)^{2\mu}\left(\xi^{2s}+\mu^{2s}\right)\left(1 - \beta d_{2n,j}^{2n}\right) > 0.$$

Thus, it follows from the spectral expansion of A that $P_j\left(i\xi;\beta;A\right)>0$ for $\xi\in R$. On the other hand, a characteristic polynomial

$$\begin{aligned}P_j\left(\lambda;\beta;\mu\right)&=\left(\left(i\lambda\right)^{2s}+\mu^{2s}\right)\left[\left(-\lambda^2+\mu^2\right)^{2n}-\beta\left(i\lambda\right)^{2j}\mu^{4-2j}\right]=\\&=\prod_{q=1}^{s}\left(\lambda-\xi_q\mu\right)\left(-\lambda-\xi_q\mu\right)\left[\left(-\lambda^2+\mu^2\right)^{2n}-\beta\left(i\lambda\right)^{2j}\mu^{4-2j}\right],\end{aligned}\quad ,$$

where $Re\xi_q<0 \quad \left(q=\overline{1,s}\right)$, moreover ξ_q is a root of equation $\left(-1\right)^s\lambda^s+1=0$. Denote

$$\phi_j\left(\lambda;\beta;\mu\right)=\left(-\lambda^2+\mu^2\right)^{2n}-\beta\left(i\lambda\right)^{2j}\mu^{4-2j},\quad j=\overline{1,2n-1}.$$

Then, obviously $\phi_j\left(i\xi;\beta;\mu\right)>0$ for $\beta\in\left[0;d_{2n,j}^{-2n}\right)$ and the roots of the equation $\phi_j\left(\lambda;\beta;\mu\right)=0$ are of the form $\lambda_j=\mu w_j \quad \left(j=\overline{1,2n}\right)$. On the other hand, if $\lambda_j=\mu w_j$, a root of equation $\phi_j\left(\lambda;\beta;\mu\right)=0$ is located symmetric with respect to a real axis and origin of coordinates. Hence it follows that if $Rew_j<0 \quad \left(j=\overline{1,n}\right)$ then

$$\phi_j\left(\lambda;\beta;\mu\right)=\prod_{j=1}^{n}\left(\lambda-w_j\mu\right)\prod_{j=1}^{n}\left(-\lambda-w_j\mu\right).$$

Thus

$$\psi_j\left(\lambda;\beta;\mu\right)=\prod_{j=1}^{n}\left(\lambda-\xi_q\mu\right)\prod_{j=1}^{n}\left(\lambda-w_j\mu\right)=\prod_{j=1}^{n}\left(\lambda-w_j\mu\right),\quad w_0=-1$$

It follows from the spectral expansion of the operator A that

$$P_j\left(\lambda;\beta;A\right)=\psi_j\left(\lambda;\beta;A\right)\psi_j\left(-\lambda;\beta;A\right),\tag{89}$$

moreover

$$\psi_j\left(\lambda;\beta;A\right)=\prod_{q=1}^{s}\left(\lambda-\xi_qA\right)\prod_{j=1}^{n}\left(\lambda E-w_jA\right)=\sum_{j=0}^{n+s}\alpha_j\left(\beta\right)\lambda^jA^n+1-j,\tag{90}$$

where all $\alpha_j(\beta)$ are real, $L_0(\beta)=1$, $\alpha_n(\beta)=1$.

Lemma 2.1. *The Cauchy problem*

$$\begin{cases} \psi_j\left(d/dt;\ \beta;\ A\right)u(t)=0 \\ u^{(\nu)}(0)=0, \quad \nu=\overline{0,n-1} \\ u^{(\nu)}(0)=A^{-(2n-\nu-1/2)}\varphi_\nu \end{cases}, \tag{91}$$

for all $\varphi_\nu\in H$ has a unique solution from the space $W_2^{n+s}(R_+;H)$.

Proof. Obviously $\xi_q\neq\xi_p \quad (q\neq p)$. Further, we prove that the roots of the equation, $\psi_j(\lambda;\beta;A)=0 \quad (\mu\in\sigma(A))$are prime. It suffices to prove that for $\beta\in\left[0;d_{2n,j}^{-2n}\right)$ all the roots of the equation $\left(-\lambda^2+\mu^2\right)^{2n}-\beta(-1)^j\lambda^{2j}\mu^{4-2j}=0$ are prime. Really

$$\left(-\lambda^2+\mu^2\right)^{2n}=\beta(-1)^j\lambda^{2j}\mu^{4n-2j}$$

$$2n\left(-\lambda^2+\mu^2\right)^{2n-1}=\beta(-1)^j\lambda^{2j-1}\mu^{4n-2j}$$

Since $\beta\neq 0$, then

$$\beta(-1)^j\lambda^{2j}\mu^{4n-2j}=(-1)^{j+1}\beta^{2j-2}\frac{j}{2n}\mu^{4n-2j}\left(-\lambda^2+\mu^2\right).$$

Hence, for $\beta\neq 0$ we have

$$\lambda^2=-\frac{j}{2n-j}\mu^2.$$

Substituting this expression into the first equation, we have

$$\left(\frac{j}{2n-j}\mu^2+\mu^2\right)^{2n}=\beta(-1)^j(-1)^j\left(\frac{j}{2n-j}\right)^{2j}\mu^{4n-2j}\mu^{2j}\Rightarrow\beta=d_{2n,j}^{-2n}.$$

Thus, $\psi_j(d/dt;\beta;A)=0$ has a general solution of the form

$$u(t)=\sum_{j=1}^{n+s}e^{\tau_j tA}c_j, \quad c_j\in H_{2n+s-1/2},$$

where $\tau_1=\xi_1,\tau_2=\xi_2,...,\tau_s=\xi_s,\tau_{s+1}=w_1,...,\tau_{s+n}=w_n$. Here we assume the initial conditions $u^{(\nu)}(0)=0, \quad \nu=\overline{0,n-1}$ and $u^{(\nu)}(0)=A^{2n-\nu-1/2}\varphi_\nu$ we get a system of equations

$$\sum_{j=1}^{n+s}\tau_j^\nu A^\nu c_j=\xi\nu,$$

where $\xi\nu = 0,\ \nu = \overline{0, n-1}$: $\xi\nu = A^{2n+s-\nu-1/2}\varphi_\nu$, $\nu = \overline{n, 2n+s-1}$ or

$$\sum_{j=1}^{n+s} \tau_j^{\nu} c_j = A^{-\nu}\xi\nu, \quad \nu = \overline{0, 2n+s-1}.$$

Having solved this system we easily get the vector c_j. The lemma is proved.

Corollary. *The coefficients of the polynomial $\psi_j\left(\lambda;\beta;A\right)$ satisty the following conditions*

$$\tilde{A}_m\left(\beta\right) = \sum_{\nu=-\infty}^{\infty} \alpha_{m+\nu,j}\left(\beta\right)\alpha_{m-\nu,j}\left(\beta\right) = \begin{cases} C_{2n}^m, & m \neq j \\ -\beta, & m = j \\ C_{2n}^m - \beta, & m = \overline{0, 2n-1}, \end{cases}$$

where

$$C_{2n}^m = \frac{(2n-1)(2n-2)\dots(2n-m+1)}{m!}.$$

The proof of this corollary follows from the expansion (89) by comparing the coefficients of the same degress of λ.

Lemma 2.2.*For all $u\left(t\right) \in W_2^{2n+s}\left(R_+;H\right)$ it holds the equality*

$\left\|P_0 u\left(t\right)\right\|^2_{W_2^s(R_+;H)} = \sum\limits_{q=0}^{2n} \tilde{A}_q \left\|A^{2n-q+s}u^{(q)}\right\|^2_{L_2(R_+;H)} + \sum\limits_{l=0}^{2n} \tilde{A}_l \left\|A^{2n-l}u^{(l+s)}\right\|^2_{L_2(R_+;H)} -$ $\left(Q_0\tilde{\varphi},\tilde{\varphi}\right)_{n^{2n+s}}$ *where*

$$Q_0 = \begin{pmatrix} P_{1,1}^0 & P_{1,2}^0 \dots P_{1,2n}^0 & & 0\dots0 \\ P_{2,1}^0 & P_{2,2}^0 \dots P_{2,2n}^0 & & 0\dots0 \\ \dots\dots & & & \\ P_{2n,1}^0 & P_{2n,2}^0 \dots P_{2n,2n}^0 & & 0\dots0 \\ 0 & 0\dots & 0 & 0\dots0 \\ \dots\dots & & & \\ 0 & 0\dots & 0 & \underbrace{0\dots0}_{s} \end{pmatrix} + \begin{pmatrix} \overbrace{0\dots \quad 0}^{s} & 0\dots & 0 & & 0 \\ 0\dots \quad 0 & 0\dots & 0 & & 0 \\ \dots\dots & & & & \\ 0\dots \quad 0 & P_{1,1}^0\dots & P_{1,2n-1}^0 & & P_{1,2n}^0 \\ 0\dots \quad 0 & P_{2,1}^0\dots & P_{2,2n-1}^0 & & P_{2,2n}^0 \\ \dots\dots & & & & \\ 0\dots \quad 0 & P_{2n,2}^0\dots & P_{2n,2n-1}^0 & & P_{2n,2n}^0 \end{pmatrix}$$

$$P_{j,r}^0 = \sum_{\nu=-\infty}^{\infty} \left(-1\right)^{\nu}\theta_{j+\nu}\theta_{r-\nu-1}, \quad 1 \leq r \leq j \leq n; \quad P_{j,r}^0 = P_{r,j}^0, \quad 1 \leq j \leq r \leq n$$

$$\tilde{A}_q = \sum_{\nu=-\infty}^{\infty} (-1)^{\nu} \theta_{q+\nu}\theta_{q-\nu}, \quad \theta_q = \begin{cases} C_n^m, & q = 2m \\ 0, & q \neq 2m \end{cases}$$

Proof. Let $u(t) \in W_2^{2n+s}(R_+; H)$.Then, obviously

$$\|P_0 u\|_{W_2^s}^2 = \left\| A^s \left(-\frac{d^2}{dt^2} + A^2 \right)^n u \right\|_{L_2}^2 + \left\| \left(-\frac{d^2}{dt^2} + A^2 \right)^n u^{(s)} \right\|_{L_2}^2 =$$

$$= \left\| \sum_{q=0}^{n} C_n^q A^{2n-2q+s} u^{(q)} \right\|_{L_2}^2 + \left\| \sum_{q=0}^{n} C_n^q A^{2n-2q} u^{(2q+s)} \right\|_{L_2}^2 =$$

$$= \left\| \sum_{q=0}^{2n} Q_q A^{2n-2q+s} u^{(q)} \right\|_{L_2}^2 + \left\| \sum_{q=0}^{2n} Q_q A^{2n-2q} u^{(q+s)} \right\|_{L_2}^2, \tag{92}$$

where

$$\theta_q = \begin{cases} C_n^m, & q = 2m, \quad m = \overline{0,n} \\ 0, & q \neq 2m \end{cases}.$$

On the other, using the results of the paper [1] we have

$$\left\| \sum_{q=0}^{2n} Q_q A^{2n-2q+s} u^{(q)} \right\|_{L_2}^2 = \sum_{q=0}^{2n} \tilde{A}_q \left\| A^{2n-2q+s} u^{(q)} \right\|_{L_2}^2 - (R_0 \tilde{\varphi}_0, \tilde{\varphi}_0)_{H^{2n}},$$

$$\left\| \sum_{q=0}^{2n} Q_q A^{2n-2q} u^{(q+s)} \right\|_{L_2}^2 = \sum_{q=0}^{2n} \tilde{A}_q \left\| A^{2n-2q} u^{(q+s)} \right\|_{L_2}^2 - (R_0 \tilde{\varphi}_s, \tilde{\varphi}_s)_{H^{2n}}$$

where

$$H^{2n} = \underbrace{H \oplus ... \oplus H}_{2n}, \quad \tilde{A}_q = \sum_{\nu=-\infty}^{\infty} (-1)^{\nu} \theta_{q-\nu}\theta_{q+\nu} \quad (\theta_s = 0,\ s < 0,\ s > 2n),$$

$$R_0 = \left(p_{j,r}^0\right), \quad 1 \le j \le r \le 2n, \quad \tilde{\varphi}_0 = \left(A^{2n-\nu-1/2+s} u^{(\nu)}(0)\right), \quad \nu = \overline{0, 2n-1},$$

$$\tilde{\varphi}_s = \left(A^{2n-\nu-1/2+s}u^{(\nu+s)}(0)\right), \quad \nu = \overline{0, 2n-1}.$$

Thus we proved the lemma

Lemma 3.2. *For all $\beta \in \left[0; d_{2n,j}^{-2n}\right)$ and $u(t) \in W_2^{2n+s}(R_+; H)$ it holds the inequality*

$$\begin{aligned} &\|\psi(d/dt; \beta; A) u(t)\|_{W_2^s}^2 + (Q_j(\beta)\tilde{\varphi}, \tilde{\varphi})_{H^{2n+s}} = \\ &= \|P_0 u\|_{W_2^s}^2 - \beta \left\|A^{2n-j}u^{(j)}\right\|_{W_2^s}^2 \end{aligned} \tag{93}$$

where $Q_j(\beta) = M_j(\beta) - Q_0$, $M_j(\beta) = (m_{j,r}(\beta))_{j,r=1}^{2n+s}$, $m_{j,r}(\beta) = m_{r,j}(\beta)$, $j \le r$ and Q_0 is determined from lemma 3.2.

Proof of this lemma follows from simple calculations used in the paper [2], definition of $\tilde{A}_j(\beta)$ and corollary of lemma 3.2.

It holds

Theorem 3.1. *The operator P_0, determined in the form*

$$P_0 u(t) = P_0(d/dt) u(t) \equiv \left(-\frac{d^2}{dt^2} + A^2\right)^n u(t), \quad u(t) \in \overset{\circ}{W}{}_2^{2n+s}(R_+; H)$$

realzies an isomorphism between the spaces $\overset{\circ}{W}{}_2^{2n+s}(R_+; H)$ and $W_2^s(R_+; H)$.

Proof. Let's consider the equation $P_0 u = 0$, $\overset{\circ}{W}{}_2^{2n+s}(R_+; H)$, i.e.

$$\left(-\frac{d^2}{dt^2} + A^2\right)^n u(t) = 0, \tag{94}$$

$$u^{(\nu)}(0) = 0, \quad \nu = \overline{0, n-1}. \tag{95}$$

Obviously, equation (94) has a general solution in the form

$$u_0(t) - e^{-tA}\left[\xi_0 + tA\xi_1 + ... + t^{n-1}A^{n-1}\xi_{n-1}\right], \quad \xi_0, ..., \xi_{n-1} \in H_{2n+s-1/2}.$$

It follows from condition (95) that $\xi\nu = 0$, $\nu = \overline{0, n-1}$, i.e. $u_0(t) = 0$. On the other hand, it is easy to see that a theorem on intermediate derivatives [2] yield

$$\|P_0 u\|^2_{W_2^s} = \left\| \sum_{m=0}^{2n} C_{2n}^m A^{2n-m} u^{(m)} \right\|^2_{W_2^s} \leq const \|u\|^2_{W_2^s}.$$

Now, let's show that the equation $P_0 u = f$ is solvable for all $f(t) \in W_2^s(R_+; H)$, $u(t) \in W_2^{2n+s}(R_+; H)$. As first we consider the equation

$$P_0(d/dt)\,\tilde{u}(t) = \left(-\frac{d^2}{dt^2} + A^2\right)^n \tilde{u}(t) = \tilde{f}(t), \quad t \in R = (-\infty, +\infty), \tag{96}$$

where

$$\tilde{f}(t) = \begin{cases} f(t), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

Show that equation (96) has the solution $\tilde{u}(t) \in W_2^{2n+s}(R_+; H)$ that satisfies it for all $t \in R$. Really, by means of the Fourier transformation we get

$$\tilde{u}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\xi^2 E^2 + A^2\right)^{-n} \hat{\tilde{f}}(\xi)\, e^{i\xi t} d\xi, \quad t \in R.$$

Obviously

$$\|\tilde{u}\|^2_{W_2^{2n+s}} = \left\|\tilde{u}^{(2n+s)}\right\|^2_{L_2} + \left\|A^{2n+s}\tilde{u}(s)\right\|^2_{L_2} = \left\|(-i\xi)^{2n+s} \hat{\tilde{u}}(\xi)\right\|^2_{L_2} +$$

$$+\left\|A^{2n+s}\tilde{u}(s)\right\|^2_{L_2} \leq \sup_{\xi\in R}\left\|\xi^{2n}\left(\xi^2 E + A^2\right)^{-n}\right\|^2_{H\to H}\left\|\xi^s \hat{\tilde{f}}(\xi)\right\|^2_{L_2} +$$

$$+\sup_{\xi\in R}\left\|A^{2n}\left(\xi^2 E + A^2\right)^{-n}\right\|^2_{H\to H}\left\|A^s \hat{\tilde{f}}(\xi)\right\|^2_{l_2}. \tag{97}$$

On the other hand, for $\xi \in R$

$$\left\|\xi^{2n}\left(\xi^2 E+A^2\right)^{-n}\right\|=\sup_{\mu\in\sigma(A)}\left\|\xi^{2n}\left(\xi^2+\mu^2\right)^{-n}\right\|\leq 1$$

and

$$\left\|A^{2n}\left(\xi^2 E+A^2\right)^{-n}\right\|=\sup_{\mu\in\sigma(A)}\left\|\mu^{2n}\left(\xi^2+\mu^2\right)^{-n}\right\|\leq 1$$

Therefore, it follows from inequality (97) that

$$\|\tilde{u}\|^2_{W_2^{2n+s}(R;H)}=\left\|\xi^s\overset{\wedge}{\tilde{f}}(\xi)\right\|^2_{L_2}+\left\|A^s\overset{\wedge}{\tilde{f}}(\xi)\right\|^2_{L_2}=\left\|\tilde{f}\right\|^2_{W_2^{2s}(R;H)}$$

Consequently $\tilde{u}(t)\in W_2^{2n+s}(R_+;H)$. Denote contraction of $\tilde{u}(t)$ in $R_+=[0,\infty)$ by $\upsilon(t)$. Then, obviously $\upsilon(t)\in W_2^{2n+s}(R_+;H)$. Now, let's look for the solution of the equation

$$P_0 u(t)=\left(-\frac{d^2}{dt^2}+A^2\right)^n u(t)=f(t),\quad t\in R_+$$

in the form

$$u(t)=\upsilon(t)+e^{-tA}\left[\xi_0+tA\xi_1+...+t^{n-1}A^{n-1}\xi_{n-1}\right],$$

where $\xi_0,...,\xi_{n-1}\in H_{2n+s-1/2}$. Hence we can easily find in a unique way all $\xi_k,\quad k=\overline{0,n-1}$ from the initial conditions $u^{(\nu)}(0)=0,\quad \nu=\overline{0,n-1}$. Since the mapping $\overset{\circ}{W}{}_2^{2n+s}(R_+;H)\to W_2^{2s}(R_+;H)$ is continuous and one- to-one, then it follows from the Banach theorem that there exists the inverse $P_0^{-1}:W_2^s(R_+;H)\to\overset{\circ}{W}{}_2^{2n+s}(R_+;H)$. The theorem is proved.

3.3. The basic result

It follows from the theorem on intermediate derivatives [2] that for $u(t)\in W_2^{2n+s}(R_+;H)$

$$\left\|A^{2n-j}u^{(j)}(t)\right\|^2_{W_2^s(R_+;H)}\leq const\|u\|^2_{W_2^{2n+s}(R_+;H)}.$$

On the other hand, it follows from theorem 3.1. that the norms $\|P_0u\|_{W_2^{2n+s}(R;H)}$ and $\|u\|_{W_2^{2n+s}(R;H)}$ are equivalent on the space $\overset{\circ}{W}{}_2^{2n+s}(R_+;H)$ i.e.

$$c_1\|u\|_{W_2^{2n+s}} \leq \|P_0u\|_{W_2^{2s}} \leq c_2\|u\|_{W_2^{2n+s}}, \quad (c_1, c_2 > 0)\,.$$

Then the theorem on intermediate derivatives [2] gives that the following numbers

$$\overset{\circ}{N}{}_j^{(s)}(R_+;H) = \sup_{0\neq u(t)\in \overset{\circ}{W}{}_2^{2n}(R_+;H)} \left\|A^{2n-j}u^{(j)}(t)\right\|_{W_2^s(R_+;H)} \|p_0u\|^{-1}_{W_2^s(R_+;H)}\,, \quad j=\overline{1,2n-1}$$

are finite. Now, let's prove a theorem on finding the numbers $\overset{\circ}{N}{}_j^{(s)}(R_+;H)$, $j=\overline{1,2n-1}$.

Theorem 3.2. *It holds the equality*

$$\overset{\circ}{N}{}_j^{(s)}(R_+;H) = \begin{cases} d_{2n,j}^n \ \ if \ \ \det Q\left(\beta;\{\nu\}_{\nu=0}^{n-1}\right)\neq 0,\ \beta\in\left[0,d_{2n,j}^{-2n}\right) \\ \overset{\circ}{\mu}{}_{2n,j}^{-1/2} \quad in\ \ the\ \ contrary\ \ case\,, \end{cases}.$$

where $Q\left(R;\{\nu\}_{\nu=0}^{n-1}\right)$ *is a matrix obtained from* $Q(\beta)$ *by rejecting the first* n *rows and columns, and* $\overset{\circ}{\mu}{}_{2n,j}^{-1/2}$ *is the least root of the equation* $\det Q\left(\beta;\{\nu\}_{\nu=0}^{n-1}\right) = 0$ *from the interval* $\left[0,d_{2n,j}^{-2n}\right)$.

Proof. In the previous section the proved that in the space $W_2^{2n+s}(R_+;H)$ the numbers

$$\overset{\circ}{N}{}_j^{(s)}(R;H) = \sup_{0\neq u(t)\in W_2^{2n+s}(R;H)} \left\|A^{2n-j}u^{(j)}\right\|_{W_2^s(R;H)}\|p_0u\|^{-1}_{W_2^{2s}(R;H)} = d_{2n,j}^n.$$

Then, obviously, in the space

$$\overset{\circ}{W}{}_2^{2n+s}\left(R_+;H;\ \overline{0,2n+s-1}\right) =$$

$$= \left\{u/u\in W_2^{2n+s}(R_+;H)\,,\ \ u^{(\nu)}(0)=0, \quad \nu=\overline{0,2n+s-1}\right\}$$

the numbers

$$\overset{\circ}{N}{}_j^{(s)}(R_+;H;\ 0,2n+s-1) = d_{2n,j}^n$$

Really, if $\overset{\circ}{N}{}_j^{(s)} = d_{2n,j}^{n}$ then for $\forall\varepsilon > 0$ we can find such a finite function $\upsilon_\varepsilon(t) \in W_2^{2n+s}(R_+; H)$that $\upsilon_\varepsilon(t) = 0,\ |t| > 0 \quad (\varepsilon > 0)$ therefore

$$\|P_0(d/dt)\,\upsilon_\varepsilon\|^2_{W_2^{2s}(R_+;H)} - \left(d_{2n,j}^{-2} + \varepsilon\right)\left\|A_{2n-j}\upsilon_\varepsilon^{(j)}\right\|^2_{W_2^{2s}(R_+;H)} < \varepsilon.$$

Then assuming $\tilde{\upsilon}_\varepsilon(t) = \upsilon_\varepsilon(t - 2N) \in \overset{\circ}{W}{}_2^{2n+s}\left(R_+; H; \overline{0, 2n-1}\right)$ we see that

$$\|P_0(d/dt)\,\tilde{\upsilon}_\varepsilon\|^2_{W_2^{2s}(R_+;H)} - \left(d_{2n,j}^{-2n} + \varepsilon\right)\left\|A^{2n-j}\tilde{\upsilon}_\varepsilon^{(j)}\right\|^2_{W_2^{s}} < \varepsilon, \tag{98}$$

follows from the last inequality. On the other hand, it follows from lemma 3.3. that for all $u \in \overset{\circ}{W}{}_2^{2n+s}$ it holds

$$\|P_0(d/dt)\,u\|^2_{W_2^{2s}(R_+;H)} \geq \beta\left\|A^{2n-j}u^{(j)}\right\|^2_{W_2^{s}(R_+;H)}$$

Passing to limit as $\beta \to d_{2n,j}^{-2n}$ from the last inequality we get

$$\overset{\circ}{N}{}_j^{(s)}\left(R_+; H; \overline{0, 2n+s-1}\right) > d_{2n,j}^{-n}. \tag{99}$$

It follows from (98) and (99) that

$$\overset{\circ}{N}{}_j^{(s)}(R_+; H) = d_{2n,j}^{-n} \tag{100}$$

Since $\overset{\circ}{W}{}_2^{2n+s}\left(R_+; H; \overline{0, 2n+s-1}\right) \subset \overset{\circ}{W}{}_2^{2n+s}(R_+; H)$ obviously

$$\overset{\circ}{N}{}_j^{(s)}(R_+; H) \geq d_{2n,j}^{-n} = \overset{\circ}{N}{}_j^{(s)}\left(R_+; H; \overline{0, 2n+s-1}\right).$$

Further, it follows from lemma 3.3 and equality (93), that

$$Q\left(\beta; \{\nu\}_{\nu=0}^{n-1}\tilde{\varphi}, \tilde{\varphi}\right)_{H^{2n+s}} > 0,$$

then

$$\|\psi(d/dt; \beta; A)\,u\|^2_{W_2^s} + Q\left(\beta; \{\nu\}_{\nu=0}^{n-1}\tilde{\varphi}, \tilde{\varphi}\right)_{H^{2n+s}} = \|P_0 u\|^2_{W_2^s} - \beta\left\|A^{2n-j}u^{(j)}\right\|^2_{W_2^s} > 0.$$

Passing to limit at the first part of the last relation w have that for all $u \in \overset{\circ}{W}{}_2^{2n+s}(R_+; H)$ it holds the inequality

$$\left\|A^{2n-j}u^{(j)}\right\|_{W_2^s(R_+;H)} \leq d_{2n,j}^{-n}\|P_0 u\|^2_{W_2^s(R_+;H)},$$

i.e.

$\overset{\circ}{N}{}_j^{(s)}(R_+;H) \leq d_{2n,j}^{-n}$. But since $\overset{\circ}{N}{}_j^{(s)}(R_+;H) \geq d_{2n,j}^{-n}$ then hence it follows that $\overset{\circ}{N}{}_j^{(s)}(R_+;H) = d_{2n,j}^{-n}$ if $Q\left(\beta;\{\nu\}_0^{n-1}\right) > 0$ for $\beta \in \left[0, d_{2n,j}^{-2n}\right)$. On the other hand obviously, $\overset{\circ}{N}{}_j^{(s)}(R_+;H) \geq d_{2n,j}^{-n}$ therefore $\left(\overset{\circ}{N}{}_j^{(s)}(R_+;H)\right)^{-2} \in \left(0; d_{2n,j}^{-2n}\right)$. Show that for $\beta \in \left(0; \left(\overset{\circ}{N}{}_j^{(s)}(R_+;H)\right)^{-2}\right)$

$$Q\left(\beta;\{\nu\}_0^{n-1}\right) > 0.$$

Really, for $\beta \in \left(0; \left(\overset{\circ}{N}{}_j^{(s)}(R_+;H)\right)^{-2}\right)$

$$\|\psi(d/dt;\beta;A)u\|_{W_2^s}^2 + Q\left(\beta;\{\nu\}_{\nu=0}^{n-1}\tilde{\varphi},\tilde{\varphi}\right)_{H^{2n+s}} \geq \|P_0u\|_{W_2^s}^2\left(1-\beta\left(\overset{\circ}{N}{}_j^{(s)}(R_+;H)\right)^{-2}\right) > 0.$$

Then, applying lemma 3.1 we get that for the solution of the Cauchy problem for all $\tilde{\varphi}$ it holds the inequality

$$Q\left(\beta;\{\nu\}_{\nu=0}^{n-1}\tilde{\varphi},\tilde{\varphi}\right) > 0,$$

i.e. for $\beta \in \left(0; \left(\overset{\circ}{N}{}_j^{(s)}(R_+;H)\right)^{-2}\right)$ the matrix $Q\left(\beta;\{\nu\}_{\nu=0}^{n-1}\right) > 0$. Thus $\overset{\circ}{N}{}_j^{(s)}(R_+;H) = d_{2n,j}^{-2n}$ if $Q\left(\beta;\{\nu\}_{\nu=0}^{n-1}\right) > 0$ and this means that $\det Q\left(\beta;\{\nu\}_{\nu=0}^{n-1}\right) \neq 0$. On the other hand, if $Q\left(\beta;\{\nu\}_{\nu=0}^{n-1}\right) > 0$ for all $\beta \in \left[0, d_{2n,j}^{-n}\right)$, then $\overset{\circ}{N}{}_j^{(s)}(R_+;H) < d_{2n,j}^{-2n}$ therefore $\overset{\circ}{N}{}_j^{(s)}(R_+;H) \in \left(d_{2n,j}^{-n}\right)$. Then for $\beta \in \left(\left(\overset{\circ}{N}{}_j^{(s)}(R_+;H)\right)^{-2}; d_{2n,j}^{-2n}\right)$ from the definition of the number $\overset{\circ}{N}{}_j^{(s)}(R_+;H)$ it follows that there exists such $w_\beta(t)$ vector-function that

$$\|P_0(d/dt)w_\beta\|_{W_2^s}^2 < \beta\left\|A^{2n-j}w_\beta^{(j)}\right\|_{W_2^s}^2.$$

Then, it follows from lemma 3.3. that for $\beta \in \left(\left(\overset{\circ}{N}{}_j^{(s)}(R_+;H)\right)^{-2}; d_{2n,j}^{-2n}\right)$

$$\|\psi(d/dt;\beta;A)w_\beta\|_{W_2^s}^2 + \left(Q\left(\beta;\{\nu\}_{\nu=0}^{n-1}\right)\tilde{\varphi},\tilde{\varphi}\right) < 0.$$

Consequently, for $\beta \in \left(\left(\overset{\circ}{N}{}_j^{(s)} (R_+; H) \right)^{-2} ; d_{2n,j}^{-2n} \right)$

$$\left(Q \left(\beta; \{\nu\}_{\nu=0}^{n-1} \right) \tilde{\varphi}_\beta, \tilde{\varphi}_\beta \right) < 0.$$

Thus, the $\lambda_1 (\beta)$ least eigen value changes its sign for $\beta_0 = \left(\overset{\circ}{N}{}_j^{(s)} (R_+; H) \right)^{-2}$. Since this point is the last root of the equation $\det Q \left(\beta; \{\nu\}_{\nu=0}^{n-1} \right) = 0$, the theorem is proved.

Now, let's formulate a theorem on solvability of the problem (85), (86).

Theorem 3.3. *Let A be a self-adjoint positive- definite operator, the operators $B_j = A_j A^{-j}$ and $D_j = A^s A_j A^{-(j+s)}$ $\left(j = \overline{1, 2n-1} \right)$ be bounded in H and it hold the inequality*

$$\alpha = \sum_{j=1}^{2n-1} \max \left(\|B_{2n-j}\| ; \|D_{2n-j}\| \overset{\circ}{N}{}_j^{(s)} (R_+; H) \right) < 1$$

where the numbers $\overset{\circ}{N}{}_j^{(s)}$ are defined from theorem 3.2. Then for any $t \in R_+$ there exists a vector-function $u(t) \in W_2^{2n+s} (R; H)$ that satisfies the equation (85) for all $f(t) \in W_2^{2s} (R; H)$ and boundary conditions in the sense of convergence

$$\lim_{t \to 0} \left\| u^{(\nu)} \right\|_{2n+s-\nu-1/2} = 0, \ \nu = \overline{0, n-1}$$

and it holds the inequality $\|u\|_{W_2^{2n+s}(R_+;H)} \leq const \|f\|_{W_2^s(R_+;H)}$.

The proof of this theorem word by word repeats the proof of the theorem from the previous section and we don't cite it here.

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