Introduction to the Geometry of Classical Dynamics

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Preface

The aim of this paper is to lead (in most elementary terms) an undergraduate student of Mathematics or Physics from the historical Newtonian-d'Alembertian dynamics up to the border with the modern (*geometrical*) Lagrangian-Hamiltonian dynamics, without making any use of the traditional (*analytical*) formulation of the latter. [1]

Our expository method will in principle adopt a rigorously coordinate-free language, apt to gain – from the very historical formulation – the 'consciousness' (at an early stage) of the geometric structures that are 'intrinsic' to the very nature of classical dynamics. The coordinate formalism will be confined to the ancillary role of providing simple proofs for some geometric results (which would otherwise require more advanced geometry), as well as re-obtaining the *local* analytical formulation of the theory from the *global* geometrical one. [2]

The main conceptual tool of our approach will be the simple and general notion of *differential equation in implicit form*, which, treating an equation just as a subset extracted from the tangent bundle of some manifold through a geometric or algebraic property, will directly allow us to capture the structural core underlying the evolution law of classical dynamics. [3]

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[1] Such an Introduction will cover the big gap existing in the current literature between the (empirical) elementary presentation of Newtonian-d'Alembertian dynamics and the (abstract) differential-geometric formulation of Lagrangian-Hamiltonian dynamics. Standard textbooks on the latter are [1][2][3][4], and typical research articles are [5][6][7][8][9][10].

[2] The differential-geometric techniques adopted in this paper will basically be limited to smooth manifolds embedded in Euclidean affine spaces, and are listed in Appendix (whose reading is meant to preceed that of the main text). More advanced geometry can be found in the textbooks already quoted, as well as in a number of excellent introductions, e.g. [11][12][13][14].

[3] Research articles close to the spirit of this approach are, among others, [15][16] (on implicit differential equations) and [17][18][19][20] (on their role in advanced dynamics).

Contents

Chapter 1

From Newton to d'Alembert

In this chapter, we shall recall the problem of classical particle dynamics, Newton's answer to the problem and d'Alembert's reformulation of the answer. The latter will then be shown to correspond to a differential equation in implicit form on a Euclidean space.

1.1 The data

Classical particle dynamics deals with an empirical problem, whose data – in the *simplest* cases – can be described in mathematical terms as follows.

Configuration space

A *reference space* – 'mathematical extension' of a rigid body carrying an observer – is conceived as a 3-dimensional, Euclidean, affine space $\mathcal{E}_3$, modelled on a vector space E_3 with inner product $\cdot$ (*time* will be conceived as an oriented, 1-dimensional, Euclidean, affine space, classically identified with the real line $\mathbb{R}$).

'Particle' is synonymous with 'point-like body', i.e. a body whose *position* in $\mathcal{E}_3$ is defined by a single point of $\mathcal{E}_3$.

Therefore, for a given (ordered) system of ν particles, a position – or *configuration* – in $\mathcal{E}_3$ is defined by a single point of $\mathcal{E} := {\mathcal{E}_3}^{\nu}$ (3ν-dimensional, Euclidean, affine space, modelled on $E := {E_3}^{\nu}$). [1]

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[1] Recall that the inner product in E is defined by putting, for all $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_\nu)$ and $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_\nu)$ belonging to E,

$$\mathbf{v} \cdot \mathbf{w} := \sum_{i=1}^{\nu} \mathbf{v}_i \cdot \mathbf{w}_i$$

The particle system may generally be subject in $\mathcal{E}_3$ to some *holonomic* (or *positional*) *constraints*, owing to which it is virtually allowed to occupy only the positions belonging to an embedded submanifold

$$Q \subset \mathcal{E}$$

called *configuration space* of the system in $\mathcal{E}_3$.

Q generally consists of all the points $\mathrm{p} \in \mathcal{E}$ satisfying a number $\kappa < 3\nu$ of independent scalar equalities $f_\alpha(\mathrm{p}) = 0$ $(\alpha = 1, \ldots, \kappa)$, called *two-sided constraints*, and/or some strict scalar inequalities $g_\beta(\mathrm{p}) > 0$ $(\beta = 1, \ldots, \mu)$, called strict *one-sided constraints*. Under usual hypotheses of regularity on the constraints, Q is an embedded submanifold of $\mathcal{E}$, whose dimension $n := \dim Q = 3\nu - \kappa$ –called *number of the degrees of freedom* of the system– is given by the dimension of the Euclidean environment minus the number of the two-sided constraints (in absence of two-sided constraints, Q is an open submanifold). [2]

Mass distribution

The response of the system to any internal or external influence, will generally depend on how 'massive' its particles are, the *inertial mass* of a particle being conceived as a positive scalar quantity.

The *inertial mass distribution* carried by the system will be denoted by

$$m := (m_1, \ldots, m_\nu)$$

Force field

The 'force' –$\delta \acute{\upsilon} \nu \alpha \mu \iota \varsigma$– resultant of all the internal and/or external influences acting in $\mathcal{E}_3$ on the particles and not depending on their being constrained or unconstrained, is generally described as a smooth vector-valued mapping, called *force field*,

$$\mathbf{F} : U \times E \subset T\mathcal{E} \to E : (\mathrm{p}, \mathbf{v}) \longmapsto \mathbf{F}(\mathrm{p}, \mathbf{v}) = (\mathbf{F}_1(\mathrm{p}, \mathbf{v}), \ldots, \mathbf{F}_\nu(\mathrm{p}, \mathbf{v}))$$

where U denotes an open subset of $\mathcal{E}$ containing Q.

Remark that, if $\mathbf{F}|_{\{\mathrm{p}\} \times E} = \text{const.}$ for all $\mathrm{p} \in U$, then $\mathbf{F}$ –called *positional* force field– can be regarded as a smooth mapping $\mathbf{f} : U \subset \mathcal{E} \to E : \mathrm{p} \mapsto \mathbf{f}(\mathrm{p})$, where $\mathbf{f}(\mathrm{p})$ denotes the constant value of $\mathbf{F}|_{\{\mathrm{p}\} \times E}$. [3]

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[2] We shall not consider non-strict one-sided constraints, which would give rise to a manifold Q with boundary (nor shall we consider time-dependent constraints, which would give rise to a manifold Q fibred over the real line).

[3] We shall not consider time-dependent force fields, which would later give rise to time-dependent differential equations.

Mechanical system

The above empirical *mechanical system* will briefly be denoted by the triplet

$$\mathcal{S} := (Q, m, \mathbf{F})$$

1.2 The question

With reference to such a mechanical system $\mathcal{S}$, the basic problem of dynamics can be expressed in the following terms.

Smooth motions

A *smooth motion* of the particle system in the reference space $\mathcal{E}_3$ is described as a smooth curve of $\mathcal{E}$, say

$$\gamma : I \subset \mathbb{R} \to \mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

Such a curve γ is meant to establish a configuration $\mathrm{p}(t)$ at each *instant* t of a *time interval* $I \subset \mathbb{R}$, along which the positions $\mathrm{p}(t) = (\mathrm{p}_1(t), \ldots, \mathrm{p}_\nu(t))$ of the particles, their *velocities* $\dot{\mathrm{p}}(t) = (\dot{\mathrm{p}}_1(t), \ldots, \dot{\mathrm{p}}_\nu(t))$ and their *accelerations* $\ddot{\mathrm{p}}(t) = (\ddot{\mathrm{p}}_1(t), \ldots, \ddot{\mathrm{p}}_\nu(t))$ (as well as the derivatives of any order) vary continuously.

Dynamically possible motions

Smooth *dynamics* basically deals with the 'time-evolution problem' – in the unknown γ – expressed by the following question:

"For the above constrained point-mass system, what are the smooth motions – in the chosen reference space – that are *possible* under the action of the given $\delta\acute{\upsilon}\nu\alpha\mu\iota\varsigma$? "

Such motions will briefly be called the *dynamically possible motions* (DPMs) of $\mathcal{S}$ (whereas the smooth motions which would be possible in absence of force, i.e. $\mathbf{F} = \mathbf{0}$, will be said to be the *inertial motions* of $\mathcal{S}$).

1.3 The answer after Newton

After Newton, the answer to the above 'predictive' question is given by the following law.

Newton's law of constrained dynamics

A smooth curve

$$\gamma : I \to \mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

is a *DPM* of $\mathcal{S}$, iff, for all $t \in I$, [4]

$$\mathrm{p}(t) \in Q$$

$$m\, \ddot{\mathrm{p}}(t) = \mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) + \Phi(t)$$
$$\Phi(t) \in T^{\perp}_{\mathrm{p}(t)}Q$$

The first condition just exhibits the 'kinematical effects' of the constraints, which only allow motions living in Q.

The second condition is the classical *Newton's law* with a right hand side encompassing the possible 'dynamical effects' of the contraints, expressed by an 'unknown' *constraint reaction* $\Phi(t) \in E$.

The third condition expresses the only 'known' empirical requisite of the constraint reaction, which – apart from possible 'frictions' tangent to Q – is always orthogonal to Q. [5]

1.4 d'Alembert's reformulation

After d'Alembert, the unknown constraint reaction can be 'cancelled' from Newton's law of constrained dynamics as follows.

d'Alembert's principle of virtual works

The last two of the above conditions can obviously be expressed in the form

$$\mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) - m\, \ddot{\mathrm{p}}(t) \in T^{\perp}_{\mathrm{p}(t)}Q$$

[4] For any $\mu = (\mu_1, \ldots, \mu_\nu) \in \mathbb{R}^\nu$ and $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_\nu) \in E$, we shall put $\mu\,\mathbf{w} := (\mu_1 \mathbf{w}_1, \ldots, \mu_\nu \mathbf{w}_n)$. Moreover, for any $\mathrm{p} \in Q$, $T^{\perp}_{\mathrm{p}}Q$ will denote the orthogonal complement in E of the tangent vector space $T_{\mathrm{p}}Q$.

[5] If the empirical law of friction is to be taken into consideration, then you will embody it in $\mathbf{F}$.

that is,

$$\Big(\mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) - m\,\ddot{\mathrm{p}}(t)\Big) \cdot \delta\mathrm{p} = 0\,, \quad \forall\, \delta\mathrm{p} \in T_{\mathrm{p}(t)}Q$$

called *d'Alembert's principle of virtual works* (since the inner product therein defines the *work* of *active force* $\mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t))$ and *inertial force* $-m\ddot{\mathrm{p}}(t)$ along any *virtual displacement* $\delta\mathrm{p}$, i.e. any 'infinitesimal' displacement tangent to Q and therefore virtually allowed by the constraints). [6]

So a smooth curve

$$\gamma : I \to \mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

is a *DPM* of $\mathcal{S}$, iff it satisfies, for all $t \in I$ the *time-evolution law*

$$\mathrm{p}(t) \in Q\,, \quad m\,\ddot{\mathrm{p}}(t) \cdot \delta\mathrm{p} = \mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) \cdot \delta\mathrm{p}\,, \quad \forall\, \delta\mathrm{p} \in T_{\mathrm{p}(t)}Q \qquad (\diamond)'$$

1.5 d'Alembert's implicit equation

From the mathematical point of view, condition $(\diamond)'$ shows that determining the *DPM*s of $\mathcal{S}$ is a *second-order* differential problem, whose unknown is a smooth curve of $\mathcal{E}$. It turns into a *first-order* differential problem, whose unknown is a smooth curve of $T\mathcal{E}$, as follows.

Tangent dynamically possible motions

If

$$\gamma : I \to \mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

is *DPM* of $\mathcal{S}$, its *tangent lift*

$$\dot{\gamma} : I \to T\mathcal{E} : t \longmapsto \dot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t))$$

will be called a *tangent dynamically possible motion* (*TDPM*) of $\mathcal{S}$.

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[6] Owing to footnotes [1] and [4], d'Alembert's principle reads

$$\sum_{i=1}^{\nu} \Big(\mathbf{F}_i(\mathrm{p}(t), \dot{\mathrm{p}}(t)) - m_i \ddot{\mathrm{p}}_i(t)\Big) \cdot \delta\mathrm{p}_i = 0\,, \quad \forall\, \delta\mathrm{p} = (\delta\mathrm{p}_1, \ldots, \delta\mathrm{p}_\nu) \in T_{\mathrm{p}(t)}Q$$

Remark that, if Q is an open submanifold of $\mathcal{E}$ (absence of two-sided constraints), one has $T^{\perp}_{\mathrm{p}}Q = E^{\perp} = \{\mathbf{0}\}$ for all $\mathrm{p} \in Q$ (absence of constraint reaction) and then d'Alembert's principle simply reads

$$m_i \ddot{\mathrm{p}}_i(t) = \mathbf{F}_i(\mathrm{p}(t), \dot{\mathrm{p}}(t))\,, \quad i = 1, \ldots, \nu$$

*DPM*s and *TDPM*s bijectively correspond to one another, since the *tangent lift operator* $\gamma \mapsto \dot\gamma$ is obviously inverted by the *base projection operator* $\dot\gamma \mapsto \gamma$. Trough such a bijection, the problem of determining the *DPM*s proves to be naturally equivalent to that of determining the *TDPM*s.

Owing to $(\diamond)'$, a smooth curve

$$c : I \to T\mathcal{E} : t \longmapsto c(t) = (\mathrm{p}(t), \mathbf{v}(t))$$

is a *TDPM* of $\mathcal{S}$, iff it satisfies, for all $t \in I$, the *time-evolution law*

$$\mathrm{p}(t) \in Q\,,\ \dot{\mathrm{p}}(t) = \mathbf{v}(t)\,,\ m\,\dot{\mathbf{v}}(t)\cdot\delta\mathrm{p} = \mathbf{F}(\mathrm{p}(t), \mathbf{v}(t))\cdot\delta\mathrm{p}\,,\ \forall\,\delta\mathrm{p} \in T_{\mathrm{p}(t)}Q \qquad (\diamond)$$

d'Alembert equation

Condition $(\diamond)$ will now be seen to correspond to a first-order differential equation in implicit form on $T\mathcal{E}$ (second-order on $\mathcal{E}$), [7] namely *d'Alembert equation*

$$\begin{aligned}\mathcal{D}_{d'Al} &:= \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TT\mathcal{E} \mid \mathrm{p} \in Q\,,\ \ \mathbf{u} = \mathbf{v}\,,\ \ \mathbf{F}(\mathrm{p}, \mathbf{v}) - m\,\mathbf{w} \in T_{\mathrm{p}}^{\perp}Q\,\}\\ &= \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TT\mathcal{E} \mid \mathrm{p} \in Q\,,\ \ \mathbf{u} = \mathbf{v}\,,\ \ m\,\mathbf{w}\cdot\delta\mathrm{p} = \mathbf{F}(\mathrm{p}, \mathbf{v})\cdot\delta\mathrm{p}\,,\ \forall\,\delta\mathrm{p} \in T_{\mathrm{p}}Q\,\}\\ &\subset T^2\mathcal{E}\end{aligned}$$

Proposition 1 *The* TDPM*s of* $\mathcal{S}$ *are the integral curves of* $\mathcal{D}_{d'Al}$ *(and then the* DPM*s are its base integral curves).*

Proof Recall that a smooth curve

$$c : I \to T\mathcal{E} : t \longmapsto c(t) = (\mathrm{p}(t), \mathbf{v}(t))$$

is an integral curve of $\mathcal{D}_{d'Al}$, iff its tangent lift

$$\dot c : I \to TT\mathcal{E} : t \longmapsto \dot c(t) = (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t))$$

satisfies condition

$$\mathrm{Im}\,\dot c \subset \mathcal{D}_{d'Al}$$

that is, for all $t \in I$,

$$\dot c(t) = (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t)) \in \mathcal{D}_{d'Al}$$

which is exactly condition $(\diamond)$, characterizing the *TDPM*s.

[7] Recall that $T\mathcal{E} = \mathcal{E} \times E$ is a Euclidean affine space modelled on $E \times E$. Its tangent bundle is therefore $TT\mathcal{E} = (\mathcal{E} \times E) \times (E \times E)$.

Also recall that a smooth curve

$$\gamma : I \to T\mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

is a base integral curve of $\mathcal{D}_{d'Al}$, iff its second tangent lift

$$\ddot{\gamma} : I \to TT\mathcal{E} : t \longmapsto \ddot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t))$$

satisfies condition

$$\mathrm{Im}\, \ddot{\gamma} \subset \mathcal{D}_{d'Al}$$

that is, for all $t \in I$,

$$\ddot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t)) \in \mathcal{D}_{d'Al}$$

which is exactly condition $(\diamond)'$, characterizing the *DPM*s.

$\square$

Chapter 2

From d'Alembert to Lagrange

Dynamics is now a problem of *integration*, i.e. determination and/or qualitative analysis of the integral curves of d'Alembert equation (implicit differential equation on Euclidean space $T\mathcal{E}$). In this connection, the latter will be shown to be equivalent to a 'Lagrange equation' (implicit differential equation on manifold TQ), which will naturally be obtained and thoroughly discussed.

2.1 Integrable part of d'Alembert equation

As to the integration of $\mathcal{D}_{d'Al}$, the first step is to extract its *integrable part*, i.e. the region

$$\mathcal{D}^{(i)}_{d'Al} \subset \mathcal{D}_{d'Al}$$

swept by the tangent lifts of all its integral curves (i.e. covered by the orbits of such lifts). As to the extraction of $\mathcal{D}^{(i)}_{d'Al}$, it is quite natural to start from the following remark.

Owing to Prop.1, the base integral curves (if any) of $\mathcal{D}_{d'Al}$ are constrained to live in Q (see condition $(\diamond)'$) and then their tangent lifts, i.e. the integral curves, live in TQ . As a consequence, the tangent lifts of the integral curves live in TTQ , that is to say,

$$\mathcal{D}^{(i)}_{d'Al} \subset TTQ$$

So we obtain

$$\mathcal{D}^{(i)}_{d'Al} \subset \mathcal{D}_{d'Al} \cap TTQ$$

Restriction of d'Alembert equation

The above result suggests focusing on the 'restriction' of $\mathcal{D}_{d'Al}$ obtained via intersection with TTQ, i.e. on the first-order differential equation in implicit form on TQ (second-order on Q)

$$\mathcal{D}_{Lagr} := \mathcal{D}_{d'Al} \cap TTQ$$

which will be called *Lagrange equation.*

Note that Lagrange equation is an effective restriction of d'Alembert equation (i.e. $\mathcal{D}_{Lagr} \subsetneqq \mathcal{D}_{d'Al}$) in presence, and only in presence, of two-sided constraints (when $\dim Q < \dim \mathcal{E}$), as is shown by the following proposition.

Proposition 2 $\mathcal{D}_{Lagr} \subsetneqq \mathcal{D}_{d'Al}$, *iff* $\dim Q < \dim \mathcal{E}$.

Proof Let $\dim Q < \dim \mathcal{E}$ (whence $T_{\mathrm{p}}Q \subsetneqq E$ for all $\mathrm{p} \in Q$). We shall prove that, under the above hypothesis, $\mathcal{D}_{Lagr} \subsetneqq \mathcal{D}_{d'Al}$. Indeed, we can choose $\mathrm{p} \in Q$, $\mathbf{v} \in E - T_{\mathrm{p}}Q$ (whence $(\mathrm{p}, \mathbf{v}) \notin TQ$), $\mathbf{u} = \mathbf{v}$ and $\mathbf{w} = \frac{1}{m}(\mathbf{F}(\mathrm{p}, \mathbf{v}) + \Phi)$ with $\Phi \in T_{\mathrm{p}}^{\perp}Q$ (whence $\mathbf{F}(\mathrm{p}, \mathbf{v}) - m\mathbf{w} \in T_{\mathrm{p}}^{\perp}Q$) and clearly we obtain $(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in \mathcal{D}_{d'Al}$, $(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \notin TTQ$ and then $(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \notin \mathcal{D}_{Lagr}$.
Conversely, let $\dim Q = \dim \mathcal{E}$ (whence $T_{\mathrm{p}}Q = T^2_{(\mathrm{p},\mathbf{v})}Q = E$ for all $(\mathrm{p}, \mathbf{v}) \in TQ$). We shall prove that, under the above hypothesis, $\mathcal{D}_{d'Al} \subset \mathcal{D}_{Lagr}$. Indeed, for any $(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in \mathcal{D}_{d'Al}$, we have $\mathrm{p} \in Q$, $\mathbf{v} \in E = T_{\mathrm{p}}Q$, $\mathbf{u} = \mathbf{v}$ and $\mathbf{w} \in E = T^2_{(\mathrm{p},\mathbf{v})}Q$, that is, $(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in T^2Q \subset TTQ$ and then $(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in \mathcal{D}_{Lagr}$.

□

Extraction of the integrable part

By extracting $\mathcal{D}_{Lagr}$ from $\mathcal{D}_{d'Al}$, via intersection of the latter with TTQ, we just obtain $\mathcal{D}^{(i)}_{d'Al} \neq \emptyset$, as is shown by the following proposition.

Proposition 3 $\mathcal{D}^{(i)}_{d'Al} = \mathcal{D}_{Lagr} \neq \emptyset$

Proof First we remark that $\mathcal{D}_{d'Al}$ and $\mathcal{D}_{Lagr}$ are equivalent equations – i.e. they have the same integral curves – since, owing to inclusions $\mathcal{D}^{(i)}_{d'Al} \subset \mathcal{D}_{Lagr} \subset \mathcal{D}_{d'Al}$, condition $\operatorname{Im} \dot{c} \subset \mathcal{D}_{d'Al}$ (i.e. $\operatorname{Im} \dot{c} \subset \mathcal{D}^{(i)}_{d'Al}$) implies $\operatorname{Im} \dot{c} \subset \mathcal{D}_{Lagr}$ and, conversely, $\operatorname{Im} \dot{c} \subset \mathcal{D}_{Lagr}$ implies $\operatorname{Im} \dot{c} \subset \mathcal{D}_{d'Al}$.
That amounts to saying $\mathcal{D}^{(i)}_{d'Al} = \mathcal{D}^{(i)}_{Lagr}$.
Then we anticipate that $\mathcal{D}_{Lagr}$ is integrable, i.e. $\emptyset \neq \mathcal{D}_{Lagr} = \mathcal{D}^{(i)}_{Lagr}$.
Hence our claim.

□

So the focal point is now to prove the integrability of $\mathcal{D}_{Lagr}$. That will follow from the stronger property of $\mathcal{D}_{Lagr}$ being reducible to normal form, as will be shown in the sequel.

2.2 Lagrange equation

In order to prove the reducibility of Lagrange equation to normal form, we shall need to give a deeper insight into its algebraic formulation and the underlying geometric structures. Its integral curves will then be given a *global* characterization in terms of the above geometric structures (the traditional *local* characterization in coordinate formalism will finally be deduced).

Covector formulation

From the set-theoretical point of view, Lagrange equation can be expressed in the form

$$\mathcal{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,, \;\; m\,\mathbf{w}\cdot\delta\mathrm{p} = \mathbf{F}(\mathrm{p}, \mathbf{v})\cdot\delta\mathrm{p}\,, \; \forall\, \delta\mathrm{p} \in T_{\mathrm{p}}Q\,\} \subset T^2Q$$

From the algebraic point of view, the condition on virtual works characterizing $\mathcal{D}_{Lagr}$ can be given the form of a *covector* equality, as will now be shown.

Associated with the inertial mass distribution m, there is a semi-basic 1-form

$$[m] : T^2Q \to T^*Q : (\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \longmapsto (\mathrm{p}, [m](\mathrm{p}, \mathbf{v}, \mathbf{w}))$$

on T^2Q, called *covector inertial field*, whose value

$$[m](\mathrm{p}, \mathbf{v}, \mathbf{w}) := (m\,\mathbf{w})\cdot|_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$$

at any $(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q$ is the virtual work of $m\,\mathbf{w}$. [1]

Associated with the force field $\mathbf{F}$, there is a semi-basic 1-form

$$F : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \longmapsto (\mathrm{p}, F(\mathrm{p}, \mathbf{v}))$$

on TQ, called *covector force field*, whose value

$$F(\mathrm{p}, \mathbf{v}) := \mathbf{F}(\mathrm{p}, \mathbf{v})\cdot|_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$$

at any $(\mathrm{p}, \mathbf{v}) \in TQ$ is the virtual work of $\mathbf{F}(\mathrm{p}, \mathbf{v})$. [2]

[1] For any $\mathbf{u} \in E$, we define $\mathbf{u}\cdot \in E^*$ by putting $\mathbf{u}\cdot : \mathbf{v} \in E \mapsto \mathbf{u}\cdot\mathbf{v} \in \mathbb{R}$. Then, for any $\mathrm{p} \in Q$, the restriction of $\mathbf{u}\cdot$ to $T_{\mathrm{p}}Q$ yields $\mathbf{u}\cdot|_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$.

[2] The virtual work of a positional force field $\mathbf{f}$ can be regarded as an ordinary 1-form on Q, namely $f : \mathrm{p} \in Q \mapsto (\mathrm{p}, f(\mathrm{p})) \in T^*Q$, $\quad f(\mathrm{p}) := \mathbf{f}(\mathrm{p})\cdot|_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$.

Proposition 4 *Lagrange equation can be given the covector formulation*

$$\mathcal{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,, \;\; [m](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})\}$$

Proof Just notice that condition

$$m\,\mathbf{w} \cdot \delta\mathrm{p} = \mathbf{F}(\mathrm{p}, \mathbf{v}) \cdot \delta\mathrm{p}\,, \; \forall\, \delta\mathrm{p} \in T_{\mathrm{p}}Q$$

means

$$(m\,\mathbf{w}) \cdot |_{T_{\mathrm{p}}Q} = \mathbf{F}(\mathrm{p}, \mathbf{v}) \cdot |_{T_{\mathrm{p}}Q}$$

that is,

$$[m](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})$$

□

Riemannian geodesic curvature field

The reducibility of $\mathcal{D}_{Lagr}$ to normal form requires that, for any choice of the data $(\mathrm{p}, \mathbf{v}) \in TQ$, the algebraic equation $[m](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})$ should be uniquely solvable with respect to the unknown $\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q$. As the latter only appears in the left hand side of the equation, the above property is to be checked through a thorough investigation of $[m]$.

The following geometric considerations −showing that such a semi-basic 1-form on T^2Q is the transformed of a suitable semi-basic 1-form on TQ through a distinguished semi-spray− will prove to be crucial.

Remark that $[m]$ is the semi-basic 1-form 'induced' on T^2Q by the Euclidean metric

$$g_m : E \to E^* : \mathbf{u} \longmapsto g_m(\mathbf{u}) := (m\,\mathbf{u}) \cdot$$

(positive definite, symmetric, linear map), since its value at any $(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q$ is

$$[m](\mathrm{p}, \mathbf{v}, \mathbf{w}) = g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$$

In the same way, there is a semi-basic 1-form

$$g : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \longmapsto (\mathrm{p}, g_{\mathrm{p}}(\mathbf{v}))$$

induced on TQ by g_m, whose value at any $(\mathrm{p}, \mathbf{v}) \in TQ$ is

$$g_{\mathrm{p}}(\mathbf{v}) := g_m(\mathbf{v})\Big|_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$$

g is a Riemannian metric on Q, characterized by the quadratic form – or *Lagrangian function* –

$$K : TQ \to \mathbb{R} : (\mathrm{p}, \mathbf{v}) \mapsto K(\mathrm{p}, \mathbf{v}) := \frac{1}{2} \langle g_{\mathrm{p}}(\mathbf{v}) \mid \mathbf{v} \rangle = \frac{1}{2} m\, \mathbf{v} \cdot \mathbf{v}$$

(the *kinetic energy* of the mechanical system).

Riemannian manifold (Q, K) carries a distinguished semi-spray

$$\Gamma_K : TQ \to T^2Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \mathbf{v}; \mathbf{v}, \Gamma_K(\mathrm{p}, \mathbf{v}))$$

called *Riemannian spray*, uniquely determined by the following property.

Proposition 5 *There exists one, and only one, semi-spray* Γ_K *on* TQ *s.t., for all* $(\mathrm{p}, \mathbf{v}) \in TQ$,

$$g_m(\Gamma_K(\mathrm{p}, \mathbf{v}))\Big|_{T_{\mathrm{p}}Q} = 0$$

Proof (i) *Unicity* First remark that, for any $(\mathrm{p}, \mathbf{v}) \in TQ$, the map

$$\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q \overset{(\alpha)}{\longmapsto} g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$$

is injective, since

$$g_m(\mathbf{w}_1)\Big|_{T_{\mathrm{p}}Q} = g_m(\mathbf{w}_2)\Big|_{T_{\mathrm{p}}Q}$$

– with $\mathbf{w}_1, \mathbf{w}_2 \in T^2_{(\mathrm{p},\mathbf{v})}Q$ and then $\mathbf{w}_1 - \mathbf{w}_2 \in T_{\mathrm{p}}Q$ – implies

$$\begin{aligned} g_{\mathrm{p}}(\mathbf{w}_1 - \mathbf{w}_2) &= g_m(\mathbf{w}_1 - \mathbf{w}_2)\Big|_{T_{\mathrm{p}}Q} \\ &= g_m(\mathbf{w}_1)\Big|_{T_{\mathrm{p}}Q} - g_m(\mathbf{w}_2)\Big|_{T_{\mathrm{p}}Q} \\ &= 0 \end{aligned}$$

that is, recalling that $g_{\mathrm{p}} : T_{\mathrm{p}}Q \to T^*_{\mathrm{p}}Q$ is a linear isomorphism,

$$\mathbf{w}_1 = \mathbf{w}_2$$

So, if there exists a vector in $T^2_{(\mathrm{p},\mathbf{v})}Q$ whose image through (α) is zero, it is unique.

(ii) *Existence* For any $(\mathrm{p}, \mathbf{v}) \in TQ$, put

$$\Gamma_K(\mathrm{p}, \mathbf{v}) := \mathbf{w} + \mathbf{u} \in T^2_{(\mathrm{p},\mathbf{v})}Q$$

with

$$\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q \ , \quad \mathbf{u} := -\, g_{\mathrm{p}}^{-1}\Big(g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} \Big) \in T_{\mathrm{p}}Q$$

The image of $\Gamma_K(\mathrm{p},\mathbf{v})$ through (α) is zero, since

$$\begin{aligned} g_m(\Gamma_K(\mathrm{p},\mathbf{v}))\Big|_{T_{\mathrm{p}}Q} &= g_m(\mathbf{w}+\mathbf{u})\Big|_{T_{\mathrm{p}}Q} \\ &= g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} + g_m(\mathbf{u})\Big|_{T_{\mathrm{p}}Q} \\ &= g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} + g_{\mathrm{p}}(\mathbf{u}) \\ &= g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} - g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} \\ &= 0 \end{aligned}$$

□

Γ_K transforms g (semi-basic 1-form on TQ) into

$$[K] : T^2Q \to T^*Q : (\mathrm{p},\mathbf{v};\mathbf{v},\mathbf{w}) \mapsto (\mathrm{p},[K](\mathrm{p},\mathbf{v},\mathbf{w}))$$

(semi-basic 1-form on T^2Q) by putting, for any $(\mathrm{p},\mathbf{v};\mathbf{v},\mathbf{w}) \in T^2Q$,

$$[K](\mathrm{p},\mathbf{v},\mathbf{w}) := g_{\mathrm{p}}(\mathbf{w} - \Gamma_K(\mathrm{p},\mathbf{v})) \in T^*_{\mathrm{p}}Q$$

$[K]$ will be called *Riemannian geodesic curvature field.*

Actually $[K]$ does not differ from $[m]$, as is shown in the following proposition.

Proposition 6 *Lagrange equation, in covector formulation, reads*

$$\mathcal{D}_{Lagr} = \{(\mathrm{p},\mathbf{v};\mathbf{u},\mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\ , \ \ [K](\mathrm{p},\mathbf{v},\mathbf{w}) = F(\mathrm{p},\mathbf{v})\}$$

Proof Owing to Prop. 4, it will suffice to show that

$$[K] = [m]$$

To this end, note that, for any $(\mathrm{p},\mathbf{v};\mathbf{v},\mathbf{w}) \in T^2Q$, from Prop. 5 we obtain

$$\begin{aligned} [K](\mathrm{p},\mathbf{v},\mathbf{w}) &:= g_{\mathrm{p}}(\mathbf{w} - \Gamma_K(\mathrm{p},\mathbf{v})) \\ &= g_m(\mathbf{w} - \Gamma_K(\mathrm{p},\mathbf{v}))\Big|_{T_{\mathrm{p}}Q} \\ &= g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} - g_m(\Gamma_K(\mathrm{p},\mathbf{v}))\Big|_{T_{\mathrm{p}}Q} \\ &= g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} \\ &= [m](\mathrm{p},\mathbf{v},\mathbf{w}) \end{aligned}$$

□

Normal form

The reducibility of Lagrange equation to normal form immediately follows from Prop. 6.

Consider the *vertical force field*

$$\Delta_F := g^{-1} \circ F : TQ \to TQ : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \Delta_F(\mathrm{p}, \mathbf{v}))$$
$$\Delta_F(\mathrm{p}, \mathbf{v}) := g_{\mathrm{p}}^{-1}(F(\mathrm{p}, \mathbf{v})) \in T_{\mathrm{p}}Q$$

and the semi-spray

$$\Gamma := \Gamma_K + \Delta_F : TQ \to T^2Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \mathbf{v}; \mathbf{v}, \Gamma(\mathrm{p}, \mathbf{v}))$$
$$\Gamma(\mathrm{p}, \mathbf{v}) := \Gamma_K(\mathrm{p}, \mathbf{v}) + \Delta_F(\mathrm{p}, \mathbf{v}) \in T^2_{(\mathrm{p},\mathbf{v})}Q$$

Proposition 7 *Lagrange equation can be put in the normal form*

$$\mathcal{D}_{Lagr} = \operatorname{Im}\Gamma = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,, \quad \mathbf{w} = \Gamma(\mathrm{p}, \mathbf{v})\}$$

Proof Just notice that covector equality

$$[K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})$$

reads

$$g_{\mathrm{p}}(\mathbf{w} - \Gamma_K(\mathrm{p}, \mathbf{v})) = F(\mathrm{p}, \mathbf{v})$$
$$\mathbf{w} - \Gamma_K(\mathrm{p}, \mathbf{v}) = g_{\mathrm{p}}^{-1}(F(\mathrm{p}, \mathbf{v}))$$
$$\mathbf{w} = \Gamma_K(\mathrm{p}, \mathbf{v})) + \Delta_F(\mathrm{p}, \mathbf{v})$$
$$\mathbf{w} = \Gamma(\mathrm{p}, \mathbf{v})$$

□

Integral curves

The condition characterizing the integral curves of Lagrange equation can now be formulated as follows.

Let

$$c : I \to TQ : t \mapsto c(t) = (\mathrm{p}(t), \mathbf{v}(t))$$

be a smooth curve of TQ and

$$\tau_Q \circ c : I \to Q : t \mapsto (\tau_Q \circ c)(t) = \mathrm{p}(t)$$

its projection onto Q. [3]

[3] The tangent lift of $\tau_Q \circ c$ will be denoted by $(\tau_Q \circ c)^{\boldsymbol{\cdot}}$.

Proposition 8 *c is an integral curve of $\mathcal{D}_{Lagr}$, iff*

$$(\tau_Q \circ c)^{\boldsymbol{\cdot}} = c\,, \quad [K] \circ \dot{c} = F \circ c \tag{$\circ$}$$

or, in normal form,

$$\dot{c} = \Gamma \circ c \tag{$\bullet$}$$

Proof As is known, c is an integral curve of $\mathcal{D}_{Lagr}$, iff

$$\operatorname{Im} \dot{c} \subset \mathcal{D}_{Lagr}$$

that is to say, for all $t \in I$,

$$(*) \qquad \dot{c}(t) = (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t)) \in \mathcal{D}_{Lagr}$$

(i) Owing to Prop. 6, condition $(*)$ reads

$$\dot{\mathrm{p}}(t) = \mathbf{v}(t)\,, \quad [K](\mathrm{p}(t), \mathbf{v}(t), \dot{\mathbf{v}}(t)) = F(\mathrm{p}(t), \mathbf{v}(t))$$

that is,

$$\begin{aligned} (\tau_Q \circ c)^{\boldsymbol{\cdot}}(t) &= (\mathrm{p}(t), \dot{\mathrm{p}}(t)) \\ &= (\mathrm{p}(t), \mathbf{v}(t)) \\ &= c(t) \end{aligned}$$

and

$$\begin{aligned} ([K] \circ \dot{c})(t) &= (\,\mathrm{p}(t),\, [K](\mathrm{p}(t), \mathbf{v}(t), \dot{\mathbf{v}}(t))\,) \\ &= (\,\mathrm{p}(t),\, F(\mathrm{p}(t), \mathbf{v}(t))\,) \\ &= (F \circ c)(t) \end{aligned}$$

That proves our first claim.

(ii) Owing to Prop. 7, condition $(*)$ also reads

$$\dot{\mathrm{p}}(t) = \mathbf{v}(t)\,, \quad \dot{\mathbf{v}}(t) = \Gamma(\mathrm{p}(t), \mathbf{v}(t))$$

that is,

$$\begin{aligned} \dot{c}(t) &= (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t)) \\ &= (\mathrm{p}(t), \mathbf{v}(t); \mathbf{v}(t), \Gamma(\mathrm{p}(t), \mathbf{v}(t))) \\ &= (\Gamma \circ c)(t) \end{aligned}$$

That proves our second claim.

$\square$

As a consequence, the condition characterizing the base integral curves of Lagrange equation will be formulated as follows.

Let
$$\gamma : I \to Q : t \mapsto \gamma(t) = \mathrm{p}(t)$$
be a smooth curve of Q and
$$[K] \circ \ddot{\gamma} : I \to T^*Q$$
its *Riemannian geodesic curvature*, whose vector (rather than covector) expression is the *covariant derivative* [4]
$$\frac{\nabla \dot{\gamma}}{dt} := g^{-1} \circ [K] \circ \ddot{\gamma} : I \to TQ$$

Proposition 9 *γ is a base integral curve of $\mathcal{D}_{Lagr}$, iff*
$$[K] \circ \ddot{\gamma} = F \circ \dot{\gamma} \qquad (\circ)'$$
or, in vector formulation,
$$\frac{\nabla \dot{\gamma}}{dt} = \Delta_F \circ \dot{\gamma}$$
or, in normal form,
$$\ddot{\gamma} = \Gamma \circ \dot{\gamma} \qquad (\bullet)'$$

Proof Recall that a base integral curve of Lagrange equation is the projection $\gamma = \tau_Q \circ c$ of an integral curve c (smooth curve of TQ satisfying condition $(\circ)$ or the equivalent $(\bullet)$).

(i) Now, if γ is a base integral curve, condition $(\circ)$ implies $\dot{\gamma} = (\tau_Q \circ c)^{\boldsymbol{\cdot}} = c$ – whence $\ddot{\gamma} = \dot{c}$ – and then $(\circ)'$. Conversely, if γ satisfies condition $(\circ)'$, then it is obviously a base integral curve (projection of $c := \dot{\gamma}$ satisfying $(\circ)$). Clearly, condition $(\circ)'$ is equivalent to
$$g^{-1} \circ [K] \circ \ddot{\gamma} = g^{-1} \circ F \circ \dot{\gamma}$$
that is,
$$\frac{\nabla \dot{\gamma}}{dt} = \Delta_F \circ \dot{\gamma}$$
which is the above mentioned vector formulation of $(\circ)'$.

[4] Covariant derivative is also related to an important geometric structure, called *Levi-Civita connection* of Riemannian manifold (Q, K).

(ii) In the same way, through condition $(\bullet)$, one can show that the base integral curves are characterized by $(\bullet)'$.
Alternatively, from

$$\begin{aligned}\frac{\nabla\dot{\gamma}}{dt} : t \in I \quad &\xmapsto{\ddot{\gamma}} \quad (\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t)) \in T^2Q \\ &\xmapsto{[K]} \quad \Big(\mathrm{p}(t),\, g_{\mathrm{p}(t)}\left(\ddot{\mathrm{p}}(t) - \Gamma_K(\mathrm{p}(t), \dot{\mathrm{p}}(t))\right)\Big) \in T^*Q \\ &\xmapsto{g^{-1}} \quad \Big(\mathrm{p}(t),\, \ddot{\mathrm{p}}(t) - \Gamma_K(\mathrm{p}(t), \dot{\mathrm{p}}(t))\Big) \in TQ\end{aligned}$$

and

$$\ddot{\gamma} - \Gamma_K \circ \dot{\gamma} : t \in I \mapsto (\mathrm{p}(t), \ddot{\mathrm{p}}(t) - \Gamma_K(\mathrm{p}(t).\dot{\mathrm{p}}(t))) \in TQ$$

that is,

$$\frac{\nabla\dot{\gamma}}{dt} = \ddot{\gamma} - \Gamma_K \circ \dot{\gamma}$$

we deduce that the vector formulation of $(\circ)'$ (which has already been seen to characterize the base integral curves) is equivalent to

$$\ddot{\gamma} = \Gamma_K \circ \dot{\gamma} + \Delta_F \circ \dot{\gamma}$$

which is condition $(\bullet)'$.

$\square$

From the dynamical point of view, some remarks are now in order.

For $F = 0$, the base integral curves – characterized by a vanishing Riemannian geodesic curvature $[K] \circ \ddot{\gamma} = 0$ – coincide with the inertial motions of $\mathcal{S}$ (i.e. the motions which would be possible if $\mathbf{F}$ were zero).

The effect of a covector force field $F \neq 0$ is then that of deviating the *DPM*s of $\mathcal{S}$ from the inertial trend, by giving them a non-vanishing Riemannian geodesic curvature, namely $[K] \circ \ddot{\gamma} = F \circ \dot{\gamma}$.

Classical Lagrange equations

The scalar equations obtained with the aid of a chart by orderly equalling the components of the covector or vector-valued functions which appear in the left and right hand sides of $(\circ)'$ or $(\bullet)'$, are the classical 'Lagrange equations' of Analytical Dynamics.

They will prove to be only *locally* equivalent to the geometric Lagrange equation, in the sense that they only characterize the base integral curves of the latter which live in the coordinate domain of the given chart.

Preliminaries

Consider the coordinate domain $\mathcal{U} = \mathrm{Im}\,\xi$ of a chart $\xi : q \in W \mapsto \xi(q) \in \mathcal{U}$, expressing the points $\mathrm{p} \in \mathcal{U} \subset Q$ in function of coordinates $q \in W \subset \mathbb{R}^n$ (with $n := \dim Q$). [5]

To any smooth curve $\gamma : t \in I \mapsto \gamma(t) = \mathrm{p}(t) \in Q$ living in $\mathcal{U}$, i.e. satisfying

$$\mathrm{p} = \mathrm{p}(t) \in \mathcal{U}$$

for all $t \in I$, there corresponds in ξ a smooth coordinate expression

$$q = q(t) \in W$$

related to γ by $\mathrm{p}(t) = \xi(q(t))$, also denoted

$$\mathrm{p} = \xi(q) \tag{1}$$

(dependence from time t is understood).

To the first tangent lift $c = \dot{\gamma}$, that is,

$$\mathrm{p} = \mathrm{p}(t) \in \mathcal{U}\ , \quad \mathbf{v} = \mathbf{v}(t) \in T_{\mathrm{p}(t)}Q$$

with

$$\mathbf{v} = \dot{\mathrm{p}}$$

there corresponds in ξ a smooth coordinate expression

$$q = q(t) \in W\ , \quad v = v(t) \in \mathbb{R}^n$$

related to $\dot{\gamma}$ by (1) and the time derivative of (1) [6]

$$\mathbf{v} = \mathbf{v}(q, v) := v^h \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \tag{2}$$

where

$$v = \dot{q}$$

is the n-tuple of linear components of $\mathbf{v}$ in ξ. Remark that, for all $h = 1, \ldots, n$,

$$\frac{\partial \mathbf{v}}{\partial v^h}\Big|_{(q,v)} = \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q\ , \quad \frac{\partial \mathbf{v}}{\partial q^h}\Big|_{(q,v)} = v^k \frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_q = \frac{d}{dt}\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \tag{3}$$

---

[5] Recall that, for any $q = \xi^{-1}(\mathrm{p}) \in W$, the partial derivatives $\left(\frac{\partial \mathrm{p}}{\partial q^1}\Big|_q, \ldots, \frac{\partial \mathrm{p}}{\partial q^n}\Big|_q\right)$ provide a basis of $T_{\mathrm{p}}Q$.

[6] A repeated index, in upper and lower position, denotes summation over $(1, \ldots, n)$.

To the second tangent lift $\dot{c} = \ddot{\gamma}$, that is,

$$\mathrm{p} = \mathrm{p}(t) \in \mathcal{U} \ , \quad \mathbf{v} = \mathbf{v}(t) \in T_{\mathrm{p}(t)}Q \ , \quad \mathbf{w} = \mathbf{w}(t) \in T^2_{(\mathrm{p}(t),\mathbf{v}(t))}Q$$

with

$$\mathbf{v} = \dot{\mathrm{p}} \ , \quad \mathbf{w} = \dot{\mathbf{v}} = \ddot{\mathrm{p}}$$

there corresponds in ξ a smooth coordinate expression

$$q = q(t) \in W \ , \quad v = v(t) \in \mathbb{R}^n \ , \quad w = w(t) \in \mathbb{R}^n$$

related to $\ddot{\gamma}$ by (1), (2) and the time derivative of (2)

$$\mathbf{w} = \mathbf{w}(q, v, w) := w^h \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q + v^h v^k \frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_q$$

where

$$w = \dot{v} = \ddot{q}$$

is the n-tuple of affine components of $\mathbf{w}$ in ξ.

In the sequel, the components of $F \circ \dot{\gamma}$ in ξ will be denoted by

$$F_h(q, v) := (F \circ \dot{\gamma})_h = (F(\mathrm{p}, \mathbf{v}))_h = \left\langle F(\mathrm{p}, \mathbf{v}) \ \Big| \ \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle = \mathbf{F}(\mathrm{p}, \mathbf{v}) \cdot \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q$$

and the components of $\Delta_F \circ \dot{\gamma} = g^{-1} \circ F \circ \dot{\gamma}$ will be denoted by

$$F^i(q, v) := (\Delta_F \circ \dot{\gamma})^i = (\Delta_F(\mathrm{p}, \mathbf{v}))^i = (g_{\mathrm{p}}^{-1}(F(\mathrm{p}, \mathbf{v})))^i = g^{ih}(q)\, (F(\mathrm{p}, \mathbf{v}))_h = g^{ih}(q)\, F_h(q, v)$$

where $[g^{hk}(q)]$ is the inverse of the nonsingular matrix $[g_{hk}(q)]$ defined by

$$g_{hk}(q) := \left\langle g_{\mathrm{p}}\Big(\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q\Big) \ \Big| \ \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q \right\rangle = m\, \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \cdot \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q$$

Lagrange equations

The above coordinate formalism will now be adopted for the characterization of the base integral curves of $\mathcal{D}_{Lagr}$ living in the given coordinate domain.

Proposition 10 *A smooth curve* $\gamma : t \in I \mapsto \mathrm{p} = \xi(q(t)) \in \mathcal{U} \subset Q$ *–living in the coordinate domain* $\mathcal{U}$ *of a chart* ξ *of* Q *– is a base integral curve of* $\mathcal{D}_{Lagr}$ *, iff its coordinate expression* $q = q(t)$ *satisfies (for all* $h, i = 1, \ldots, n$ *) the classical* Lagrange equations

$$\dot{q} = v \ , \quad \frac{d}{dt}\frac{\partial K}{\partial v^h}\Big|_{(q,v)} - \frac{\partial K}{\partial q^h}\Big|_{(q,v)} = F_h(q,v) \tag{$\circ$}_h$$

or, in normal form, [7]

$$\dot{q} = v \ , \quad \dot{v}^i = -\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_q v^j v^k + F^i(q,v) \tag{$\bullet^i$

Proof (i) Recall that γ is a base integral curve of $\mathcal{D}_{Lagr}$, iff it satisfies equation $(\circ)'$. As γ lives in the coordinate domain of ξ , equation $(\circ)'$ is equivalent to the n scalar equations obtained by orderly equalling the components in ξ of its left and right hand sides, i.e.

$$([K] \circ \ddot{\gamma})_h = (F \circ \dot{\gamma})_h \tag{$\circ$}'_h$$

The components $F_h(q,v) := (F \circ \dot{\gamma})_h$ have already been shown in the above preliminaries, where we have put $v = \dot{q}$. The components $([K] \circ \ddot{\gamma})_h$ will now be evaluated. To this end, by making use of (3) and usual rules of derivation, we obtain

$$\begin{aligned}
([K] \circ \ddot{\gamma})_h &:= ([K](\mathrm{p}, \mathbf{v}, \mathbf{w}))_h \\
&= \left\langle [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) \ \Big| \ \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle \\
&= m\mathbf{w}(q,v,w) \cdot \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \\
&= \frac{d}{dt}\left(m\mathbf{v}(q,v) \cdot \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right) - m\mathbf{v}(q,v) \cdot \frac{d}{dt}\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \\
&= \frac{d}{dt}\left(m\mathbf{v}(q,v) \cdot \frac{\partial \mathbf{v}}{\partial v^h}\Big|_{(q,v)} \right) - m\mathbf{v}(q,v) \cdot \frac{\partial \mathbf{v}}{\partial q^h}\Big|_{(q,v)}
\end{aligned}$$

[7] Here we encounter the *Christhoffel symbols* (of the Levi-Civita connection) associated with a Riemannian manifold, defined on the coordinate domain of any chart by

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} := \frac{1}{2} g^{ih}(\partial_j g_{kh} + \partial_k g_{hj} - \partial_h g_{jk})$$

with

$$\partial_i g_{hk} := \frac{\partial g_{hk}}{\partial q^i}$$

$$
\begin{aligned}
&= \frac{d}{dt}\frac{\partial}{\partial v^h}\Big|_{(q,v)}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) - \frac{\partial}{\partial q^h}\Big|_{(q,v)}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) \\
&= \frac{d}{dt}\frac{\partial K}{\partial v^h}\Big|_{(q,v)} - \frac{\partial K}{\partial q^h}\Big|_{(q,v)}
\end{aligned}
$$

So equations $(\circ)'_h$ just take the form $(\circ)_h$.

(ii) Also recall that γ is a base integral curve of $\mathcal{D}_{Lagr}$, iff it satisfies equation $(\bullet)'$. As γ lives in the coordinate domain of ξ , equation $(\bullet)'$ is equivalent to the n scalar equations obtained by orderly equalling the affine components in ξ of its left and right hand sides, i.e.

$$\ddot{\gamma}^i = (\Gamma\circ\dot{\gamma})^i \qquad (\bullet)'^{\,i}$$

where

$$
\begin{aligned}
\ddot{\gamma}^i &= w^i \\
&= \dot{v}^i
\end{aligned}
$$

and

$$
\begin{aligned}
(\Gamma\circ\dot{\gamma})^i &= (\Gamma_K\circ\dot{\gamma} + \Delta_F\circ\dot{\gamma})^i \\
&= (\Gamma_K\circ\dot{\gamma})^i + (\Delta_F\circ\dot{\gamma})^i \\
&= \Gamma^i_K(q,v) + F^i(q,v)
\end{aligned}
$$

The components $F^i(q,v) := (\Delta_F\circ\dot{\gamma})^i$ have already been shown in the above preliminaries. The components $\Gamma^i_K(q,v) := (\Gamma_K\circ\dot{\gamma})^i$ will now be evaluated. To that purpose we need the coordinate expression of K , which is given by

$$
\begin{aligned}
K(\mathrm{p},\mathbf{v}) &= \frac{1}{2}\left\langle\, g_{\mathrm{p}}\Big(v^h\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q\Big) \;\mid\; v^k\frac{\partial \mathrm{p}}{\partial q^k}\Big|_q\right\rangle \\
&= \frac{1}{2}\, g_{hk}(q)\, v^h v^k
\end{aligned}
$$

whence

$$
\begin{aligned}
\frac{\partial K}{\partial q^h}\Big|_{(q,v)} &= \frac{1}{2}(\partial_h g_{jk})_q\, v^j v^k \\
\frac{\partial K}{\partial v^h}\Big|_{(q,v)} &= g_{hk}(q) v^k
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 K}{\partial v^h\partial q^k}\Big|_{(q,v)} &= (\partial_k g_{hj})_q\, v^j \\
\frac{\partial^2 K}{\partial v^h\partial v^k}\Big|_{(q,v)} &= g_{hk}(q)
\end{aligned}
$$

As a consequence, the above components $([K] \circ \ddot{\gamma})_h$ can more explicitly be expressed in the form

$$\begin{aligned}([K] \circ \ddot{\gamma})_h &= ([K](\mathrm{p}, \mathbf{v}, \mathbf{w}))_h \\ &= \frac{\partial^2 K}{\partial v^h \partial v^k}\Big|_{(q,v)} w^k + \frac{\partial^2 K}{\partial v^h \partial q^k}\Big|_{(q,v)} v^k - \frac{\partial K}{\partial q^h}\Big|_{(q,v)} \\ &= g_{hk}(q) w^k + (\partial_k g_{hj})_q \, v^j v^k - \frac{1}{2}(\partial_h g_{jk})_q \, v^j v^k \\ &= g_{hk}(q) w^k + \frac{1}{2}(\partial_k g_{hj})_q \, v^j v^k + \frac{1}{2}(\partial_j g_{hk})_q \, v^j v^k - \frac{1}{2}(\partial_h g_{jk})_q \, v^j v^k \\ &= g_{hk}(q) w^k + \frac{1}{2}(\partial_k g_{hj} + \partial_j g_{kh} - \partial_h g_{jk})_q \, v^j v^k\end{aligned}$$

whence

$$\begin{aligned}\left(\frac{\nabla \dot{\gamma}}{dt}\right)^i &= (g^{-1} \circ ([K] \circ \ddot{\gamma}))^i \\ &= (g_{\mathrm{p}}^{-1}([K](\mathrm{p}, \mathbf{v}, \mathbf{w})))^i \\ &= g^{ih}(q) \, ([K](\mathrm{p}, \mathbf{v}, \mathbf{w}))_h \\ &= w^i + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_q v^j v^k\end{aligned}$$

Therefore, identities [8]

$$\begin{aligned}\left(\frac{\nabla \dot{\gamma}}{dt}\right)^i &= (\ddot{\gamma} - \Gamma_K \circ \dot{\gamma})^i \\ &= \ddot{\gamma}^i - (\Gamma_K \circ \dot{\gamma})^i\end{aligned}$$

read

$$w^i + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_q v^j v^k = w^i - \Gamma^i_K(q, v)$$

Hence we obtain

$$\Gamma^i_K(q, v) = - \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_q v^j v^k$$

So equations $(\bullet)'^{\,i}$ just take the form $(\bullet)^i$.

$\square$

[8] See the proof, part (ii), of Prop. 9.

2.3 Euler-Lagrange equation

A special mention, for its primary role in both mathematical and theoretical physics, is to be given to the dynamics of a 'conservative system'.

Conservative system

Such a name refers to a system $\mathcal{S} = (Q, m, \mathbf{f})$ carrying a *conservative field* $\mathbf{f}$, i.e. a positional force field whose virtual work

$$f = -dV$$

is an exact 1-form, deriving from a smooth *potential energy*

$$V : Q \to \mathbb{R}$$

(determined up to a locally constant function).

The name of 'conservative' is due to the following 'conservation law' of *mechanical energy*

$$\mathbb{E} := K + V : TQ \to \mathbb{R} : (\mathrm{p}, \mathbf{v}) \mapsto \mathbb{E}(\mathrm{p}, \mathbf{v}) := K(\mathrm{p}, \mathbf{v}) + V(\mathrm{p})$$

(kinetic energy *plus* potential energy).

Proposition 11 *Along each integral curve c of*

$$\mathcal{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,, \ \ [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = -d_{\mathrm{p}}V\}$$

the mechanical energy $\mathbb{E}$ keeps constant, i.e.

$$\mathbb{E} \circ c = const.$$

Proof If $c = (\mathrm{p}, \mathbf{v})$ denotes an integral curve and $\mathbf{w} := \dot{\mathbf{v}}$, from $\dot{\mathrm{p}} = \mathbf{v}$ and $[K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = -d_{\mathrm{p}}V$ it follows that

$$\begin{aligned}
\frac{d}{dt}(\mathbb{E} \circ c) &= \frac{d}{dt}\,\mathbb{E}(\mathrm{p}, \mathbf{v}) \\
&= \frac{d}{dt}K(\mathrm{p}, \mathbf{v}) + \frac{d}{dt}V(\mathrm{p}) \\
&= \frac{d}{dt}(\frac{1}{2}m\,\mathbf{v} \cdot \mathbf{v}) + \frac{d}{dt}V(\mathrm{p}) \\
&= m\,\mathbf{w} \cdot \mathbf{v} + \frac{d}{dt}V(\mathrm{p}) \\
&= \langle [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) \mid \mathbf{v} \rangle + \langle d_{\mathrm{p}}V \mid \mathbf{v} \rangle \\
&= -\langle d_{\mathrm{p}}V \mid \mathbf{v} \rangle + \langle d_{\mathrm{p}}V \mid \mathbf{v} \rangle \\
&= 0
\end{aligned}$$

Hence our claim.

□

Lagrangian geodesic curvature field

In conservative dynamics, the kinetic energy and the potential energy – which are the two ingredients 'generating' $\mathcal{D}_{Lagr}$ – can be merged into a unique object, as follows.

Define a new *Lagrangian function* by putting

$$\mathbb{L} := K - V : TQ \to \mathbb{R} : (\mathrm{p}, \mathbf{v}) \mapsto \mathbb{L}(\mathrm{p}, \mathbf{v}) := K(\mathrm{p}, \mathbf{v}) - V(\mathrm{p})$$

(kinetic energy *minus* potential energy).

Associated with $\mathbb{L}$, there is a *Lagrangian geodesic curvature field* given by

$$[\mathbb{L}] := [K] + dV : T^2Q \to T^*Q : (\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \mapsto (\mathrm{p}, [L](\mathrm{p}, \mathbf{v}, \mathbf{w}))$$

with

$$[\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) := [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) + d_{\mathrm{p}}V \in T^*_{\mathrm{p}}Q$$

(Riemannian geodesic curvature field *minus* conservative field).

From the above definitions, it follows that

$$\mathcal{D}_{Lagr} = \mathbb{D}_{Eul-Lagr}$$

that is, $\mathcal{D}_{Lagr}$ does not differ from the *Euler-Lagrange equation*

$$\mathbb{D}_{Eul-Lagr} := \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,, \;\; [\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) = 0\}$$

'generated' by $\mathbb{L}$.

Integral curves

The conditions characterizing the integral curves and the base integral curves of Euler-Lagrange equation can now be formulated as follows.

Proposition 12 *A smooth curve c of TQ is an integral curve of $\mathbb{D}_{Eul-Lagr}$, iff*

$$(\tau_Q \circ c)^{\boldsymbol{\cdot}} = c\,, \quad [\mathbb{L}] \circ \dot{c} = 0 \qquad (\Diamond)$$

Proof The same as the proof, part (i), of Prop.8.

□

Proposition 13 *A smooth curve γ of Q is a base integral curve of $\mathbb{D}_{Eul-Lagr}$, iff its* Lagrangian geodesic curvature *vanishes, i.e.*

$$[\mathbb{L}] \circ \ddot{\gamma} = 0 \qquad (\Diamond)'$$

Proof The same as the proof, part (i), of Prop.9.

□

Classical Euler-Lagrange equations

The *local* scalar equations obtained with the aid of a chart by equalling to zero the components of the covector-valued function which appears in the left hand side of $(\Diamond)'$, are the classical 'Euler-Lagrange equations' of Analytical Dynamics.

Proposition 14 *A smooth curve $\gamma : t \in I \mapsto \mathrm{p} = \xi(q(t)) \in \mathcal{U} \subset Q$ –living in the coordinate domain $\mathcal{U}$ of a chart ξ of Q– is a base integral curve of $\mathbb{D}_{Eul-Lagr}$, iff its coordinate expression $q = q(t)$ satisfies (for all $h = 1, \ldots, n$) the classical* Euler-Lagrange equations

$$\dot{q} = v \ , \quad \frac{d}{dt}\frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} - \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)} = 0 \qquad (\Diamond)_h$$

Proof Recall that γ is a base integral curve of $\mathbb{D}_{Eul-Lagr}$, iff it satisfies equation $(\Diamond)'$. As γ lives in the coordinate domain of ξ, equation $(\Diamond)'$ is equivalent to the n scalar equations obtained by equalling to zero the components in ξ of its left hand side, i.e.

$$([\mathbb{L}] \circ \ddot{\gamma})_h = 0 \qquad (\Diamond)'_h$$

The above components are given by

$$\begin{aligned}
([\mathbb{L}] \circ \ddot{\gamma})_h \; &:= \; ([\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}))_h \\
&= \; \left\langle [\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) \;\mid\; \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle \\
&= \; \left\langle \, [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) \;\mid\; \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle + \left\langle d_{\mathrm{p}} V \;\mid\; \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle \\
&= \; \frac{d}{dt}\frac{\partial K}{\partial v^h}\Big|_{(q,v)} - \frac{\partial K}{\partial q^h}\Big|_{(q,v)} + \frac{\partial V}{\partial q^h}\Big|_q \\
&= \; \frac{d}{dt}\frac{\partial (K-V)}{\partial v^h}\Big|_{(q,v)} - \frac{\partial (K-V)}{\partial q^h}\Big|_{(q,v)} \\
&= \; \frac{d}{dt}\frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} - \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)}
\end{aligned}$$

where we have put $v = \dot{q}$.

So equations $(\Diamond)'_h$ just take the form $(\Diamond)_h$.

$\square$

Remark For both $\mathbb{L} := K$ and $\mathbb{L} := K - V$, the semi-basic 1-form $[\mathbb{L}] : T^2Q \to T^*Q$ is a mapping which takes any $(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q$ to the covector $(\mathrm{p}, [\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w})) \in T^*Q$ characterized, in a chart, by components

$$([\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}))_h = \frac{\partial^2 \mathbb{L}}{\partial v^h \partial v^k}\Big|_{(q,v)} w^k + \frac{\partial^2 \mathbb{L}}{\partial v^h \partial q^k}\Big|_{(q,v)} v^k - \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)}$$

Such a coordinate technique can be used to define $[\mathbb{L}]$ for an *arbitrary* Lagrangian function $\mathbb{L} : S \subset TQ \to \mathbb{R}$ (smooth on an open subset S of TQ), since – also in such a case – the value $[\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w})$ (for any $(\mathrm{p}, \mathbf{v}) \in S$), defined by the above components $([\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}))_h$ in a given chart, turns out to be an 'invariant' covector. [9]

2.4 Variational principle of stationary action

Euler-Lagrange equation takes conservative dynamics into the classical area of 'variational principles', as will now be shown.

Variational calculus

Let

$$\gamma : I \to Q : t \mapsto \mathrm{p}(t)$$

be a smooth curve of Q.

A *smooth variaton of* γ *with fixed end-points in a closed sub-interval of* I, is a smooth mapping

$$\chi : (-\epsilon, \epsilon) \times J \to Q : (s, t) \mapsto \mathrm{p}(s, t)$$

(with $\epsilon > 0$ and $J \subset I$) satisfying

$$\mathrm{p}(0, t) = \mathrm{p}(t) \, , \quad \forall \, t \in J$$

and, at the end-points of a closed interval $[t_1, t_2] \subset J$,

$$\mathrm{p}(s, t_1) = \mathrm{p}(t_1) \, , \; \mathrm{p}(s, t_2) = \mathrm{p}(t_2) \, , \quad \forall \, s \in (-\epsilon, \epsilon)$$

χ can be thought of as a one-parameter family

$$\{ \, \chi_s : J \to Q : t \mapsto \mathrm{p}_s(t) := \mathrm{p}(s, t) \, \}_{s \in (-\epsilon, \epsilon)}$$

[9] Actually, through higher geometric methods, $[\mathbb{L}]$ can be given a 'coordinate-free' definition.

of 'varied' curves near $\gamma|_J = \chi_o$ with fixed end-points in $[t_1, t_2] \subset J \subset I$, whose tangent lifts $\{\dot\chi_s\}_{s\in(-\epsilon,\epsilon)}$ are all included in

$$\dot\chi : (-\epsilon, \epsilon) \times J \to TQ : (s,t) \mapsto \left(\mathrm{p}(s,t), \left.\frac{\partial \mathrm{p}}{\partial t}\right|_{(s,t)}\right)$$

χ also define a one-parameter family

$$\{\chi_t : (-\epsilon, \epsilon) \to Q : s \mapsto \mathrm{p}_t(s) := \mathrm{p}(s,t)\}_{t\in J}$$

of 'isocronous' curves, whose tangent vectors at the points of $\gamma|_J$ are the values of

$$\chi'_o : J \to TQ : t \mapsto \left(\mathrm{p}(t), \left.\frac{\partial \mathrm{p}}{\partial s}\right|_{(0,t)}\right)$$

Now consider, in a closed sub-interval $[t_1, t_2]$ of I, the *action* of γ, i.e. the integral [10]

$$\mathcal{I}_\gamma := \int_{t_1}^{t_2} (\mathbb{L} \circ \dot\gamma)\, dt$$

For any smooth variation χ of γ with fixed end-points in $[t_1, t_2]$, the action of χ is then the real-valued function

$$\mathcal{I}_\chi := \int_{t_1}^{t_2} (\mathbb{L} \circ \dot\chi)\, dt : (-\epsilon, \epsilon) \to \mathbb{R} : s \mapsto \mathcal{I}_{\chi_s} := \int_{t_1}^{t_2} (\mathbb{L} \circ \dot\chi_s)\, dt$$

At $s = 0$, the value of $\mathcal{I}_\chi$ is $\mathcal{I}_\gamma$ and its derivative

$$\delta_\gamma \mathcal{I}_\chi := \left.\frac{d\mathcal{I}_\chi}{ds}\right|_0$$

is called the *first variation* of $\mathcal{I}_\chi$ at γ.

Proposition 15 *For every smooth variation χ of γ with fixed end-points in a closed sub-interval $[t_1, t_2]$ of I, the first variation of $\mathcal{I}_\chi$ at γ is related to the Lagrangian geodesic curvature of γ by*

$$\delta_\gamma \mathcal{I}_\chi = -\int_{t_1}^{t_2} \langle [\mathbb{L}] \circ \ddot\gamma \mid \chi'_o \rangle\, dt$$

[10] All of the following considerations and results of variational theory, hold true for an arbitrary smooth Lagrangian function as well (on this matter, also recall the final Remark of the previous section).

Proof First remark that

$$\delta_\gamma \mathcal{I}_\chi := \frac{d}{ds}\Big|_0 \int_{t_1}^{t_2} (\mathbb{L}\circ\dot\chi)\, dt = \int_{t_1}^{t_2} \frac{\partial(\mathbb{L}\circ\dot\chi)}{\partial s}\Big|_{s=0} dt$$

Now, for any given $t_o \in J$, consider a chart ξ on a neighbourhood $\mathcal{U}$ of the point $\mathrm{p}(t_o) \in \operatorname{Im}\gamma$. In a suitably small neighbourhood of $(0,t_o)\in(-\epsilon,\epsilon)\times J$, χ takes values in $\mathcal{U}$ (by continuity) and will then have a local coordinate expression $(q^h(s,t))$. As a consequence, $\dot\chi$ will have local coordinate expression $\left(q^h(s,t), \frac{\partial q^h}{\partial t}\Big|_{(s,t)}\right)$, denoted, for $s=0$, by $(q=q(t), v=v(t))$ with $v=\dot q$, and χ'_o will have components $\chi'_o{}^h := \frac{\partial q^h}{\partial s}\Big|_{s=0}$. With the aid of ξ, we obtain

$$\begin{aligned}
\frac{\partial(\mathbb{L}\circ\dot\chi)}{\partial s}\Big|_{(0,t_o)} &= \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q(t_o),v(t_o))}\frac{\partial q^h}{\partial s}\Big|_{(0,t_o)} + \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q(t_o),v(t_o))}\frac{\partial^2 q^h}{\partial s\partial t}\Big|_{(0,t_o)} \\
&= \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q(t_o),v(t_o))}\frac{\partial q^h}{\partial s}\Big|_{(0,t_o)} + \\
&\quad \frac{d}{dt}\Big|_{t_o}\left(\frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)}\frac{\partial q^h}{\partial s}\Big|_{s=0}\right) - \left(\frac{d}{dt}\Big|_{t_o}\frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)}\right)\frac{\partial q^h}{\partial s}\Big|_{(0,t_o)} \\
&= \left(\frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q(t_o),v(t_o))} - \frac{d}{dt}\Big|_{t_o}\frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)}\right)\chi'_o{}^h(t_o) + \frac{d}{dt}\Big|_{t_o}\left(\frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)}\chi'_o{}^h\right)
\end{aligned}$$

that is, [11]

$$\frac{\partial(\mathbb{L}\circ\dot\chi)}{\partial s}\Big|_{(0,t_o)} = -\langle[\mathbb{L}]\circ\ddot\gamma \mid \chi'_o\rangle(t_o) + \frac{d}{dt}\Big|_{t_o}\langle F\mathbb{L}\circ\dot\gamma \mid \chi'_o\rangle$$

So, on the whole J, we have

$$\frac{\partial(\mathbb{L}\circ\dot\chi)}{\partial s}\Big|_{s=0} = -\langle[\mathbb{L}]\circ\ddot\gamma \mid \chi'_o\rangle + \frac{d}{dt}\langle F\mathbb{L}\circ\dot\gamma \mid \chi'_o\rangle$$

As $\chi'_o(t_1)=0$ and $\chi'_o(t_2)=0$, we also have

$$\int_{t_1}^{t_2}\frac{d}{dt}\langle F\mathbb{L}\circ\dot\gamma \mid \chi'_o\rangle\, dt = \langle F\mathbb{L}\circ\dot\gamma \mid \chi'_o\rangle(t_2) - \langle F\mathbb{L}\circ\dot\gamma \mid \chi'_o\rangle(t_1) = 0$$

Hence our claim.

□

[11] $F\mathbb{L}\circ\dot\gamma : I \to T^*Q$ will denote the (invariant) covector-valued map whose components in ξ are given by

$$(F\mathbb{L}\circ\dot\gamma)_h = \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)}$$

(see the final Remark of 'Legendre transformation' in Chap. 3).

Variational principle

A smooth curve $\gamma : I \to Q$ is said to be a *geodesic curve* of $(Q, \mathbb{L})$, if it satisfies Hamilton's *variational principle of stationary action*, owing to which, for every smooth variation χ of γ with fixed end-points in a closed sub-interval of I, $\mathcal{I}_\chi$ is required to be stationary at γ, that is to say,

$$\delta_\gamma \mathcal{I}_\chi = 0 \,, \quad \forall \chi$$

The above variational principle is completely equivalent to Euler-Lagrange equation.

Proposition 16 *The geodesic curves of* $(Q, \mathbb{L})$ *are the base integral curves of* $\mathbb{D}_{Eul-Lagr}$.

Proof (i) If $\gamma : I \to Q$ is a base integral curve of $\mathbb{D}_{Eul-Lagr}$, i.e.

$$[\mathbb{L}] \circ \ddot{\gamma} = 0$$

we clearly obtain, for every smooth variation χ of γ with fixed end-points in a closed sub-interval $[t_1, t_2]$ of I ,

$$\int_{t_1}^{t_2} \langle [\mathbb{L}] \circ \ddot{\gamma} \mid \chi'_o \rangle \, dt = 0$$

and hence, owing to Prop.15, γ is a geodesic curve of $(Q, \mathbb{L})$.

(ii) Conversely, if $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$ is not a base integral curve of $\mathbb{D}_{Eul-Lagr}$, say

$$([\mathbb{L}] \circ \ddot{\gamma})(t_o) \neq 0 \,, \quad t_o \in I$$

we shall prove that, for a suitable smooth variation χ of γ with fixed end-points in a closed sub-interval $[t_1, t_2]$ of I ,

$$\int_{t_1}^{t_2} \langle [\mathbb{L}] \circ \ddot{\gamma} \mid \chi'_o \rangle \, dt \neq 0$$

which, owing to Prop.15, means that γ is not a geodesic curve of $(Q, \mathbb{L})$.

To this end, consider a chart on a neighbourhood

$$\mathcal{U} \ni \mathrm{p}(t_o)$$

Owing to our hypothesis, at least one of the components $([\mathbb{L}] \circ \ddot{\gamma})_h$ is non-null at t_o , say

$$([\mathbb{L}] \circ \ddot{\gamma})_1(t_o) > 0$$

By continuity, there exists a suitably small open interval $J \subset I$ containing t_o s.t., for all $t \in J$,

$$\mathrm{p}(t) \in \mathcal{U}\,, \quad ([\mathbb{L}] \circ \ddot{\gamma})_1(t) > 0$$

Now, for each $s \in (-\epsilon, \epsilon)$ $-$with a suitably small $\epsilon > 0-$ and each $t \in J$, we consider the point $\mathrm{p}(s,t) \in \mathcal{U}$ of coordinates

$$\begin{aligned} q^1(s,t) &:= q^1(t) + s(cos(t - t_o) - cos\alpha) \\ q^2(s,t) &:= q^2(t) \\ \ldots \quad & \ldots \\ q^n(s,t) &:= q^n(t) \end{aligned}$$

where $(q^h(t))$ is the n-tuple of coordinates of $\mathrm{p}(t)$ and

$$0 < \alpha < \frac{\pi}{2} \quad \text{s.t.} \quad [t_o - \alpha, t_o + \alpha] \subset J$$

By doing so, we define a map

$$\chi : (-\epsilon, \epsilon) \times J \to Q : (s,t) \mapsto \mathrm{p}(s,t)$$

which is immediately seen to be a smooth variation of γ with fixed end-points in

$$[t_1, t_2] := [t_o - \alpha, t_o + \alpha]$$

The components of χ'_o are , for all $t \in J$,

$$\begin{aligned} {\chi'_o}^1(t) &= cos(t - t_o) - cos\alpha \\ {\chi'_o}^2(t) &= 0 \\ \ldots \quad & \ldots \\ {\chi'_o}^n(t) &= 0 \end{aligned}$$

and, for all $t \in (t_1, t_2)$,

$${\chi'_o}^1(t) > 0$$

Hence

$$\int_{t_1}^{t_2} \langle [\mathbb{L}] \circ \ddot{\gamma} \mid \chi'_o \rangle \, dt = \int_{t_1}^{t_2} ([\mathbb{L}] \circ \ddot{\gamma})_h \, {\chi'_o}^h \, dt = \int_{t_1}^{t_2} ([\mathbb{L}] \circ \ddot{\gamma})_1 \, {\chi'_o}^1 \, dt \; > 0$$

which is our claim.

$\square$

Riemannian case

For $\mathbb{L} = K$, the above variational theory characterizes the geodesic curves of Riemannian manifold (Q, K) through condition

$$[K] \circ \ddot{\gamma} = 0$$

Owing to the positive definiteness of K , from the conservation law of energy $\mathbb{E} = K$ it follows that, on the one hand, a geodesic curve satisfying

$$K \circ \dot{\gamma} = \text{const.} = 0$$

degenerates into a singleton and, on the other hand, a *non-degenerate* geodesic curve is a *uniform motion*, that is,

$$K \circ \dot{\gamma} = \text{const.} > 0$$

The geometric meaning of non-degenerate geodesic curves will now be shown.

Let us consider one more Lagrangian function, namely

$$\Lambda := \sqrt{2K}$$

which is clearly smooth on $TQ \setminus K^{-1}(0)$. [12]

If $\gamma : t \in I \to Q : t \mapsto \mathrm{p}(t) \in Q$ is a smooth curve satisfying

$$\operatorname{Im} \dot{\gamma} \subset TQ \setminus K^{-1}(0)$$

(i.e. $\dot{\mathrm{p}}(t) \neq \mathbf{0}$ for all $t \in I$) and $[t_1, t_2] \subset I$, the integral

$$\mathcal{L}_\gamma := \int_{t_1}^{t_2} (\Lambda \circ \dot{\gamma})\, dt$$

defines the *length* of the arc of γ with end-points $(\mathrm{p}(t_1), \mathrm{p}(t_2))$. [13]

---

[12] Λ is smooth only on $TQ \setminus K^{-1}(0)$, since, in any chart, its partial derivatives

$$\left.\frac{\partial \Lambda}{\partial q^h}\right|_{(q,v)} = \frac{1}{\sqrt{2K(q,v)}} \left.\frac{\partial K}{\partial q^h}\right|_{(q,v)} , \quad \left.\frac{\partial \Lambda}{\partial v^h}\right|_{(q,v)} = \frac{1}{\sqrt{2K(q,v)}} \left.\frac{\partial K}{\partial v^h}\right|_{(q,v)}$$

are not even defined for $v = 0$, i.e. for $\mathbf{v} = \mathbf{0}$ or, equivalently, $(\mathrm{p}, \mathbf{v}) \in K^{-1}(0)$.

[13] For any $dt > 0$, $\Lambda(\mathrm{p}(t), \dot{\mathrm{p}}(t))\, dt = \sqrt{\langle g_{\mathrm{p}(t)}(d\mathrm{p}) \mid d\mathrm{p} \rangle} > 0$ defines the length, in the given Riemannian metric, of the 'infinitesimal arc' $d\mathrm{p} := \dot{\mathrm{p}}(t) dt$.

The variational theory previously sketched, applied to Λ , [14] shows that γ is a curve of *stationary length*, that is,

$$\delta_\gamma \mathcal{L}_\chi = 0 \ , \quad \forall \chi$$

iff

$$[\Lambda] \circ \ddot{\gamma} = 0$$

Proposition 17 γ *is a non-degenerate geodesic curve of* (Q, K) *, iff it is a uniform motion of stationary length.*

Proof The above result is an immediate consequence of the following
Lemma If γ is a uniform motion, that is,

$$K \circ \dot{\gamma} = \kappa = \text{const.} > 0$$

then

$$[\Lambda] \circ \ddot{\gamma} = \frac{1}{\sqrt{2\kappa}} \, [K] \circ \ddot{\gamma}$$

In order to prove the lemma, recall that, along the uniform motion $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$, we have (for any given $t \in I$ and any chart at $\mathrm{p}(t)$) [15]

$$\begin{aligned}
([\Lambda] \circ \ddot{\gamma})_h &= \frac{d}{dt} \frac{\partial \Lambda}{\partial v^h}\Big|_{(q,v)} - \frac{\partial \Lambda}{\partial q^h}\Big|_{(q,v)} \\
&= \frac{d}{dt} \left(\frac{1}{\sqrt{2\kappa}} \frac{\partial K}{\partial v^h}\Big|_{(q,v)} \right) - \frac{1}{\sqrt{2\kappa}} \frac{\partial K}{\partial q^h}\Big|_{(q,v)} \\
&= \frac{1}{\sqrt{2\kappa}} \left(\frac{d}{dt} \frac{\partial K}{\partial v^h}\Big|_{(q,v)} - \frac{\partial K}{\partial q^h}\Big|_{(q,v)} \right) \\
&= \frac{1}{\sqrt{2\kappa}} \, ([K] \circ \ddot{\gamma})_h
\end{aligned}$$

which is our claim.

□

[14] See footnote [10].
[15] See footnote [12].

Chapter 3

From Lagrange to Hamilton

The Lagrangian dynamics of a conservative system $\mathcal{S}$ –whose geometrical arena is the *velocity phase space* TQ, supporting Euler-Lagrange equation– will now be taken into the range of Hamiltonian dynamics, where the geometrical arena is the *momentum phase space* T^*Q, supporting a 'Hamilton equation' which, defined in terms of the canonical symplectic structure of T^*Q, directly arises in normal form.

3.1 Legendre transformation

The classical transition from velocity to momentum phase space is provided by the well known map which takes any *velocity* $(\mathrm{p}, \mathbf{v}) \in TQ$ onto the corresponding *momentum* $(\mathrm{p}, m\,\mathbf{v} \cdot |_{T_\mathrm{p}Q}) \in T^*Q$.

Lagrangian function and Legendre transformation

The above map is indeed the vector bundle isomorphism

$$g : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, g_\mathrm{p}(\mathbf{v}))\ , \quad g_\mathrm{p}(\mathbf{v}) := m\,\mathbf{v} \cdot |_{T_\mathrm{p}Q}$$

owing to which the configuration space of the system has been given the structure of a Riemannian manifold (Q, K). [1]

Such an isomorphism is also said to be the *Legendre transformation* determined by Lagrangian function $\mathbb{L} = K - V$ (in terms of which it can be expressed) [2] and is denoted by

$$F\mathbb{L} : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, F_\mathrm{p}\mathbb{L}(\mathbf{v}))\ , \quad F_\mathrm{p}\mathbb{L}(\mathbf{v}) := g_\mathrm{p}(\mathbf{v})$$

(whence $\pi_Q \circ F\mathbb{L} = \tau_Q$).

[1] See 'Riemannian geodesic curvature field' in Chap. 2.
[2] See the next 'Coordinate formalism'.

Coordinate formalism

Recall that T^*Q is a $2n$-dimensional manifold, where each element (p, π) is completely characterized, in a natural chart, by two n-tuples of coordinates (q, p), namely the coordinates $q = (q^1, \dots, q^n)$ of $\mathrm{p} = \xi(q)$, in a chart ξ of Q, and the components $p = (p_1, \dots, p_n)$ of π in ξ, given by $p_h = \langle \pi \mid \frac{\partial \mathrm{p}}{\partial q^h}\big|_q \rangle$.

In natural coordinate formalism, Legendre transformation

$$(\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \pi), \quad \pi = F_{\mathrm{p}}\mathbb{L}(\mathbf{v})$$

is expressed by [3]

$$(q, v) \mapsto (q, p), \quad p_h = g_{hk}(q)\, v^k = \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} \tag{1}$$

The inverse transformation

$$(\mathrm{p}, \pi) \mapsto (\mathrm{p}, \mathbf{v}), \quad \mathbf{v} = (F_{\mathrm{p}}\mathbb{L})^{-1}(\pi)$$

is in turn expressed by

$$(q, p) \mapsto (q, v), \quad v^h = g^{hk}(q)\, p_k =: \nu^h(q, p) \tag{2}$$

Remark Such a coordinate technique can be used to define a *Legendre morphism* $F\mathbb{L}$ for an *arbitrary* Lagrangian function $\mathbb{L} : S \subset TQ \to T^*Q$ (smooth on an open subset S of TQ), since – also in such case – the value $\pi = F_{\mathrm{p}}\mathbb{L}(\mathbf{v}) \in T^*_{\mathrm{p}}Q$ (for any $(\mathrm{p}, \mathbf{v}) \in S$), defined by the above components $p_h = \frac{\partial \mathbb{L}}{\partial v^h}\big|_{(q,v)}$ in a given chart, turns out to be an 'invariant' covector. [4]

$\mathbb{L}$ is then said to be a *regular* or a *singular* Lagrangian function, according to whether $F\mathbb{L}$ is injective or not.

[3] Recall that

$$p_h = \langle g_{\mathrm{p}}(\mathbf{v}) \mid \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \rangle = \langle g_{\mathrm{p}}(\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q) \mid \mathbf{v} \rangle = \langle g_{\xi(q)}(\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q) \mid \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q \rangle\, v^k = g_{hk}(q)\, v^k = \frac{\partial K}{\partial v^h}\Big|_{(q,v)} = \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)}$$

[4] Actually, through higher geometric methods, $F\mathbb{L}$ can be given a 'coordinate-free' definition.

3.2 Hamiltonian function

Legendre transformation also determines transition from the Lagrangian function (on TQ) to a 'Hamiltonian function' (on T^*Q).

Energy and Hamiltonian function

Recall that, associated with Lagrangian function $\mathbb{L} = K - V$, there is the energy function [5]

$$\mathbb{E} := K + V = 2K - \mathbb{L} = \langle F\mathbb{L} \mid \mathrm{id}_{\mathrm{TQ}} \rangle - \mathbb{L} : TQ \to \mathbb{R}$$

The *Hamiltonian function*

$$H := \mathbb{E} \circ (F\mathbb{L})^{-1} : T^*Q \to \mathbb{R}$$

is the 'push forward' of $\mathbb{E}$ by Legendre transformation $F\mathbb{L}$.

Coordinate formalism

In natural coordinate formalism, the Hamiltonian function

$$\begin{aligned}(\mathrm{p}, \pi) \mapsto H(\mathrm{p}, \pi) &= \mathbb{E}((F\mathbb{L})^{-1}(\mathrm{p}, \pi)) = \mathbb{E}(\mathrm{p}, (F_{\mathrm{p}}\mathbb{L})^{-1}(\pi)) = \mathbb{E}(\mathrm{p}, \mathbf{v}) = \langle F_{\mathrm{p}}\mathbb{L}(\mathbf{v}) \mid \mathbf{v} \rangle - \mathbb{L}(\mathrm{p}, \mathbf{v}) \\ &= \langle \pi \mid \mathbf{v} \rangle - \mathbb{L}(\mathrm{p}, \mathbf{v}) \,, \quad \mathbf{v} = (F_{\mathrm{p}}\mathbb{L})^{-1}(\pi)\end{aligned}$$

is expressed by

$$(q, p) \mapsto H(q, p) = p_k \, v^k - \mathbb{L}(q, v) \,, \quad v = \nu(q, p)$$

Hence, owing to (1) and (2), we obtain

$$\frac{\partial H}{\partial q^h}\Big|_{(q,p)} = p_k \frac{\partial v^k}{\partial q^h}\Big|_{(q,p)} - \frac{\partial \mathbb{L}}{\partial v^k}\Big|_{(q,v)} \frac{\partial v^k}{\partial q^h}\Big|_{(q,p)} - \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)} = -\frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)} \tag{3}$$

and

$$\frac{\partial H}{\partial p_h}\Big|_{(q,p)} = v^h + p_k \frac{\partial v^k}{\partial p_h}\Big|_{(q,p)} - \frac{\partial \mathbb{L}}{\partial v^k}\Big|_{(q,v)} \frac{\partial v^k}{\partial p_h}\Big|_{(q,p)} = \nu^h(q, p) \tag{4}$$

---

[5] We put

$$\langle F\mathbb{L} \mid \mathrm{id}_{\mathrm{TQ}} \rangle : (\mathrm{p}, \mathbf{v}) \in TQ \longmapsto \langle F_{\mathrm{p}}\mathbb{L}(\mathbf{v}) \mid \mathrm{id}_{\mathrm{TQ}}(\mathbf{v}) \rangle = \langle F_{\mathrm{p}}\mathbb{L}(\mathbf{v}) \mid \mathbf{v} \rangle = \langle g_{\mathrm{p}}(\mathbf{v}) \mid \mathbf{v} \rangle = 2K(\mathrm{p}, \mathbf{v}) \in \mathbb{R}$$

3.3 Hamiltonian vector field

The canonical symplectic geometry of T^*Q will now be taken into consideration.

Canonical symplectic structure and Hamiltonian vector field

Let

$$\omega : TM \to T^*M : (\pi, X) \mapsto (\pi, \omega_\pi(X))$$

be the canonical symplectic structure of cotangent bundle $M := T^*Q$. [6]

As ω is a vector bundle isomorphism of TM onto T^*M, it transforms the differential

$$dH : M \to T^*M$$

of Hamiltonian function H into the *Hamiltonian vector field*

$$X_H := \omega^{-1} \circ dH : M \to TM$$

Coordinate formalism

Recall that, on the $2n$-dimensional manifold $M = T^*Q$, the canonical symplectic structure ω is characterized, in any natural chart, by the $2n \times 2n$ matrix of components

$$\begin{bmatrix} \omega_{(1)(1)} & \omega_{(1)(2)} \\ \omega_{(2)(1)} & \omega_{(2)(2)} \end{bmatrix}$$

with $\omega_{(1)(1)} = \omega_{(2)(2)} = 0$ ($n \times n$ zero matrix) and $\omega_{(1)(2)} = -\omega_{(2)(1)} = \delta$ ($n \times n$ identity matrix).

If, for any $\pi \in M$,

$$\begin{bmatrix} d_\pi H_{(1)} \; d_\pi H_{(2)} \end{bmatrix}$$

and

$$\begin{bmatrix} X_H(\pi)^{(1)} \\ X_H(\pi)^{(2)} \end{bmatrix}$$

are the $2n$-tuples of components of $d_\pi H \in T^*_\pi M$ and $X_H(\pi) \in T_\pi M$, respectively, the linear mapping

$$d_\pi H = \omega_\pi(X_H(\pi))$$

is expressed by [7]

$$d_\pi H_{(1)} = \omega_{(\alpha)(1)} X_H(\pi)^{(\alpha)} \ , \quad d_\pi H_{(2)} = \omega_{(\alpha)(2)} X_H(\pi)^{(\alpha)}$$

[6] Any 'point' $(\mathrm{p}, \pi) \in T^*Q$ is now referred to as $\pi \in M$.
[7] Summation over $(\alpha) = (1), (2)$ is understood.

that is,

$$d_\pi H_{(1)} = -X_H(\pi)^{(2)}\,, \quad d_\pi H_{(2)} = X_H(\pi)^{(1)}$$

As

$$d_\pi H_{(1)} = \left[\frac{\partial H}{\partial q^h}\Big|_{(q,p)}\right]\,, \quad d_\pi H_{(2)} = \left[\frac{\partial H}{\partial p^h}\Big|_{(q,p)}\right]$$

we obtain

$$X_H(\pi)^{(1)} = \left[\frac{\partial H}{\partial p_h}\Big|_{(q,p)}\right]\,, \quad X_H(\pi)^{(2)} = \left[-\frac{\partial H}{\partial q^h}\Big|_{(q,p)}\right] \tag{5}$$

3.4 Hamilton equation

Remark that Legendre transformation $F\mathbb{L} : TQ \to T^*Q$ transforms each smooth curve $c : I \to TQ$ of velocity phase space into a smooth curve $F\mathbb{L} \circ c : I \to T^*Q$ of momentum phase space.

Cotangent dynamically possible motions

If c is an integral curve of $\mathbb{D}_{Eul-Lagr}$, i.e. a *TDPM* of $\mathcal{S}$, then $F\mathbb{L} \circ c$ will be said to be a *cotangent dynamically possible motion* (*CDPM*) of $\mathcal{S}$.

As $F\mathbb{L}$ is an isomorphism, *TDPM*s and *CDPM*s bijectively correspond to one another. Through such a bijection, the problem of determining the *TDPM*s proves to be naturally equivalent to that of determining the *CDPM*s.

Hamilton equation

Proposition 18 *The* CDPM*s are the integral curves of* Hamilton equation

$$\mathbb{D}_{Ham} := \operatorname{Im} X_H$$

Proof Let

$$k : I \to T^*Q\,, \quad c : I \to TQ$$

be smooth curves corresponding to each other through Legendre transformation, i.e.

$$k = F\mathbb{L} \circ c$$

and then both projecting down onto the same curve

$$\pi_Q \circ k = \pi_Q \circ F\mathbb{L} \circ c = \tau_Q \circ c : I \to Q$$

We shall prove that

$$(\lozenge) \qquad (\tau_Q \circ c)^{\boldsymbol{\cdot}} = c\,,\ [\mathbb{L}] \circ \dot{c} = 0 \quad \Longleftrightarrow \quad \dot{k} = X_H \circ k \qquad (\blacklozenge)$$

For any $t \in I$, the natural coordinates $(q,p) = (q(t), p(t))$ of $k = k(t)$ are then related to the natural coordinates $(q,v) = (q(t), v(t))$ of $c = c(t)$ by (1) and (2), i.e.

$$p_h = \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)}, \qquad v^h = \nu^h(q,p).$$

So condition ($\lozenge$), that is,

$$\dot{q}^h = v^h, \qquad \frac{d}{dt}\frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} - \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)} = 0$$

is equivalent, owing to (1) and (2), to

$$\dot{q}^h = \nu^h(q,p), \qquad \dot{p}_h = \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)}$$

and then, owing to (3) and (4), to

$$\dot{q}^h = \frac{\partial H}{\partial p_h}\Big|_{(q,p)}, \qquad \dot{p}_h = -\frac{\partial H}{\partial q^h}\Big|_{(q(p)} \qquad (\blacklozenge)_h$$

which, owing to (5), is just condition ($\blacklozenge$). [8]

□

Remark the 'second-order' character of the above Hamilton equation, exhibeted by the fact that any integral curve k of $\mathbb{D}_{Ham}$ is the *Legendre lift* of its own projection $\pi_Q \circ k$, i.e.

$$k = F\mathbb{L} \circ c = F\mathbb{L} \circ (\tau_Q \circ c)^{\bullet} = F\mathbb{L} \circ (\pi_Q \circ k)^{\bullet}$$

The integral curves and the base integral curves of $\mathbb{D}_{Ham}$ are then bijectively related to one another by projection π_Q and, owing to the above bijection, the first-order problem of determining the integral curves of $\mathbb{D}_{Ham}$ proves to be naturally equivalent to the second-order problem of determining its base integral curves, i.e. the solution curves γ of

$$(F\mathbb{L} \circ \dot{\gamma})^{\bullet} = X_H \circ (F\mathbb{L} \circ \dot{\gamma})$$

Classical Hamilton equations

The *local* scalar equations $(\blacklozenge)_h$, obtained in a natural chart by orderly equalling the components of the left and right hand side of ($\blacklozenge$), are the classical 'Hamilton equations' of Analytical Dynamics.

[8] Recall that $\begin{bmatrix} \dot{q}^h \\ \dot{p}_h \end{bmatrix}$ are the components of $\dot{k}$.

Chapter 4

Concluding remarks

Some comments on the previous results and some perspectives on further developments, now follow.

4.1 Inertia and force

Propositions 1, 3 and 6 have shown that the mathematical model actually underlying classical dynamics is given by the triplet

$$\mathcal{M} := (Q, K, F)$$

which will still be called *mechanical system.*

The corresponding Lagrange equation

$$\mathcal{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,, \ \ [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})\}$$

is then the *dynamics* of $\mathcal{M}$, the *DPM*s of $\mathcal{M}$ being defined as the base integral curves γ of $\mathcal{D}_{Lagr}$ and hence characterized by condition

$$[K] \circ \ddot{\gamma} = F \circ \dot{\gamma}$$

If $F = 0$, the *DPM*s γ of $\mathcal{M}$ – characterized by a Riemannian geodesic curvature $[K] \circ \ddot{\gamma}$ identically vanishing – coincide with the geodesic curves of Riemannian manifold (Q, K), which can therefore be called *inertial motions* in (Q, K).

If $F \neq 0$, the *DPM*s γ of $\mathcal{M}$ – characterized by a Riemannian geodesic curvature $[K] \circ \ddot{\gamma}$ differing from zero as much as is imposed by the covector force $F \circ \dot{\gamma}$ – are then to be called *forced motions* in (Q, K), since they appear to be perturbed with respect to the above inertial trend.

So K and F seem to correspond to the *empirical* notions of 'inertia' (defining inertial motions) and 'force' (defining forced motions), respectively.

4.2 Gauge transformations

However the above notions of inertia and force are *not* uniquely determined by dynamics.

In fact, there exist 'gauge transformations' of $\mathcal{M}$, which alter the geometrical ingredient K and the dynamical ingredient F without altering the dynamics itself, as will now be shown.

For any real-valued smooth function $V : Q \to \mathbb{R}$, consider the *gauge transformation*

$$K \mapsto \mathbb{L} := K - V \ , \quad F \mapsto \mathbb{F} := F + dV$$

On the one hand, such a transformation gives rise to a new *mechanical system*

$$\mathbb{M} := (Q, \mathbb{L}, \mathbb{F})$$

with *Lagrange equation*

$$\mathbb{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q \mid \mathbf{u} = \mathbf{v} \ , \ \ [\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) = \mathbb{F}(\mathrm{p}, \mathbf{v})\}$$

and *DPM*s – the base integral curves of $\mathbb{D}_{Lagr}$ – characterized by condition

$$[\mathbb{L}] \circ \ddot{\gamma} = \mathbb{F} \circ \dot{\gamma}$$

If $\mathbb{F} = 0$, the *DPM*s γ of $\mathbb{M}$ – characterized by a Lagrangian geodesic curvature $[\mathbb{L}] \circ \ddot{\gamma}$ identically vanishing – coincide with the geodesic curves of Lagrangian manifold $(Q, \mathbb{L})$, which will therefore be called *inertial motions* in $(Q, \mathbb{L})$.

If $\mathbb{F} \neq 0$, the *DPM*s γ of $\mathbb{M}$ – characterized by a Lagrangian geodesic curvature $[\mathbb{L}] \circ \ddot{\gamma}$ differing from zero as much as is imposed by the covector force $\mathbb{F} \circ \dot{\gamma}$ – are then to be called *forced motions* in $(Q, \mathbb{L})$, since they appear to be perturbed with respect to the above inertial trend.

On the other hand, from the very definitions of

$$[\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) := [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) + d_{\mathrm{p}}V$$

and

$$\mathbb{F}(\mathrm{p}, \mathbf{v}) := F(\mathrm{p}, \mathbf{v}) + d_{\mathrm{p}}V$$

for all $(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q$, it follows that

$$[\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) = \mathbb{F}(\mathrm{p}, \mathbf{v})$$

iff

$$[K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})$$

whence

$$\mathbb{D}_{Lagr} = \mathcal{D}_{Lagr}$$

So transition from $\mathcal{M}$ to $\mathbb{M}$ just leads to different specifications of the *conventional* notions of inertia and force, without altering the *observable* class of *DPM*s.

4.3 Geometrizing physical fields

From the physical point of view, the meaning of the above kind of gauge transformations can be illustrated as follows.

Think of V as the potential energy of a conservative component of $\mathbf{F}$, that is to say, put $\mathbf{F} = \bar{\mathbf{F}} + \mathbf{f}$, $\mathbf{f}$ being a conservative field with virtual work $f = -dV$.

Now look at the ingredients $\mathbb{L}$ and $\mathbb{F}$ of $\mathbb{M}$.

On the one hand, $\mathbb{F} = F - f$ is the virtual work of $\bar{\mathbf{F}} = \mathbf{F} - \mathbf{f}$, that is to say, the conservative field does not appear any more in the dynamical ingredient of $\mathbb{M}$.

On the other hand, $\mathbb{L}$ embodies $-V$, that is to say, the conservative field is encompassed in the geometrical ingredient of $\mathbb{M}$ as a *potential* function, whose effect is that of transforming the 'natural' Riemannian geometry K of the particle system into a 'perturbed' Lagrangian geometry $\mathbb{L} = K - V$.

That anticipates, in a sense, Einstein's idea of 'geometrizing physical fields'.

4.4 Geometrical dynamics

From the mathematical point of view, gauge transformations suggest a generalized *geometrical dynamics*.

In such a theory, a *mechanical system* will be conceived as an arbitrary triplet $\mathbb{M}$ consisting of a smooth manifold Q equipped with a (regular or a singular) Lagrangian function $\mathbb{L}$ and a semibasic 1-form $\mathbb{F}$ (both smooth on an open subset of TQ).

The *dynamics* of $\mathbb{M}$ will then be defined by the corresponding Lagrange equation $\mathbb{D}_{Lagr}$ (which will or will not be reducible to normal form, according to whether $\mathbb{L}$ is a regular or a singular Lagrangian function, respectively).

The global analysis of $\mathbb{D}_{Lagr}$, its Hamiltonian formulation (or Hamilton-Dirac formulation, in the general case of $\mathbb{L}$ being a singular Lagrangian) and further generalizations, as well as applications to classical and relativistic dynamics, are the object of (part of) the current research work on *geometrical dynamics*.

Chapter 5

Appendix

The geometry of smooth manifolds embedded in Euclidean affine spaces, is the main topic of this Appendix. [1] Tangent bundles and differential equations, cotangent bundles and differential forms, will be made to fall into its range. The above geometry will finally be re-read, so as to lead to the modern approach to smooth manifolds (which does away with any reference to embeddings into Euclidean environments).

5.1 Submanifolds of a Euclidean space

A submanifold of a Euclidean affine space will be presented as a 'locus' which *locally* exhibits the same differential-topological behaviour as an affine subspace (that is suggested from the familiar circumstance of 'smooth' curves or surfaces locally resembling straight lines or planes, respectively).

Affine subspaces

Let $\mathcal{E}$ be an m-dimensional Euclidean affine space, modelled on a vector space E (acting freely and transitively on $\mathcal{E}$ as an Abelian group of operators). [2]

A subset $\mathcal{A} \subset \mathcal{E}$ is said to be an n-dimensional Euclidean *affine subspace* of $\mathcal{E}$, if it is the orbit of a point $\mathrm{o} \in \mathcal{E}$ under the action of an n-dimensional vector subspace $A \subset E$, i.e. $\mathcal{A} = \mathrm{o} + A := \{\mathrm{o} + \mathbf{v} : \mathbf{v} \in A\}$. [3]

[1] The main (linear, metric, topological) properties of Euclidean affine spaces and the fundamentals of differential calculus on such spaces are the prerequisites for reading this Appendix.

[2] E is meant to be a vector space on the field $\mathbb{R}$ of real numbers. The elements of E will be called *free* vectors, whereas – for any *point* $\mathrm{p} \in \mathcal{E}$ – the elements of $\{\mathrm{p}\} \times E$ will be said to be vectors *attached* at p. The action of E on $\mathcal{E}$ will be denoted by $+$.

[3] The name 'affine subspace' is clearly due to the fact that condition $\mathcal{A} = \mathrm{o} + A$ is

Such a subspace $\mathcal{A}$ can be described in $\mathcal{E}$ by means of *global* affine parametrizations (or systems of affine coordinates), say

$$\mathcal{A} = \operatorname{Im} \xi$$

with [4]

$$\xi : \mathbb{R}^n \to \mathcal{E} : q \mapsto \mathrm{p} = \xi(q) = \mathrm{o} + \Xi(q) = \mathrm{o} + q^h \mathbf{e}_h$$

(where $\mathrm{o} \in \mathcal{A}$ and $\Xi : v \in \mathbb{R}^n \mapsto \Xi(v) = v^h \mathbf{e}_h \in E$ is the injective linear map determined by a basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of A).

From the differential-topological point of view, the above kind of parametrization exhibits the following well known properties:

(1) $\xi : \mathbb{R}^n \to \mathcal{A}$ is a homeomorphism; [5]

(2) $\xi : \mathbb{R}^n \to \mathcal{E}$ is C^∞- differentiable, with injective differential $d_q\xi = \Xi$ at each $q \in \mathbb{R}^n$. [6]

Embedded submanifolds

Let $\mathcal{E}$ be an m-dimensional Euclidean affine space (modelled on E).

A subset $Q \subset \mathcal{E}$ carrying the structure of an n-dimensional *embedded submanifold* of $\mathcal{E}$ (or *smooth manifold embedded* in $\mathcal{E}$), is one s.t. each point $\mathrm{p} \in Q$ admits an open neighbourhood $\mathcal{U}$, in the subspace topology of Q, which is the image

$$\mathcal{U} = \operatorname{Im} \xi$$

of a *local* parametrization

$$\xi : W \subset \mathbb{R}^n \to \mathcal{E} : q \longmapsto \mathrm{p} = \xi(q)$$

defined on an open subset W of $\mathbb{R}^n$ and satisfying the following topological-differential properties:

necessary and sufficient for a subset $\mathcal{A}$ of $\mathcal{E}$ to 'inherit' from the affine environment a structure of affine space modelled on a vector subspace A of E. Familiar examples are the *straight lines* and the *planes* (i.e. the affine subspaces of dimension 1 and 2, respectively).

[4] A repeated index, in lower and upper position, will denote summation.

[5] Here we refer to the subspace topology of $\mathcal{A}$.

[6] Recall that a mapping like ξ is C^∞- differentiable, iff it is continuous and admits continuous partial derivatives of any order. Its differential at q is the linear map

$$d_q\xi : \mathbb{R}^n \to E : v = (v^h) \longmapsto \mathbf{v} = v^h \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q$$

1. $\xi : W \to \mathcal{U}$ is a homeomorphism;
2. $\xi : W \to \mathcal{E}$ is C^∞- differentiable, with injective differential $d_q\xi$ at each $q \in W$.

Remark that $d_q\xi$ is injective, iff its image

$$\operatorname{Im} d_q\xi = \operatorname{Span}\left(\frac{\partial \mathrm{p}}{\partial q^h}\bigg|_q \right)$$

(spanned by the first-order partial derivatives of ξ) is an n-dimensional vector subspace of E.

Hence $n \leq m$.

Atlas of charts

Owing to the definition, a manifold $Q \subset \mathcal{E}$ is covered by an *atlas* of open subsets $\mathcal{U}$ described in $\mathcal{E}$ by means of local parametrizations ξ. Each point $\mathrm{p} \in \mathcal{U}$ is then given a unique n-tuple of *coordinates* $q = \xi^{-1}(\mathrm{p}) \in W$ by the inverse map $\xi^{-1} : \mathcal{U} \to W$, which is called an n-dimensional *local chart* (or *local coordinate system*) on Q (it is said to be *global* in the special case of its *coordinate domain* $\mathcal{U}$ being equal to the whole Q). [7]

Locally Euclidean topology

From a topological point of view, Q is a *locally Euclidean* subspace of $\mathcal{E}$, in the sense that it is covered by open subsets (namely, the coordinate domains) which, owing to property 1, are all homeomorphic to open subsets of Euclidean space $\mathbb{R}^n$.

Smoothness

From a differential point of view, Q is a *smooth* subspace of $\mathcal{E}$, in the sense that it admits, owing to property 2, an 'n-dimensional tangent vector space' at each one of its points, as will now be shown.

Smooth curves and tangent vectors

Recall that 'tangency' is a local (infinitesimal) concept, linked to 'derivation' as follows.

[7] In the sequel, ξ too will be called a chart.

Consider a *smooth curve* of $\mathcal{E}$, i.e. a C^∞- differentiable $\mathcal{E}$-valued map

$$\gamma : I \to \mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

defined on an open interval $I \subset \mathbb{R}$ ($\mathrm{Im}\,\gamma$ will be called the *orbit* of γ).

The vector $\dot{\mathrm{p}}(t) \in E$ obtained from γ by derivation at any $t \in I$, is said to be the free vector *tangent* at $\mathrm{p}(t)$ to γ. [8]

Tangent vector spaces

Now consider the smooth curves γ of $\mathcal{E}$ with orbits lying on Q, i.e.

$$\mathrm{Im}\,\gamma \subset Q$$

(also referred to as 'smooth curves of Q' and denoted by $\gamma : I \to Q$), and passing through a given point $\mathrm{p} \in Q$, i.e.

$$\mathrm{p} \in \mathrm{Im}\,\gamma$$

A vector of E will be said to be a free vector *tangent* at p to Q, if it is tangent at p to one of the above curves.

The set

$$T_{\mathrm{p}}Q \subset E$$

of all the free vectors tangent at p to Q will be called *tangent vector space* of Q at p, by virtue of the following result.

Proposition 19 $T_{\mathrm{p}}Q$ *is an n-dimensional vector subspace of* E.

Proof Follows from property 2, owing to the following
Lemma For any chart $\xi : W \to \mathcal{U}$ at p (i.e. $\mathcal{U} \ni \mathrm{p} = \xi(q)$, with $q \in W$),

$$T_{\mathrm{p}}Q = \mathrm{Im}\, d_q\xi \qquad (\dagger)$$

(i) Let $\mathbf{v} \in T_{\mathrm{p}}Q$, i.e.

$$\mathbf{v} = \dot{\mathrm{p}}(t_o)$$

for some smooth curve $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$ through $\mathrm{p}(t_o) = \mathrm{p}$ (with $t_o \in I$).

[8] Think of the graphic representation of derivative $\dot{\mathrm{p}}(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\big(\mathrm{p}(t+\Delta t) - \mathrm{p}(t)\big)$ – when it does not vanish – as an oriented segment, limit of a secant segment, attached at $\mathrm{p}(t)$.

As $\mathcal{U}$ is an open neighbourhood of $\mathrm{p}(t_o) = \mathrm{p}$ in Q, there exists (by continuity) an open neighbourhood I_o of t_o in I s.t. $\gamma(I_o) \subset \mathcal{U}$. As a consequence, γ ca be given the local C^∞ *coordinate expression*

$$\alpha = \xi^{-1} \circ \gamma|_{I_o} : I_o \to W : t \mapsto q(t) = \xi^{-1}(\mathrm{p}(t))$$

By derivation at t_o of $\mathrm{p}(t) = \xi(q(t))$, we obtain

$$\begin{aligned} \mathbf{v} &= \dot{\mathrm{p}}(t_o) \\ &= \dot{q}^h(t_o) \frac{\partial \mathrm{p}}{\partial q^h}\Big|_{q(t_o)} \\ &= d_{q(t_o)}\xi\,(\dot{q}(t_o)) \\ &= d_q\xi\,(v)\;, \quad v := \dot{q}(t_o) \in \mathbb{R}^n \end{aligned}$$

Hence $\mathbf{v} \in \operatorname{Im} d_q\xi$.

(ii) Now let $\mathbf{v} \in \operatorname{Im} d_q\xi$, i.e.

$$\mathbf{v} = d_q\xi\,(v)\;, \quad v \in \mathbb{R}^n$$

Choose $t_o \in \mathbb{R}$ and consider the affine map

$$\alpha : \mathbb{R} \to \mathbb{R}^n : t \mapsto q(t) := q + (t - t_o)v$$

As W is an open neighbourhood of $q(t_o) = q$ in $\mathbb{R}^n$, there exists (by continuity) an open neighbourhood I_o of t_o in $\mathbb{R}$ s.t. $\alpha(I_o) \subset W$. As a consequence,

$$\gamma = \xi \circ \alpha|_{I_o} : I_o \to \mathcal{U} : t \mapsto \mathrm{p}(t) = \xi(q(t))$$

is a smooth curve of Q through

$$\mathrm{p}(t_o) = \xi(q(t_o)) = \xi(q) = \mathrm{p}$$

whose derivative at t_o is

$$\begin{aligned} \dot{\mathrm{p}}(t_o) &= \dot{q}^h(t_o) \frac{\partial \mathrm{p}}{\partial q^h}\Big|_{q(t_o)} \\ &= d_{q(t_o)}\xi\,(\dot{q}(t_o)) \\ &= d_q\xi\,(v) \\ &= \mathbf{v} \end{aligned}$$

Hence $\mathbf{v} \in T_{\mathrm{p}}Q$.

□

Remark that

$$\mathbf{v} \in T_{\mathrm{p}}Q \longmapsto v = (d_q\xi)^{-1}(\mathbf{v}) \in \mathbb{R}^n$$

is the linear isomorphism which takes any vector $\mathbf{v} \in T_{\mathrm{p}}Q$ to its components $v \in \mathbb{R}^n$ in the basis $\left(\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right)$ of $T_{\mathrm{p}}Q$ (called *linear components* of $\mathbf{v}$ in ξ).

Open submanifolds

Trivial examples of submanifolds of $\mathcal{E}$ are the open subets.

Any system of affine coordinates on $\mathcal{E}$ restricted to an open subset $U \subset \mathcal{E}$ determines an m-dimensional global chart on U, which is therefore an m-dimensional embedded submanifold of $\mathcal{E}$ and, for all $\mathrm{p} \in U$,

$$T_{\mathrm{p}} U = E$$

For instance, if

$$g = (g_1, \ldots, g_\mu) : \mathcal{E} \to \mathbb{R}^\mu : \mathrm{p} \longmapsto g(\mathrm{p}) = (g_1(\mathrm{p}), \ldots, g_\mu(\mathrm{p}))$$

is a continuous map, then

$$\begin{aligned} U &:= g^{-1}(\mathbb{R}^+)^\mu = \{\mathrm{p} \in \mathcal{E} \mid g(\mathrm{p}) \in (\mathbb{R}^+)^\mu\} \\ &= \{\mathrm{p} \in \mathcal{E} \mid g_1(\mathrm{p}) > 0, \ldots, g_\mu(\mathrm{p}) > 0\} \end{aligned}$$

(inverse image by g of the open subset $(\mathbb{R}^+)^\mu \subset \mathbb{R}^\mu$) is an open submanifold of $\mathcal{E}$.

Implicit function theorem

Non-trivial examples of submanifolds of $\mathcal{E}$ arise from the well known Implicit Function Theorem, concerning 'loci' described by means of algebraic equations.

Proposition 20 *Let*

$$f = (f_1, \ldots, f_\kappa) : U \subset \mathcal{E} \to \mathbb{R}^\kappa : \mathrm{p} \longmapsto f(\mathrm{p}) = (f_1(\mathrm{p}), \ldots, f_\kappa(\mathrm{p}))$$

be a differentiable map, defined on an open subset $U \subset \mathcal{E}$ *(say* $U = g^{-1}(\mathbb{R}^+)^\mu$ *) and taking values in* $\mathbb{R}^\kappa$ *(*$\kappa < m := \dim \mathcal{E}$ *), with surjective differential* $d_{\mathrm{p}} f : E \to \mathbb{R}^\kappa$ *at each point* p *of*

$$\begin{aligned} Q &:= f^{-1}(0) = \{\mathrm{p} \in U \mid f(\mathrm{p}) = 0\} = \{\mathrm{p} \in U \mid f_1(\mathrm{p}) = 0, \ldots, f_\kappa(\mathrm{p}) = 0\} \\ &= \{\mathrm{p} \in \mathcal{E} \mid f_1(\mathrm{p}) = 0, \ldots, f_\kappa(\mathrm{p}) = 0,\ g_1(\mathrm{p}) > 0, \ldots, g_\mu(\mathrm{p}) > 0\} \end{aligned}$$

Q *is then an* n*-dimensional embedded submanifold of* $\mathcal{E}$ *with*

$$n := m - \kappa$$

and, for all $\mathrm{p} \in Q$,

$$T_{\mathrm{p}} Q = \operatorname{Ker} d_{\mathrm{p}} f$$

Proof (i) The above mentioned theorem of Analysis states that, for any $\mathrm{p} \in Q$, the fact of $d_{\mathrm{p}} f$ being surjective implies the existence of an open neighbourhood $\mathcal{U}$ of p in Q, an open subset $W \subset \mathbb{R}^n$ and a κ-tuple of real-valued functions $\varphi^1 : W \to \mathbb{R}, \ldots, \varphi^\kappa : W \to \mathbb{R}$ s.t.

$$\mathcal{U} = \operatorname{Im} \xi$$

where

$$\xi : W \to \mathcal{E}$$

is a map defined (with a suitable choice of affine coordinates $\phi : \mathbb{R}^m \to \mathcal{E}$) by

$$(x^1, \ldots, x^n) \xmapsto{\xi} \phi(\, x^1, \ldots, x^n,\, \varphi^1(x^1, \ldots, x^n), \ldots, \varphi^\kappa(x^1, \ldots, x^n)\,)$$

and satisfying properties 1 and 2, i.e. an n-dimensional chart on Q. [9]

(ii) As ξ is composable with f and $f \circ \xi = 0$, we have

$$d_{\mathrm{p}} f \circ d_q \xi = d_q (f \circ \xi) = 0$$

(with $\mathrm{p} = \xi(q)$) and then

$$\operatorname{Im} d_q \xi \subset \operatorname{Ker} d_{\mathrm{p}} f$$

Moreover, the surjectivity of $d_{\mathrm{p}} f$ and the injectivity of $d_q \xi$ imply the dimensional result

$$\dim \operatorname{Ker} d_{\mathrm{p}} f = \dim E - \dim \operatorname{Im} d_{\mathrm{p}} f = m - \kappa = n = \dim \operatorname{Im} d_q \xi$$

So we obtain

$$\operatorname{Im} d_q \xi = \operatorname{Ker} d_{\mathrm{p}} f$$

which, owing to (†), completes our proof.

□

[9] So, on $\phi^{-1}(\mathcal{U})$, the algebraic equations

$$f_1\big(\phi(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{n+\kappa})\big) = 0, \ldots, f_\kappa\big(\phi(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{n+\kappa})\big) = 0$$

implicitly define $x^{n+1}, \ldots, x^{n+\kappa}$ as functions of $(x^1, \ldots, x^n) \in W$, i.e.

$$x^{n+1} = \varphi^1(x^1, \ldots, x^n), \ldots, x^{n+\kappa} = \varphi^\kappa(x^1, \ldots, x^n)$$

5.2 Tangent bundle and differential equations

The tangent bundle of a manifold is the first geometric arena of differential equations.

Tangent bundle and canonical projection

The *tangent bundle* of an n-dimensional smooth manifold $Q \subset \mathcal{E}$ is the disjoint union of its tangent vector spaces, i.e.

$$TQ = \bigcup_{\mathrm{p} \in Q} \{\mathrm{p}\} \times T_{\mathrm{p}}Q$$

(for any $\mathrm{p} \in Q$, the set $\{\mathrm{p}\} \times T_{\mathrm{p}}Q$ is made up of all the *attached vectors tangent at* p *to* Q).

The *canonical projection* of TQ onto its *base manifold* Q is the map

$$\tau_Q : TQ \to Q : (\mathrm{p}, \mathbf{v}) \mapsto \mathrm{p}$$

Vector fibre bundle structure

TQ is a *vector fibre bundle* over Q, in the sense that its *fibre* $\{\mathrm{p}\} \times T_{\mathrm{p}}Q$ over each $\mathrm{p} \in Q$ is a vector space (canonically isomorphic to $T_{\mathrm{p}}Q$).

Such fibres are 'glued' together by *natural charts* (arising from the charts $\xi : W \to \mathcal{U}$ of Q)

$$\xi^1 : (q, v) \in W \times \mathbb{R}^n \mapsto (\mathrm{p}, \mathbf{v}) = (\xi(q), d_q\xi\,(v)) \in \tau_Q^{-1}(\mathcal{U})$$

which give TQ the structure of a $2n$-dimensional smooth manifold embedded in $T\mathcal{E} = \mathcal{E} \times E$. [10]

Tangent lift

The *tangent lift* of a smooth curve

$$\gamma : I \to Q : t \mapsto \gamma(t) = \mathrm{p}(t)$$

living in $Q \subset \mathcal{E}$, is the smooth curve

$$\dot{\gamma} : I \to TQ : t \mapsto \dot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t))$$

living in $TQ \subset T\mathcal{E}$.

[10] Remark that, for any open subset $U \subset \mathcal{E}$ (and then for $U = \mathcal{E}$ as well), $TU = U \times E$. Recall that $T\mathcal{E} = \mathcal{E} \times E$ is a Euclidean affine space, modelled on $E \times E$.

Differential equations in implicit form

From the very definition of tangent vector spaces, it follows that TQ is the region of $T\mathcal{E}$ 'swept' by the tangent lifts of all the smooth curves of Q (i.e. covered by the orbits of such lifts).

As a consequence, if a subset

$$D \subset TQ$$

is assigned (by prescribing some geometric or algebraic property), the problem may arise of determining the smooth curves γ of Q whose tangent lifts live in D, i.e.

$$\mathrm{Im}\,\dot{\gamma} \subset D \tag{$\diamond$}$$

that is to say,

$$\dot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t)) \in D\,, \quad \forall t \in I \tag{$\diamond$}$$

Such a D will be called (*autonomous*) *first-order differential equation in implicit form on* Q. [11]

Integral curves

Any solution to the above problem, i.e. any smooth curve γ of Q satisfying condition ($\diamond$), is called an *integral curve* of D (or *maximal* integral curve, if it is *not* restriction of any other integral curve).

Integrable part

The problem of establishing the existence of integral curves is the same as that of determining the *integrable part* of D, which is the region

$$D^{(i)} \subset D$$

swept by the tangent lifts of all its integral curves.

D is said to be *integrable*, if

$$\emptyset \neq D = D^{(i)}$$

[11] We shall not consider the case of a *non-autonomous* equation $D \subset J^1Q$, subset of the *jet bundle* $J^1Q := \mathbb{R} \times TQ$, whose solutions are the smooth curves γ of Q with *jet extension* $j^1\gamma : t \in I \mapsto j^1\gamma\,(t) := (t, \mathrm{p}(t), \dot{\mathrm{p}}(t)) \in J^1Q$ living in D, i.e. $\mathrm{Im}\, j^1\gamma \subset D$.

Vector fields and normal form

The main case of integrability is that of a differential equation D reducible to *normal form*, that is, one for which there exists a smooth *vector field*

$$X : Q \to TQ : \mathrm{p} \mapsto (\mathrm{p}, X(\mathrm{p}))$$

on Q [12] s.t.

$$D = \operatorname{Im} X = \{(\mathrm{p}, \mathbf{v}) \in TQ \mid \mathbf{v} = X(\mathrm{p})\}$$

For such an equation, condition ($\diamond$) reads

$$\dot{\gamma} = X \circ \gamma$$

that is to say,

$$\dot{\mathrm{p}}(t) = X(\mathrm{p}(t)) \ , \quad \forall t \in I$$

Cauchy problems and determinism

The integrability of

$$D = \operatorname{Im} X \neq \emptyset$$

is a consequence of the following fundamental property of normal equations (here stated without proof).

Determinism theorem: For any $t_o \in \mathbb{R}$ and $\mathrm{p}_o \in Q$, there exists a unique maximal solution to *Cauchy problem* (D, t_o, p_o), i.e. a unique maximal integral curve $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$ of D, with $I \ni t_o$, satisfying *initial condition* $\mathrm{p}(t_o) = \mathrm{p}_o$ (all of the other solutions to the problem being restrictions of the maximal one).

Indeed, for any $(\mathrm{p}_o, \mathbf{v}_o) \in D$, i.e. $\mathrm{p}_o \in Q$ and $\mathbf{v}_o = X(\mathrm{p}_o)$, we obtain

$$(\mathrm{p}_o, \mathbf{v}_o) = (\mathrm{p}(t_o), X(\mathrm{p}(t_o))) = (\mathrm{p}(t_o), \dot{\mathrm{p}}(t_o)) \in \operatorname{Im} \dot{\gamma}_o \subset D^{(i)}$$

whence

$$D = D^{(i)}$$

[12] X is said to be *smooth*, if – in any chart $\xi : W \to \mathcal{U}$ – it admits C^∞ linear components

$$q \in W \longmapsto (d_q\xi)^{-1}\big(X(\xi(q))\big) \in \mathbb{R}^n$$

5.3 Second tangent bundle and second-order differential equations

The second tangent bundle of a manifold is the geometric arena of second-order differential equations.

Second tangent lift and second tangent bundle

Consider the tangent bundle

$$TTQ \subset TT\mathcal{E}$$

of $TQ \subset T\mathcal{E}$.

Recall that TTQ is the region of $TT\mathcal{E} = (\mathcal{E} \times E) \times (E \times E)$ swept by the tangent lifts of all the smooth curves of TQ, the tangent lift of any such curve

$$c : I \to TQ : t \mapsto c(t) = (\mathrm{p}(t), \mathbf{v}(t))$$

being given by

$$\dot{c} : I \to TTQ : t \mapsto \dot{c}(t) = (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t))$$

In particular, the *second tangent lift* of a smooth curve

$$\gamma : I \to Q : t \mapsto \gamma(t) = \mathrm{p}(t)$$

living in Q, is the tangent lift of

$$\dot{\gamma} : I \to TQ : t \mapsto \dot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t))$$

i.e. the smooth curve [13]

$$\ddot{\gamma} : I \to T^2Q : t \mapsto \ddot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t))$$

living in the region $T^2Q \subset TTQ$, called *second tangent bundle* of Q, defined by

$$T^2Q = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\}$$

[13] $\ddot{\mathrm{p}}(t)$ will denote the derivative of $\dot{\mathrm{p}}(t)$.

Affine fibre bundle structure

Remark that

$$T^2Q = \bigcup_{(\mathrm{p},\mathbf{v})\in TQ} \{(\mathrm{p},\mathbf{v})\} \times \left(\{\mathbf{v}\} \times T^2_{(\mathrm{p},\mathbf{v})}Q\right)$$

with

$$T^2_{(\mathrm{p},\mathbf{v})}Q := \{\mathbf{w} \in E \mid (\mathrm{p},\mathbf{v};\mathbf{v},\mathbf{w}) \in TTQ\}$$

is an *affine fibre bundle* over TQ, in the sense that its *fibre* $\{(\mathrm{p},\mathbf{v})\} \times \left(\{\mathbf{v}\} \times T^2_{(\mathrm{p},\mathbf{v})}Q\right)$ over each $(\mathrm{p},\mathbf{v}) \in TQ$ is an affine space (canonically isomorphic to $T^2_{(\mathrm{p},\mathbf{v})}Q$), as will now be shown.

Proposition 21 *For any $(\mathrm{p},\mathbf{v}) \in TQ$, $T^2_{(\mathrm{p},\mathbf{v})}Q$ is an n-dimensional affine subspace of E modelled on $\mathrm{T}_{\mathrm{p}}Q$, i.e.*

$$T^2_{(\mathrm{p},\mathbf{v})}Q = \mathbf{z} + T_{\mathrm{p}}Q$$

for some $\mathbf{z} \in E$.

Proof Follows from property (†), owing to the following
Lemma For any natural chart $\xi^1 : W \times \mathbb{R}^n \to \tau_Q^{-1}(\mathcal{U})$ at $(\mathrm{p},\mathbf{v})$ (i.e. $\tau_Q^{-1}(\mathcal{U}) \ni (\mathrm{p},\mathbf{v}) = (\xi(q), d_q\xi\,(v))$ with $(q,v) \in W \times \mathbb{R}^n$),

$$T^2_{(\mathrm{p},\mathbf{v})}Q = \mathbf{z}(q,v) + \mathrm{Im}\, d_q\xi \qquad (\dagger\dagger)$$

where $\mathbf{z}(q,v) := v^h v^k \frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_q$.

(i) Let $\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q$, that is,

$$(\mathrm{p},\mathbf{v};\mathbf{v},\mathbf{w}) \in TTQ$$

or

$$(\mathrm{p},\mathbf{v};\mathbf{v},\mathbf{w}) = \dot{c}(t_o) = (\mathrm{p}(t_o),\mathbf{v}(t_o);\dot{\mathrm{p}}(t_o),\dot{\mathbf{v}}(t_o))$$

for some smooth curve $c : t \in I \mapsto c(t) = (\mathrm{p}(t),\mathbf{v}(t)) \in TQ$ and $t_o \in I$.
As $\mathcal{U}$ is an open neighbourhood of $\mathrm{p} = \mathrm{p}(t_o)$ in Q, there exists (by continuity) an open neighbourhood I_o of t_o in I s.t. $\mathrm{p}(t) \in \mathcal{U}$ for all $t \in I_o$. Then $c|_{I_o}$ can be given the C^∞ coordinate expression

$$\mathrm{p}(t) = \xi(q(t))\,,\ \mathbf{v}(t) = d_{q(t)}\xi\,(v(t)) = v^h(t)\frac{\partial \mathrm{p}}{\partial q^h}\Big|_{q(t)}$$

with

$$q(t_o) = q\ ,\ \ \dot{q}(t_o) = v(t_o) = v$$

By derivation at t_o, we obtain

$$\begin{aligned}
\mathbf{w} &= \dot{\mathbf{v}}(t_o) \\
&= v^h(t_o)\dot{q}^k(t_o)\frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_{q(t_o)} + \dot{v}^h(t_o)\frac{\partial \mathrm{p}}{\partial q^h}\Big|_{q(t_o)} \\
&= v^h v^k \frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_q + w^h \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q , \quad w := \dot{v}(t_o) \\
&= \mathbf{z}(q,v) + d_q\xi\,(w)
\end{aligned}$$

Hence $\mathbf{w} \in \mathbf{z}(q,v) + \operatorname{Im} d_q\xi$.

(ii) Now let $\mathbf{w} \in \mathbf{z}(q,v) + \operatorname{Im} d_q\xi$, that is,

$$\mathbf{w}_{=}\mathbf{z}(q,v) + d_q\xi\,(w)$$

for some $w \in \mathbb{R}^n$.
Choose $t_o \in \mathbb{R}$ and consider the map

$$\alpha : \mathbb{R} \to \mathbb{R}^n : t \mapsto q(t) := q + v(t-t_o) + \frac{1}{2}w(t-t_o)^2$$

satisfying

$$q(t_o) = q \ , \quad \dot{q}(t_o) = v \ , \quad \ddot{q}(t_o) = w$$

As W is an open neighbourhood of $q(t_o) = q$ in $\mathbb{R}^n$, there exists (by continuity) an open neighbourhood I_o of t_o in $\mathbb{R}$ s.t. $\alpha(I_o) \subset W$. As a consequence,

$$\gamma = \xi \circ \alpha|_{I_o} : I_o \to \mathcal{U} : t \mapsto \mathrm{p}(t) = \xi(q(t))$$

is a smooth curve of Q through

$$\mathrm{p}(t_o) = \xi(q(t_o)) = \xi(q) = \mathrm{p}$$

whose derivatives at t_o are

$$\begin{aligned}
\dot{\mathrm{p}}(t_o) &= \dot{q}^h(t_o)\frac{\partial \mathrm{p}}{\partial q^h}\Big|_{q(t_o)} \\
&= d_{q(t_o)}\xi\,(\dot{q}(t_o)) \\
&= d_q\xi\,(v) \\
&= \mathbf{v}
\end{aligned}$$

and

$$\begin{aligned}
\ddot{\mathrm{p}}(t_o) &= \dot{q}^h(t_o)\dot{q}^k(t_o)\frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_{q(t_o)} + \ddot{q}^h(t_o)\frac{\partial \mathrm{p}}{\partial q^h}\Big|_{q(t_o)} \\
&= v^h v^k \frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_q + w^h \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \\
&= \mathbf{z}(q,v) + d_q\xi\,(w) \\
&= \mathbf{w}
\end{aligned}$$

So

$$(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) = (\mathrm{p}(t_o), \dot{\mathrm{p}}(t_o); \dot{\mathrm{p}}(t_o), \ddot{\mathrm{p}}(t_o)) = \ddot{\gamma}(t_o) \in \mathrm{Im}\, \ddot{\gamma} \subset T^2Q \subset TTQ$$

Hence $\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q$.

$\square$

Remark that

$$\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q \longmapsto w = (d_q\xi)^{-1}(\mathbf{w} - \mathbf{z}(q, v)) \in \mathbb{R}^n$$

is the affine isomorphism which takes any vector $\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q$ to its affine coordinates $w \in \mathbb{R}^n$ in the frame $\left(\mathbf{z}(q, v), \left(\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q\right)\right)$ of $T^2_{(\mathrm{p},\mathbf{v})}Q$ (called *affine components* of $\mathbf{w}$ in ξ).

Second-order differential equations in implicit form

Part (ii) of the above proof shows as well that T^2Q is the region of TTQ 'swept' by the second tangent lifts of all the smooth curves of Q .

As a consequence, if a subset

$$\mathcal{D} \subset T^2Q$$

is assigned (by prescribing some geometric or algebraic property), the problem may arise of determining the smooth curves $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$ whose second tangent lifts live in $\mathcal{D}$, i.e.

$$\mathrm{Im}\, \ddot{\gamma} \subset \mathcal{D} \qquad (\diamond\diamond)$$

that is to say,

$$\ddot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t)) \in \mathcal{D}\,, \quad \forall t \in I \qquad (\diamond\diamond)$$

Such a $\mathcal{D}$ will be called (*autonomous*) *second-order differential equation in implicit form on* Q .

Integral curves and base integral curves

A second-order differential equation on Q, say $\mathcal{D} \subset T^2Q \subset TTQ$, is a first-order differential equation on TQ as well, whose integral curves exhibit the following peculiar property.

Proposition 22 *Each integral curve c of $\mathcal{D} \subset T^2Q$ is the tangent lift of its own projection $\tau_Q \circ c$, i.e.*

$$(\tau_Q \circ c)^{\boldsymbol{\cdot}} = c$$

Proof Let

$$c : t \in I \mapsto c(t) = (\mathrm{p}(t), \mathbf{v}(t)) \in TQ$$

be a smooth curve of TQ, and

$$\tau_Q \circ c : t \in I \mapsto \tau_Q(c(t)) = \mathrm{p}(t) \in Q$$

its projection onto Q.
If c is an integral curve of $\mathcal{D}$, it satisfies, for all $t \in I$,

$$\dot{c}(t) = (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t)) \in \mathcal{D}$$

whence

$$\dot{\mathrm{p}}(t) = \mathbf{v}(t)$$

that is,

$$(\tau_Q \circ c)^{\boldsymbol{\cdot}}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t)) = (\mathrm{p}(t), \mathbf{v}(t)) = c(t)$$

which is our claim.

□

The *base integral curves* $\gamma = \tau_Q \circ c$ of $\mathcal{D}$ –projections of the integral curves c of $\mathcal{D}$– are the solution curves of problem ($\diamond\diamond$), as will now be shown.

Proposition 23 *A smooth curve γ of Q is a base integral curve of $\mathcal{D}$, iff it is a solution curve of problem* ($\diamond\diamond$).

Proof (i) Let $\mathrm{Im}\,\ddot{\gamma} \subset \mathcal{D}$. Putting $c := \dot{\gamma}$, whence $\tau_Q \circ c = \tau_Q \circ \dot{\gamma}$ and $\dot{c} = \ddot{\gamma}$, we obtain $\tau_Q \circ c = \gamma$ and $\mathrm{Im}\,\dot{c} \subset \mathcal{D}$.
(ii) Conversely, let $\gamma = \tau_Q \circ c$ and $\mathrm{Im}\,\dot{c} \subset \mathcal{D}$. From Prop. 22, we obtain $\dot{\gamma} = (\tau_Q \circ c)^{\boldsymbol{\cdot}} = c$, whence $\ddot{\gamma} = \dot{c}$, and then $\mathrm{Im}\,\ddot{\gamma} \subset \mathcal{D}$.

□

The integral curves and the base integral curves of $\mathcal{D}$ are then bijectively related to one another by projection τ_Q (Prop. 22) and, owing to the above bijection, the first-order problem of determining the integral curves of $\mathcal{D}$ proves to be naturally equivalent to the second-order problem of determining its base integral curves, i.e. the solution curves γ of ($\diamond\diamond$) (Prop. 23).

Semi-sprays and normal form

A second-order equation $\mathcal{D}$ is then reducible to normal form, if there exists a smooth *semi-spray*

$$\Gamma : TQ \to T^2Q \subset TTQ : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \mathbf{v}; \mathbf{v}, \Gamma(\mathrm{p}, \mathbf{v}))$$

on TQ [14] s.t.

$$\mathcal{D} = \operatorname{Im}\Gamma = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,,\ \mathbf{w} = \Gamma(\mathrm{p}, \mathbf{v})\}$$

Proposition 24 *Condition* $\operatorname{Im}\dot{c} \subset \mathcal{D}$, *characterizing the integral curves* c *of* $\mathcal{D} = \operatorname{Im}\Gamma$, *reads*

$$\dot{c} = \Gamma \circ c$$

Condition $\operatorname{Im}\ddot{\gamma} \subset \mathcal{D}$, *characterizing the base integral curves* γ *of* $\mathcal{D} = \operatorname{Im}\Gamma$, *reads*

$$\ddot{\gamma} = \Gamma \circ \dot{\gamma}$$

Proof (i) For a smooth curve $c : t \in I \mapsto c(t) = (\mathrm{p}(t), \mathbf{v}(t)) \in TQ$, condition

$$\dot{c}(t) = (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t)) \in \mathcal{D} = \operatorname{Im}\Gamma\,, \quad \forall t \in I$$

reads

$$\dot{\mathrm{p}}(t) = \mathbf{v}(t)\,, \quad \dot{\mathbf{v}}(t) = \Gamma(\mathrm{p}(t), \mathbf{v}(t))\,, \quad \forall t \in I$$

that is,

$$\begin{aligned} \dot{c}(t) &= (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t)) \\ &= (\mathrm{p}(t), \mathbf{v}(t); \mathbf{v}(t), \Gamma(\mathrm{p}(t), \mathbf{v}(t))) \\ &= (\Gamma \circ c)\,(t)\,, \quad \forall t \in I \end{aligned}$$

(ii) For a smooth curve $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$, condition

$$\ddot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t)) \in \mathcal{D} = \operatorname{Im}\Gamma\,, \quad \forall t \in I$$

reads

$$\ddot{\mathrm{p}}(t) = \Gamma(\mathrm{p}(t), \dot{\mathrm{p}}(t))\,, \quad \forall t \in I$$

that is,

$$\begin{aligned} \ddot{\gamma}(t) &= (\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t)) \\ &= (\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \Gamma(\mathrm{p}(t), \dot{\mathrm{p}}(t))) \\ &= (\Gamma \circ \dot{\gamma})\,(t)\,, \quad \forall t \in I \end{aligned}$$

□

[14] Γ (vector field on TQ taking values in T^2Q) is said to be *smooth*, if −in any chart $\xi : W \to \mathcal{U}$− it admits C^∞ affine components

$$(q, v) \in W \times \mathbb{R}^n \longmapsto (d_q\xi)^{-1}\big(\Gamma(\xi(q), d_q\,\xi(v)) - \mathbf{z}(q, v)\big) \in \mathbb{R}^n$$

Cauchy problems and determinism

If $\mathcal{D} = \operatorname{Im}\Gamma$, then – owing to determinism theorem – Cauchy problem $(\mathcal{D}, t_o, (\mathrm{p}_o, \mathbf{v}_o))$, for any $t_o \in \mathbb{R}$ and $(\mathrm{p}_o, \mathbf{v}_o) \in TQ$, admits a unique maximal solution, which amounts to saying that there exists a unique maximal base integral curve $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$ of $\mathcal{D}$, with $t_o \in I$, satisfying *initial conditions* $(\mathrm{p}(t_o), \dot{\mathrm{p}}(t_o)) = (\mathrm{p}_o, \mathbf{v}_o)$ (all of the other solutions to the problem being restrictions of the maximal one).

5.4 Cotangent bundle and differential forms

The cotangent bundle of a manifold is the geometric arena of differential forms.

Cotangent vector spaces

For any $\mathrm{p} \in Q$, to the tangent vector space $T_{\mathrm{p}}Q$ there corresponds, by duality, the *cotangent vector space* $T^*_{\mathrm{p}}Q$ made up of all the *covectors*

$$\pi : T_{\mathrm{p}}Q \to \mathbb{R} : \mathbf{v} \mapsto \langle \pi \mid \mathbf{v} \rangle$$

(linear maps of $T_{\mathrm{p}}Q$ in $\mathbb{R}$).

Covectors can naturally be summed to one another and multiplied by scalars, so that $T^*_{\mathrm{p}}Q$ turns into a vector space.

To the basis $\left(\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right)$ determined in $T_{\mathrm{p}}Q$ by a chart ξ at $\mathrm{p} = \xi(q)$, there corresponds a *dual basis* $(d_{\mathrm{p}}q^h)$ in $T^*_{\mathrm{p}}Q$, defined by

$$d_{\mathrm{p}}q^h : T_{\mathrm{p}}Q \to \mathbb{R} : \mathbf{v} = v^k \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q \longmapsto \langle d_{\mathrm{p}}q^h \mid \mathbf{v} \rangle := v^h$$

The components (p_h) which characterize any covector $\pi \in T^*_{\mathrm{p}}Q$ in the above basis – called components of π in ξ – are given by

$$p_h := \left\langle \pi \mid \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle$$

since, by *pairing* π with all $\mathbf{v} \in T_{\mathrm{p}}Q$, we obtain

$$\langle \pi \mid \mathbf{v} \rangle = \left\langle \pi \mid v^h \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle = \left\langle \pi \mid \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle v^h = p_h v^h = p_h \langle d_{\mathrm{p}}q^h \mid \mathbf{v} \rangle = \langle p_h d_{\mathrm{p}}q^h \mid \mathbf{v} \rangle$$

that is,

$$\pi = p_h \, d_{\mathrm{p}}q^h$$

Cotangent bundle and canonical projection

The *cotangent bundle* of an n-dimensional smooth manifold Q is the disjoint union of its cotangent vector spaces, i.e.

$$T^*Q := \bigcup_{\mathrm{p}\in Q} \{\mathrm{p}\} \times T^*_{\mathrm{p}}Q$$

The *canonical projection* of T^*Q onto its *base* Q is the map

$$\pi_Q : T^*Q \to Q : (\mathrm{p}, \pi) \mapsto \mathrm{p}$$

Vector bundle structure

T^*Q is a *vector fibre bundle* over Q, since its *fibre* $\{\mathrm{p}\} \times T^*_{\mathrm{p}}Q$ over each $\mathrm{p} \in Q$ is a vector space (canonically isomorphic to $T^*_{\mathrm{p}}Q$).

Such fibres are 'glued' together by *natural charts* (arising from the charts $\xi : W \to \mathcal{U}$ of Q)

$$\xi_1 : (q,p) \in W \times \mathbb{R}^n \mapsto (\mathrm{p}, \pi) = (\xi(q), p_h\, d_{\mathrm{p}}q^h) \in \pi_Q^{-1}(\mathcal{U})$$

which give T^*Q the structure of a $2n$-dimensional smooth manifold. [15]

Differential 1-forms

A smooth real-valued function $V : Q \to \mathbb{R}$ [16] gives rise to a *covector field*

$$dV : Q \to T^*Q : \mathrm{p} \mapsto (\mathrm{p}, d_{\mathrm{p}}V)$$

called *differential* of V or *exact 1-form* on Q, where the differential $d_{\mathrm{p}}V \in T^*_{\mathrm{p}}Q$ of V at p is defined by putting, for any vector $\mathbf{v} = \dot{\mathrm{p}}(t_o) \in T_{\mathrm{p}}Q$ tangent to a smooth curve $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$ at $\mathrm{p} = \mathrm{p}(t_o)$,

$$\langle d_{\mathrm{p}}V \mid \mathbf{v}\rangle := \frac{d}{dt}(V \circ \gamma)\Big|_{t_o}$$

[15] Unlike $TQ \subset T\mathcal{E} = \mathcal{E} \times E$, cotangent bundle T^*Q is *not* naturally embedded in some Euclidean space related to $\mathcal{E}$, and therefore its structure of smooth manifold is to be meant in the generalized sense explained in the last subsection 'Intrinsic geometry of smooth manifolds' of this Appendix.

[16] V is said to be *smooth*, if −in any chart $\xi : W \to \mathcal{U}$− it admits a C^∞ *coordinate expression*

$$V_\xi := V \circ \xi : q \in W \mapsto V(\xi(q)) \in \mathbb{R}$$

and then, by suitably restricting γ to the coordinate domain of a chart ξ where $\mathrm{p} = \xi(q)$ and $\mathbf{v} = d_q\xi(v)$,

$$\begin{aligned}
\langle d_{\mathrm{p}} V \mid \mathbf{v} \rangle &= \frac{d}{dt}(V_\xi \circ \xi^{-1} \circ \gamma)\Big|_{t_o} \\
&= \frac{\partial V_\xi}{\partial q^h}\Big|_{q(t_o)} \dot{q}^h(t_o) \\
&= \frac{\partial V_\xi}{\partial q^h}\Big|_q v^h \\
&= \frac{\partial V_\xi}{\partial q^h}\Big|_q \langle d_{\mathrm{p}} q^h \mid \mathbf{v} \rangle \\
&= \left\langle \frac{\partial V_\xi}{\partial q^h}\Big|_q d_{\mathrm{p}} q^h \mid \mathbf{v} \right\rangle
\end{aligned}$$

that is to say,

$$d_{\mathrm{p}} V = \frac{\partial V}{\partial q^h}\Big|_{\mathrm{p}} d_{\mathrm{p}} q^h$$

with

$$\frac{\partial V}{\partial q^h}\Big|_{\mathrm{p}} := \frac{\partial V_\xi}{\partial q^h}\Big|_q$$

The above considerations can all be extended to real-valued functions defined on open subsets of Q (check, for instance, that $d_{\mathrm{p}} q^k$ is the differential at p of the k-th projection $q^k : \mathcal{U} \to \mathbb{R}$ of ξ^{-1}).

More generally, a *differential 1-form* on Q is defined as a smooth covector field

$$f : Q \to T^*Q : \mathrm{p} \mapsto (\mathrm{p}, f(\mathrm{p}))$$

Its value $f(\mathrm{p}) \in T^*_{\mathrm{p}} Q$ is expressed in a chart ξ at $\mathrm{p} = \xi(q)$ by

$$f(\mathrm{p}) = f_h(q)\, d_{\mathrm{p}} q^h$$

in terms of components [17]

$$f_h(q) := \left\langle f(\mathrm{p}) \mid \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle$$

[17] f is said to be *smooth*, if – in any chart $\xi : W \to \mathcal{U}$ – it admits C^∞ components

$$f_h : q \in W \mapsto f_h(q) \in \mathbb{R}$$

Semi-basic differential 1-forms

The concept of 1-form furtherly generalizes into that of *semi-basic 1-form* on TQ, defined as a smooth *bundle morphism*

$$F : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, F(\mathrm{p}, \mathbf{v}))$$

Its value $F(\mathrm{p}, \mathbf{v}) \in T^*_{\mathrm{p}}Q$ is expressed in a chart ξ –where $\mathrm{p} = \xi(q)$ and $\mathbf{v} = d_q\xi(v)$– by

$$F(\mathrm{p}, \mathbf{v}) = F_h(q, v)\, d_{\mathrm{p}}q^h$$

in terms of components [18]

$$F_h(q, v) := \left\langle\, F(\mathrm{p}, \mathbf{v}) \mid \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle$$

The concept of *semi-basic 1-form* on T^2Q is defined in the same way, as a smooth *bundle morphism*

$$\Phi : T^2Q \to T^*Q : (\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \mapsto (\mathrm{p}, \Phi(\mathrm{p}, \mathbf{v}, \mathbf{w}))$$

Its value $\Phi(\mathrm{p}, \mathbf{v}, \mathbf{w}) \in T^*_{\mathrm{p}}Q$ is expressed in a chart ξ –where $\mathrm{p} = \xi(q)$, $\mathbf{v} = d_q\xi(v)$ and $\mathbf{w} = \mathbf{z}(q, v) + d_q\xi\,(w)$– by

$$\Phi(\mathrm{p}, \mathbf{v}, \mathbf{w}) = \Phi_h(q, v, w)\, d_{\mathrm{p}}q^h$$

in terms of components [19]

$$\Phi_h(q, v, w) = \left\langle\, \Phi(\mathrm{p}, \mathbf{v}, \mathbf{w}) \mid \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle$$

Special kinds of semibasic 1-forms will now be described.

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[18] F is said to be *smooth*, if –in any chart $\xi : W \to \mathcal{U}$– it admits C^∞ components

$$F_h : (q, v) \in W \times \mathbb{R}^n \mapsto F_h(q, v) \in \mathbb{R}$$

[19] Φ is said to be *smooth*, if –in any chart $\xi : W \to \mathcal{U}$– it admits C^∞ components

$$\Phi_h : (q, v, w) \in W \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \Phi_h(q, v, w) \in \mathbb{R}$$

Semi-Riemannian metric

A *semi-Riemannian metric* on an n-dimensional manifold Q is a *symmetric vector bundle isomorphism* of TQ onto T^*Q, that is, a semi-basic 1-form

$$g : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, g_{\mathrm{p}}(\mathbf{v}))$$

such that, for each $\mathrm{p} \in Q$, its restriction

$$g_{\mathrm{p}} : T_{\mathrm{p}}Q \to T^*_{\mathrm{p}}Q : \mathbf{v} \mapsto g_{\mathrm{p}}(\mathbf{v})$$

is a linear isomorphism (*non-degenerateness*) and the bilinear *inner product* $\langle g_{\mathrm{p}}(\mathbf{u}) \mid \mathbf{v}\rangle$ of any two vectors $\mathbf{u}, \mathbf{v} \in T_{\mathrm{p}}Q$ satisfies the symmetry condition (*commutativity*)

$$\langle g_{\mathrm{p}}(\mathbf{u}) \mid \mathbf{v}\rangle = \langle g_{\mathrm{p}}(\mathbf{v}) \mid \mathbf{u}\rangle$$

If non-degenerateness is strengthened by requiring *positive definiteness*, i.e. $\langle g_{\mathrm{p}}(\mathbf{v}) \mid \mathbf{v}\rangle > 0$ for all $(\mathrm{p}, \mathbf{v}) \in TQ$ with $\mathbf{v} \neq \mathbf{0}$, g is said to be a *Riemannian metric*.

Owing to *polarization identity*

$$\langle g_{\mathrm{p}}(\mathbf{u}) \mid \mathbf{v}\rangle = \frac{1}{2}\Big(\langle g_{\mathrm{p}}(\mathbf{u}+\mathbf{v}) \mid \mathbf{u}+\mathbf{v}\rangle - \langle g_{\mathrm{p}}(\mathbf{u}) \mid \mathbf{u}\rangle - \langle g_{\mathrm{p}}(\mathbf{v}) \mid \mathbf{v}\rangle\Big)$$

(for all $\mathrm{p} \in Q$ and $\mathbf{u}, \mathbf{v} \in T_{\mathrm{p}}Q$), g is characterized by its *quadratic form*

$$K : TQ \to \mathbb{R} : (\mathrm{p}, \mathbf{v}) \mapsto K(\mathrm{p}, \mathbf{v}) := \frac{1}{2}\langle g_{\mathrm{p}}(\mathbf{v}) \mid \mathbf{v}\rangle$$

A manifold equipped with a (semi-)Riemannian metric will then be denoted by (Q, K) and called (*semi-*)*Riemannian manifold.*

Remark that, any semi-spray

$$\Gamma_K : TQ \to T^2Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \mathbf{v}; \mathbf{v}, \Gamma_K(\mathrm{p}, \mathbf{v}))$$

associated with a (semi-)Riemannian manifold (Q, K), naturally transforms

$$g : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, g_{\mathrm{p}}(\mathbf{v}))$$

(semi-basic 1-form on TQ) into

$$[K] : T^2Q \to T^*Q : (\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \mapsto (\mathrm{p}, [K](\mathrm{p}, \mathbf{v}, \mathbf{w}))$$

(semi-basic 1-form on T^2Q) by putting

$$[K](\mathrm{p}, \mathbf{v}, \mathbf{w}) := g_{\mathrm{p}}(\mathbf{w} - \Gamma_K(\mathrm{p}, \mathbf{v})) \in T^*_{\mathrm{p}}Q$$

In a chart ξ at $\mathrm{p} = \xi(q)$, the inner product $\langle\, g_{\mathrm{p}}(\mathbf{u}) \mid \mathbf{v}\,\rangle$ is expressed (owing to bilinearity) by

$$\langle\, g_{\mathrm{p}}(\mathbf{u}) \mid \mathbf{v}\,\rangle = \left\langle\, g_{\xi(q)}\Big(u^h \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q\Big) \;\mid\; v^k \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q \,\right\rangle = u^h \left\langle\, g_{\xi(q)}\Big(\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q\Big) \;\mid\; \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q \,\right\rangle v^k$$

and therefore g_{p} is characterized by the (non-singular, symmetric) $n \times n$ matrix of components

$$g_{hk}(q) := \left\langle\, g_{\xi(q)}\Big(\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q\Big) \;\mid\; \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q \,\right\rangle$$

Almost-symplectic structure

An *almost-symplectic structure* on a $2n$-dimensional manifold M is a *skew symmetric vector bundle isomorphism* of TM onto T^*M, that is, a semi-basic 1-form

$$\omega : TM \to T^*M : (\pi, X) \mapsto (\pi, \omega_\pi(X))$$

such that, for each $\pi \in M$, its restriction

$$\omega_\pi : T_\pi M \to T^*_\pi M : X \mapsto \omega_\pi(X)$$

is a linear isomorphism (*non-degenerateness*) and the bilinear product $\langle \omega_\pi(X) \mid Y \rangle$ of any two vectors $X, Y \in T_\pi M$ satisfies the skew-symmetry condition (*anticommutativity*)

$$\langle \omega_\pi(X) \mid Y \rangle = -\langle \omega_\pi(Y) \mid X \rangle$$

ω is said to be a *symplectic structure*, if it is characterized –in a suitable atlas of charts of M– by the (non-singular, skew-symmetric) $2n \times 2n$ *symplectic matrix* of components

$$\begin{bmatrix} \omega_{(1)(1)} & \omega_{(1)(2)} \\ \omega_{(2)(1)} & \omega_{(2)(2)} \end{bmatrix}$$

with $\omega_{(1)(1)} = \omega_{(2)(2)} = 0$ (zero $n \times n$ matrix) and $\omega_{(1)(2)} = -\omega_{(2)(1)} = \delta$ (identity $n \times n$ matrix).

On a cotangent bundle $M = T^*Q$ there exists a *canonical* symplectic structure, characterized by the symplectic matrix in the atlas of the natural charts of M.

5.5 Intrinsic approach to smooth manifods

A deeper insight into the structure of an embedded submanifold of a Euclidean affine space, will provide useful suggestions for a generalized definition of 'smooth manifold' and an 'intrinsic' approach to its geometry (without any reference to embeddings into Euclidean environments).

Intrinsic geometry of embedded submanifolds

Let Q be an n-dimensional, smooth manifold embedded into a Euclidean affine space $\mathcal{E}$.

The *n-dimensional atlas* $\mathcal{A}$ made up of all the charts of Q, could be shown to be

1. C^∞, i.e. any two charts

$$\varphi = \xi^{-1} : \mathcal{U} \subset Q \to W \subset \mathbb{R}^n \,, \quad \varphi' = \xi'^{-1} : \mathcal{U}' \subset Q \to W' \subset \mathbb{R}^n$$

belonging to $\mathcal{A}$ are bijections (between subsets of Q and $\mathbb{R}^n$, respectively) related to each other by C^∞ *transition functions*

$$q \in \varphi(\mathcal{U} \cap \mathcal{U}') \xmapsto{\varphi^{-1}} \mathrm{p} \in \mathcal{U} \cap \mathcal{U}' \xmapsto{\varphi'} q' \in \varphi'(\mathcal{U} \cap \mathcal{U}')$$

$$q' \in \varphi'(\mathcal{U} \cap \mathcal{U}') \xmapsto{\varphi'^{-1}} \mathrm{p} \in \mathcal{U} \cap \mathcal{U}' \xmapsto{\varphi} q \in \varphi(\mathcal{U} \cap \mathcal{U}')$$

2. *complete*, i.e. any bijection

$$\tilde{\varphi} : \tilde{\mathcal{U}} \subset Q \to \tilde{W} \subset \mathbb{R}^n$$

which is C^∞-related to all the charts of $\mathcal{A}$, belongs to $\mathcal{A}$.

From the topological point of view, the domains of the charts belonging to a complete, C^∞, n-dimensional atlas (on any set) turn out to be the base of a 'locally Euclidean' *manifold topology*, where all the charts are homeomorphisms between open subsets (actually, the manifold topology determined by $\mathcal{A}$ on Q coincides with the subspace topology of Q).

From the differential point of view, for each $\mathrm{p} \in Q$, the tangent vector space $T_{\mathrm{p}}Q$ can be identified –through a canonical isomorphism– with the quotient vector space $\mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}}$, where

$$\mathcal{A}_{\mathrm{p}} \subset \mathcal{A}$$

denotes the set of all the charts with domains containing p and

$$(\varphi, v) \sim_{\mathrm{p}} (\varphi', v') \quad \text{iff} \quad v' = d_{\varphi(\mathrm{p})}(\varphi' \circ \varphi^{-1})\,(v) \tag{$\star$}$$

is the equivalence relation defined in $\mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n$ by the transformation law for the components of tangent vectors. [20]

The vector space structure of the quotient is the invariant one that makes an isomorphism of the bijection [21]

$$\varphi'_{\mathrm{p}} : \mathbb{R}^n \to \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}} : \; v \longmapsto \varphi'_{\mathrm{p}}(v) := [(\varphi, v)]_{\sim_{\mathrm{p}}} \tag{$\circ$}$$

determined by any $\varphi = \xi^{-1} \in \mathcal{A}_{\mathrm{p}}$.

The above mentioned identification is due to the invariant isomorphism

$$\iota_{\mathrm{p}} : T_{\mathrm{p}}Q \to \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}} : \; \mathbf{v} = d_{\varphi(\mathrm{p})}\xi\,(v) \longmapsto \iota_{\mathrm{p}}(\mathbf{v}) := \varphi'_{\mathrm{p}}(v)$$

In particular, for the basis of $T_{\mathrm{p}}Q$ determined by a chart $\varphi = \xi^{-1} \in \mathcal{A}_{\mathrm{p}}$, we put [22]

$$\left.\frac{\partial \mathrm{p}}{\partial q^h}\right|_{\varphi(\mathrm{p})} = d_{\varphi(\mathrm{p})}\xi\,(\delta_h) \xmapsto{\iota_{\mathrm{p}}} \left.\frac{\partial}{\partial q^h}\right|_{\mathrm{p}} := \varphi'_{\mathrm{p}}(\delta_h)$$

As a consequence, for the vector $\dot{\mathrm{p}}(t)$ tangent to a smooth curve $\gamma : t \in I \to \gamma(t) \in Q$ at $\mathrm{p}(t) = \gamma(t)$, we have – in any chart $\varphi = \xi^{-1} \in \mathcal{A}_{\gamma(t)}$ (where $q(t) = \varphi(\gamma(t))$) –

$$\begin{aligned} \dot{\mathrm{p}}(t) = d_{q(t)}\xi\,(\dot{q}(t)) \xmapsto{\iota_{\mathrm{p}}} \dot{\gamma}(t) \;&:=\; \varphi'_{\gamma(t)}(\dot{q}(t)) \\ &=\; \varphi'_{\gamma(t)}(\dot{q}^h(t)\;\delta_h) \\ &=\; \dot{q}^h(t)\;\varphi'_{\gamma(t)}(\delta_h) \\ &=\; \dot{q}^h(t)\;\left.\frac{\partial}{\partial q^h}\right|_{\gamma(t)} \end{aligned}$$

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[20] For any two charts $\varphi = \xi^{-1}$ and $\varphi' = \xi'^{-1}$ in $\mathcal{A}_{\mathrm{p}}$ – where $\mathrm{p} = \xi(q) = \xi'(q')$ – and any tangent vector $\mathbf{v} = d_q\xi\,(v) = d_{q'}\xi'\,(v') \in T_{\mathrm{p}}Q$, we have

$$d_{q'}\xi'\,(v') = d_q\xi\,(v) = d_q(\xi' \circ \varphi' \circ \varphi^{-1})\,(v) = d_{q'}\xi'\left(d_q(\varphi' \circ \varphi^{-1})\,(v)\right)$$

whence, owing to the injectivity of $d_{q'}\xi'$,

$$v' = d_q(\varphi' \circ \varphi^{-1})\,(v)$$

[21] $[(\varphi, v)]_{\sim_{\mathrm{p}}}$ will denote the complete equivalence class of $(\varphi, v) \in \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n$ under $\sim_{\mathrm{p}}$.

[22] $(\delta_h)_{h=1,\ldots,n}$ will denote the canonical basis of $\mathbb{R}^n$.

Intrinsic geometry of smooth manifolds

On a non-empty set M, a structure of m-dimensional *smooth manifold* will then be defined as a complete, C^∞, m-dimensional atlas $\mathcal{A}$.

From the topological point of view, the manifold topology of M – i.e. the topology determined by the domains of the charts belonging to $\mathcal{A}$ – is locally Euclidean.

From the differential point of view, the smoothness of M – i.e. the differentiability of the transition functions of $\mathcal{A}$ – determines, at each $\pi \in M$, an m-dimensional *tangent vector space* given by

$$T_\pi M := \mathcal{A}_\pi \times \mathbb{R}^m / \sim_\pi$$

(where the equivalence relation $\sim_\pi$ is defined as in $(\star)$).

Each chart φ (or coordinate system $x = (x^i)_{i=1,\ldots,m}$) belonging to $\mathcal{A}_\pi$ determines an isomorphism $\varphi'_\pi : \mathrm{R}^m \to T_\pi M$ (defined as in $(\circ)$) and then a basis in $T_\pi M$ given by

$$\left.\frac{\partial}{\partial x^i}\right|_\pi := \varphi'_\pi(\delta_i)$$

Moreover, an invariant vector $\dot\kappa(t)$ tangent to a smooth curve $\kappa : t \in I \mapsto \kappa(t) \in M$ at $\kappa(t)$ is defined by putting – in any chart $\varphi \in \mathcal{A}_{\kappa(t)}$ (where the coordinate expression $x(t) = \varphi(\kappa(t))$ is meant to be C^∞)

$$\begin{aligned}\dot\kappa(t) &:= \varphi'_{\kappa(t)}(\dot x(t)) \\ &= \varphi'_{\kappa(t)}(\dot x^i(t)\ \delta_i) \\ &= \dot x^i(t)\ \varphi'_{\kappa(t)}(\delta_i) \\ &= \dot x^i(t)\ \left.\frac{\partial}{\partial x^i}\right|_{\kappa(t)}\end{aligned}$$

From the above definitions, all of the further developments of the theory (tangent bundles and differential equations, cotangent bundle and differential forms) follow in quite a natural way.

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