

Eigenvalue Problems with Turning Points

A. Neamaty¹, A. Dabbaghian² and Y. Khalili

¹ Department of Mathematics
University of Mazandaran, Babolsar, Iran
neamaty@math.com

² Islamic Azad University Neka Branch, Neka, Iran
a.dabbaghian@umz.ac.ir

Abstract

We consider eigenvalue problems for second-order differential equation on a finite interval having a turning point. As a basic result we derive asymptotic estimates for a special fundamental system of solutions of the corresponding differential equation and determine the asymptotic distribution of the eigenvalues.

Keywords: Turning point; Asymptotic form; Sturm-Liouville

1 Introduction

This paper deals with the boundary value problem L for the differential equation

$$y'' + (u^2 r(x) + iuq(x))y = 0, \quad 0 \leq x \leq 1, \quad (i)$$

with nonlinear dependence on the spectral parameter u and with the boundary conditions

$$u(y) := y'(0) - i\beta uy(0) = 0, \quad v(y) := y'(1) + i\beta uy(1) = 0. \quad (ii)$$

Here β is complex number. Let $a, \omega > 0$ and

$$r(x) = \begin{cases} -\omega^2 & 0 \leq x < a, \\ 1 & a \leq x \leq 1, \end{cases} \quad (iii)$$

i.e., the weight-function $r(x)$ changes the sign in an interior point, which is called the turning point. The function $q(x)$ is complex-valued and absolutely continuous. Differential equations with nonlinear dependence on the spectral parameter and with turning points arise in various problems of mathematics as well as in applications (see [4-8] for details). In section 2 we determine the asymptotic dependence of the solutions. Using these asymptotic estimates we derive a formula for the asymptotic distribution of the eigenvalues.

2 Properties of the spectral characteristics

we consider

$$y'' + f(x, \rho)y = 0 \quad (1)$$

where $f(x, \rho)$ has an asymptotic expansion of the form

$$f(x, \rho) \sim \rho^2 \phi_0(x) + \rho \phi_1(x) + \phi_2(x) + \dots = \sum_{n=0}^{\infty} \rho^{2-n} \phi_n(x) \quad , \quad \rho \longrightarrow \infty \quad (2)$$

and the $\phi_n(x)$, $n=0,1,\dots$, are continuous twice-differentiable functions of x . For the moment we shall assume $\phi_0 \neq 0$ for any x in the range in equation. We consider y be the form

$$y(x, \rho) \sim \exp\left\{\sum_{n=0}^{\infty} g_n(\rho)\psi_n(x)\right\}, \quad \rho \longrightarrow \infty \quad (3)$$

where $\psi(x)$, $n=0,1,\dots$, and the sequence $\{g_n(\rho)\}$, $n=0,1,\dots$, which is an asymptotic one as $\rho \longrightarrow \infty$, have to be found. It is shown in [3] that we finally get the asymptotic solutions to (1), with f as in (2), as

$$y(x, \rho) \sim \frac{1}{\phi_0^{1/4}(x)} \exp\left(\pm i \int^x \left\{\rho \phi_0^{1/2}(t) + \frac{1}{2} \frac{\phi_1(t)}{\phi_0^{1/2}(x)}\right\} dt\right) [1], \quad (4)$$

where $[1] = 1 + O(\rho^{-1})$. Now, let $iu = \rho$. So

$$y'' + (-\rho^2 r(x) + \rho q(x))y = 0. \quad (5)$$

$$u(y) := y'(0) - \beta \rho y(0) = 0, \quad v(y) := y'(1) + \beta \rho y(1) = 0. \quad (6)$$

Denote

$$\phi_0(x) := -r(x) = \begin{cases} \omega^2 & 0 \leq x < a, \\ -1 & a \leq x \leq 1, \end{cases}$$

$$\phi_1(x) := q(x) .$$

By using (4), we may write the general solution $y(x, \rho)$, in the form

$$y_k(x, \rho) = \begin{cases} \frac{1}{\sqrt{\omega}} \exp((-1)^k i(\rho\omega x + \int_0^x \frac{q(t)}{2\omega} dt))[1], & 0 \leq x < a, \\ \exp((-1)^k(\rho(x - a(1 - \omega)) + \int_0^a \frac{q(t)}{2\omega} dt + \int_a^x \frac{q(t)}{2} dt))[1], & a < x \leq 1. \end{cases} \tag{7}$$

$$y'_k(x, \rho) = \begin{cases} \frac{1}{\sqrt{\omega}}((-1)^k(i\rho\omega + \frac{i}{2} \frac{q(x)}{\omega})) \exp((-1)^k i(\rho\omega x + \int_0^x \frac{q(t)}{2\omega} dt))[1], & 0 \leq x < a, \\ (-1)^k(\rho + \frac{q(x)}{2}) \exp((-1)^k(\rho(x - a(1 - \omega)) + \int_0^a \frac{q(t)}{2\omega} dt + \int_a^x \frac{q(t)}{2} dt))[1], & a < x \leq 1. \end{cases} \tag{8}$$

where $k=1,2$.

Let $\varphi(x, \rho)$ and $\psi(x, \rho)$ be solutions of (5) under the initial conditions $\varphi(0, \rho) = 1$, $\varphi'(0, \rho) = \beta\rho$, $\psi(1, \rho) = 1$ and $\psi'(1, \rho) = -\beta\rho$. Then $U(\varphi) = V(\psi) = 0$. Applying the FSS $\{y_1(x, \rho), y_2(x, \rho)\}$ we have

$$\varphi(x, \rho) = c_1 y_1(x, \rho) + c_2 y_2(x, \rho) \tag{9}$$

that using of Cramer's rule (for example, see [2])we obtain

$$c_1 = \frac{\sqrt{\omega}}{2}(1 + \frac{\beta}{\omega}i)[1], c_2 = \frac{\sqrt{\omega}}{2}(1 - \frac{\beta}{\omega}i)[1]. \tag{10}$$

Now, by using of (7), (9) and (10) we have

$$\varphi(x, \rho) = \begin{cases} \frac{1}{2}(1 + \frac{\beta}{\omega}i) \exp(-i(\rho\omega x + \int_0^x \frac{q(t)}{2\omega} dt))[1] \\ + \frac{1}{2}(1 - \frac{\beta}{\omega}i) \exp(i(\rho\omega x + \int_0^x \frac{q(t)}{2\omega} dt))[1], & 0 \leq x < a, \\ \frac{\sqrt{\omega}}{2}(1 + \frac{\beta}{\omega}i) \exp(-(\rho(x - a(1 - \omega)) + \int_0^a \frac{q(t)}{2\omega} dt + \int_a^x \frac{q(t)}{2} dt))[1] \\ + \frac{\sqrt{\omega}}{2}(1 - \frac{\beta}{\omega}i) \exp(\rho(x - a(1 - \omega)) + \int_0^a \frac{q(t)}{2\omega} dt + \int_a^x \frac{q(t)}{2} dt))[1], & a < x \leq 1, \end{cases} \tag{11}$$

$$\varphi'(x, \rho) = \begin{cases} \frac{1}{2}(1 + \frac{\beta}{\omega}i)(-i(\rho\omega + \frac{1}{2} \frac{q(x)}{\omega})) \exp(-i(\rho\omega x + \int_0^x \frac{q(t)}{2\omega} dt))[1] \\ + \frac{1}{2}(1 - \frac{\beta}{\omega}i)(i(\rho\omega + \frac{1}{2} \frac{q(x)}{\omega})) \exp(i(\rho\omega x + \int_0^x \frac{q(t)}{2\omega} dt))[1] & 0 \leq x < a, \\ \frac{\sqrt{\omega}}{2}(1 + \frac{\beta}{\omega}i)(-\rho + \frac{q(x)}{2}) \exp(-(\rho(x - a(1 - \omega)) + \int_0^a \frac{q(t)}{2\omega} dt + \int_a^x \frac{q(t)}{2} dt))[1] \\ + \frac{\sqrt{\omega}}{2}(1 - \frac{\beta}{\omega}i)(\rho + \frac{q(x)}{2}) \exp(\rho(x - a(1 - \omega)) + \int_0^a \frac{q(t)}{2\omega} dt + \int_a^x \frac{q(t)}{2} dt))[1] & a < x \leq 1. \end{cases} \tag{12}$$

Denote

$$\Delta(\rho) = \langle \psi(x, \rho) \varphi(x, \rho) \rangle. \quad (13)$$

By virtue of Liouville for the Wronskian [3], $\langle \psi(x, \rho), \varphi(x, \rho) \rangle$ does not depend on x . Moreover, the function $\Delta(\rho)$ is entire in ρ , and it has at most a countable set of zeros $\{\rho_k\}$. The function $\Delta(\rho)$ is called the characteristic function of L . Clearly,

$$\Delta(\rho) = V(\varphi) = -U(\psi). \quad (14)$$

Theorem. The boundary value problem L has a countable set of eigenvalues $\{\rho_k\}_{k>0}$. For $k \rightarrow \infty$

$$\rho_k = \frac{1}{(1 - a(1 - \omega))} \left[k\pi i + \frac{1}{2} \ln \frac{\omega + \beta i}{\omega - \beta i} + \frac{1}{2} \ln \frac{1 - \beta}{1 + \beta} + \kappa_1 + \kappa_2 + O\left(\frac{1}{k}\right) \right], \quad (15)$$

$$\kappa_1 = -\frac{1}{2} \int_0^a \frac{q(t)}{2\omega} dt \quad \kappa_2 = -\frac{1}{2} \int_a^1 \frac{q(t)}{2} dt.$$

proof. Substituting (10-11) into (13) we calculate

$$\begin{aligned} \Delta(\rho) = & \rho \left(\frac{\sqrt{\omega}}{2} \left(1 - \frac{\beta}{\omega} i \right) (\beta + 1) \exp(\rho(1 - a(1 - \omega))) + \int_0^a \frac{q(t)}{2\omega} dt + \int_a^1 \frac{q(t)}{2} dt \right) [1] \\ & + \frac{\sqrt{\omega}}{2} \left(1 + \frac{\beta}{\omega} i \right) (\beta - 1) \exp(-(\rho(1 - a(1 - \omega))) + \int_0^a \frac{q(t)}{2\omega} dt + \int_a^1 \frac{q(t)}{2} dt) [1] \end{aligned}$$

Hence, the equation $\Delta(\rho) = 0$ has a countable set of roots ρ_k (14).

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