

# On $C^h$ -Recurrent and $C^v$ -Recurrent Finsler Spaces of Second Order

C. K. Mishra and Gautam Lodhi

Dept. of Maths. and Stats.  
Dr. R. M. L. Avadh University  
Faizabad-224001 (U.P.), India  
chayankumarmishra@yahoo.com  
lodhi\_gautam@rediffmail.com

## Abstract

The concept of  $C^h$ -recurrent Finsler space have been studied by Makoto Matsumoto[1]. The purpose of present paper is to study the properties of  $C^h$ -recurrent ( $C^v$ -recurrent) torsion tensor field of second order in the Finsler spaces.

**Mathematics Subject Classification:** 53B41

**Keywords:** Finsler spaces,  $C^h$ -recurrent Finsler spaces,  $C^v$ -recurrent Finsler spaces

## 1 INTRODUCTION

Let us considered an n-dimensional Finsler space of class atleast  $C^7$  equipped with a metric function  $F^*$  satisfying the required conditions, the corresponding symmetric tensor  $g$  and the cartan's connection  $\Gamma$ . The relation between the metric function  $F$  and the components  $g_{ij}$  of the corresponding metric tensor  $g$  are given by

$$\begin{cases} a) & g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2, \\ b) & g_{ij} \dot{x}^i \dot{x}^j = F. \end{cases} \quad (1)$$

$$\begin{cases} a) & g_{ij} = \frac{1}{2} \dot{\partial}_i y_j, \\ b) & y_i \dot{x}^i = F^2. \end{cases} \quad (2)$$

The tensor  $C_{ijk}$ , defined as  $C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$ , is symmetric in its all indices and satisfies

$$C_{kij} \dot{x}^k = C_{ikj} \dot{x}^k = C_{ijk} \dot{x}^k = 0 \quad (3)$$

due to (3) and symmetry of  $C_{ijk}$  in its indices the tensor  $C_{jk}^i$  which is defined as  $C_{jk}^i = g^{ip} C_{jpk}$ , satisfies

$$\begin{cases} a) & C_{jk}^i = C_{kj}^i, \\ b) & C_{jk}^i \dot{x}^k = C_{kj}^i \dot{x}^k = 0. \end{cases} \quad (4)$$

The commutation formulae for a tensor  $T_j^i$  will show the role of curvature tensor and torsion tensor as follows

$$T_{j^i k l}^i - T_{j^i l k}^i = T_j^h R_{hkl}^i - T_h^i R_{jkl}^h - T_{j|h}^i R_{kl}^h, \quad (5)$$

$$T_{j|k|l}^i - T_{j^i |l|k} = T_j^h S_{hkl}^i - T_h^i S_{jkl}^i. \quad (6)$$

Where  $R_{hkl}^i$  is called Cartan third curvature and  $S_{hkl}^i$  is called Cartan first curvature tensor.

**Definition.** A Finsler space  $F^n$  is called  $C^h$ -recurrent, if torsion tensor  $C_{jk}^i$  satisfies the equation

$$C_{jkl}^i = k_l C_{jk}^i. \quad (7)$$

**Definition.** If  $F^n$  is  $C^h$ -recurrent then, the v-curvature  $S_{jkh}^i$  is also recurrent with respect to h-covariant differentiation, that is

$$S_{jkhil}^i = 2k_l S_{jkh}^i, \quad (8)$$

we called, the recurrent v-curvature  $S_{jkh}^i$  with respect to h-covariant differentiation is  $S^h$ -recurrent. Where  $_{|l}$  denote the h-covariant differentiation and  $k_l = k_l(x, \dot{x})$  is covariant vector field.

**Definition.** A Finsler space  $F^n$  is called  $C^v$ -recurrent, if the torsion tensor  $C_{jk}^i$  satisfies the equation,

$$C_{jk|l}^i = a_l C_{jk}^i. \tag{9}$$

**Definition.** If  $F^n$  is  $C^v$ -recurrent then, the v-curvature  $S_{jkh}^i$  is also recurrent with respect to v-covariant differentiation, That is

$$S_{jkh|l}^i = 2a_l S_{jkh}^i, \tag{10}$$

we called, the recurrent v-curvature  $S_{jkh}^i$  with respect to v-covariant differentiation is  $S^v$ -recurrent. Where  $|_l$  denote the v-covariant differentiation and  $a_l = a_l(x, \dot{x})$  is covariant vector field.

The  $(v)hv$ - torsion tensor is given by

$$P_{jkh}^i y^j = P_{kh}^i = C_{khr}^i y^r, \tag{11}$$

where  $P_{jkh}^i$  are component of  $hv$ -curvature tensor.

## 2 $C^h$ -recurrent torsion tensor field of second order

**Definition.** A n-dimentional Finsler space  $F^n$ , in which the torsion tensor field satisfies the relation

$$C_{jkl|m}^i = v_{lm} C_{jk}^i, \tag{12}$$

where

$$C_{jk}^i \neq 0, \tag{13}$$

is said to be a  $C^h$ -recurrent Finsler space of second order and  $v_{lm}$  is recurrence tensor field. The torsion tensor field of this space is defined as the recurrent torsion tensor field of second order.

where  $v_{lm} = (k_{ilm} + k_l k_m)$  is a recurrence tensor field.

**Definition.** A n-dimentional Finslar space  $F^n$ , in which the v-curvature tensor field satisfies the relation.

$$S_{jkh|l|m}^i = 2w_{lm} S_{jkh}^i, \tag{14}$$

where

$$S_{jkh}^i \neq 0, \quad (15)$$

is said to be a  $S^h$ -recurrent Finsler space of second order and  $w_{lm}$  is recurrence tensor field. The v-curvature tensor field of this space is defined as the recurrent v-curvature tensor field of second order.

where  $w_{lm} = (k_{ilm} + 2k_l k_m)$  is a recurrence tensor field.

Let us considered a  $C^h$ -recurrent Finsler space characterised by the condition

$$C_{jkilm}^i = v_{lm} C_{jk}^i, \quad (16)$$

Thus a  $C^h$ -recurrent Finsler space satisfies (12). Therefore the space is called  $C^h$ -recurrent Finsler space of second order. Since the metric tensor  $g_{ij}$  is h-covariant constant, then the equation (12) written as

$$C_{ijkilm} = v_{lm} C_{ijk}. \quad (17)$$

Conversely, If we assume the above equation (17) is the charectrising equation of (12), it does not imply the equation (7) in general, therefore the equation of (12) is more general then equation (7). In this case the recurrent tensor need not be of form field  $k_{ilm} + k_l k_m$ .

Transvecting equation (12) by  $y^l$  in the view of (11), gives

$$P_{jkim}^i = v_{lm} y^l C_{jk}^i. \quad (18)$$

Let us considered the space  $C^h$ -recurrent Finsler space of second order is  $*P$ -Finsler space for such space we have the equation

$$P_{jk}^i = \lambda C_{jk}^i, \quad (19)$$

from equation (18) and (19), we get

$$C_{jkim}^i = \left( \frac{v_{lm} y^l - \lambda_{lm}}{\lambda} \right) C_{jk}^i, \quad (20)$$

which shows that the space is  $C^h$ -recurrent provided  $v_{lm} y^l - \lambda_{lm} \neq 0$ .

Thus, we have:

**Theorem 1** *A  $C^h$ -recurrent Finsler space of second order is  $C^h$ -recurrent, if it is a  $*P$ -Finsler space and  $v_{lm} y^l - \lambda_{lm} \neq 0$ .*

**Theorem 2** *Every  $C^h$ -recurrent Finsler space for which the recurrence vector field  $k_l$  satisfies*

$$k_{ilm} + k_l k_m \neq 0, \tag{21}$$

*is a  $C^h$ -recurrent Finsler space of second order but the converse is not true in general.*

**Proof.** The covariant differentiation of (7), yields

$$C_{jkilm}^i = (k_{ilm} + k_l k_m) C_{jk}^i. \tag{22}$$

From (12) and (13), we have

$$v_{lm} = k_{ilm} + k_l k_m, \tag{23}$$

which proves the statement. ■

**Corollary 3** *Every  $S^h$ -recurrent Finsler space for which the recurrence vector field  $k_l$  satisfies*

$$k_{ilm} + 2k_l k_m \neq 0, \tag{24}$$

*is a  $S^h$ -recurrent Finsler space of second order but the converse is not true in general.*

**Proof.** It is obvious from (8), (14) and (15). ■

Commutating (12) with respect to indices  $l, m$ , we have

$$C_{ijklm} - C_{ijkml} = (v_{lm} - v_{ml}) C_{jk}^i. \tag{25}$$

From commutation formula (5), it gives

$$C_{jk}^r R_{rlm}^i - C_{rk}^i R_{jlm}^r - C_{jr}^i R_{klm}^r - C_{jk|r}^i R_{lm}^r = (v_{lm} - v_{ml}) C_{jk}^i, \tag{26}$$

Contracting the indices  $i$  and  $j$  in (26) and putting  $C_k$  for  $C_{ik}^i$ , we get

$$(v_{lm} - v_{ml}) C_k = -C_r R_{klm}^r - C_{k|r} R_{lm}^r. \tag{27}$$

Due to skew symmetry of  $R_{rklm}$  in its first two indices, we have

$$C_r R_{klm}^r C^k = R_{hklm} C^h C^k = 0, \tag{28}$$

where  $C^k = g^{ik}C_i$ .

Transvecting (27) by  $C^k$  and using (28), we have

$$(v_{lm} - v_{ml})C^k C_k = -C_{k|r}R_{lm}^r C^k, \quad (29)$$

above equation also written as

$$(v_{lm} - v_{ml})C^k C_k = -C_{k|r}R_{hlm}^r C^k y^h. \quad (30)$$

again, transvecting equation (30) by  $C_h$  and using (28), we have

$$(v_{lm} - v_{ml})C^k C_k C_h = 0. \quad (31)$$

This implies either

$$\begin{cases} a) & C^k C_k C_h = 0, \\ b) & (v_{lm} - v_{ml}) = 0. \end{cases} \quad (32)$$

The equation  $C^k C_k C_h = 0$  implies  $C_k = 0$ , then Deick's theorem[3] shows that  $F^n$  is essentially Riemannian. If  $(v_{lm} - v_{ml}) = 0$  implies that the recurrence tensor field  $v_{lm}$  is symmetric.

Hence we can state:

**Theorem 4** *A  $C^h$ -recurrent Finsler space of second order, is either Riemannian or its recurrence tensor field is symmetric.*

**Corollary 5** *A  $S^h$ -recurrent Finsler space of second order, is either Riemannian or its recurrence tensor field is symmetric.*

**Proof.** The proof is obvious from (14) and (5). ■

### 3 $C^v$ -recurrent torsion tensor field of second order

**Definition.** A n-dimensional Finsler space  $F^n$ , in which the torsion tensor field satisfies the relation

$$C_{jk|l|m}^i = b_{lm}C_{jk}^i, \quad (33)$$

where

$$C_{jk}^i \neq 0. \quad (34)$$

is said to be a  $C^v$ -recurrent Finsler space of second order and  $b_{lm}$  is recurrence tensor field. The torsion tensor field of this space is defined as the recurrent torsion tensor field of second order.

Where  $b_{lm} = (a_{|l}a_{|m} - a_l a_m)$  is recurrence tensor field.

**Definition.** A n-dimensional Finsler space  $F^n$ , in which the v-curvature tensor field satisfies the relation

$$S^i_{jkh|l|m} = 2(u_{lm})S^i_{jkh}, \tag{35}$$

where

$$S^i_{jkh} \neq 0. \tag{36}$$

is said to be a  $S^v$ -recurrent Finsler space of second order and  $u_{lm}$  is recurrence tensor field. The v-curvature tensor field of this space is defined as the recurrent v-curvature tensor field of second order.

Where  $u_{lm} = (a_{|l}a_{|m} - 2a_l a_m)$  is recurrence tensor field.

Commutating (33) with respect to indices  $l$  and  $m$ , we have

$$C^i_{jk|l|m} - C^i_{jk|m|l} = (b_{lm} - b_{ml})C^i_{jk}. \tag{37}$$

From commutational formula (6), it gives

$$C^r_{jk}S^i_{rlm} - C^i_{rk}S^r_{jlm} - C^i_{jr}S^r_{klm} = (b_{lm} - b_{ml})C^i_{jk}. \tag{38}$$

Transvecting above equation (38) by  $y^r$ , we get

$$(b_{lm} - b_{ml})C^i_{jk}y^r = 0. \tag{39}$$

Since  $y^r$  is non zero, this implies either

$$\left\{ \begin{array}{l} a) \quad C^i_{jk} = 0, \\ b) \quad (b_{lm} - b_{ml}) = 0. \end{array} \right. \tag{40}$$

If  $C^i_{jk} = 0$  shows that the space is Riemannian, if the space is not Riemannian we have  $(b_{lm} - b_{ml}) = 0$ , implies that the recurrence tensor field is symmetric.

Hence we can state:

**Theorem 6** *A  $C^v$ -recurrent Finsler space of second order is either Riemannian or its recurrence tensor is symmetric.*

**Corollary 7** *A  $S^v$ -recurrent Finsler space of second order is either Riemannian or its recurrence tensor is symmetric.*

**Proof.** The proof is analogous to theorem (33). ■

**Theorem 8** *Every  $C^v$ -recurrent Finsler space for which the recurrence vector field  $a_m$  satisfies*

$$a_{l|m} - a_l a_m \neq 0, \quad (41)$$

*is  $C^v$ -recurrent Finsler space of second order but the converse is not true in general.*

**Proof.** The covariant differentiation of (9), yields

$$C_{jk|l|m}^i = (a_{l|m} - a_l a_m) C_{jk}^i. \quad (42)$$

From (33) and (34), we have

$$b_{lm} = (a_{l|m} - a_l a_m), \quad (43)$$

which proves the statement. ■

**Corollary 9** *Every  $S^v$ -recurrent Finsler space for which the recurrence vector field  $a_m$  satisfies*

$$a_{l|a_m} - 2a_l a_m \neq 0, \quad (44)$$

*is  $S^v$ -recurrent Finsler space of second order but the converse is not true in general.*

**Proof.** It is obvious from (35) and (36). ■

**Theorem 10** *A  $C^v$ -recurrent Finsler space of second order satisfies the relation,*

$$\begin{aligned} & (b_{lm} - b_{ml})_{|n} + (b_{mn} - b_{nm})_{|l} + (b_{nl} - b_{ln})_{|m} \\ &= 2a_n(b_{lm} - b_{ml}) + 2a_l(b_{mn} - b_{nm}) + 2a_m(b_{nl} - b_{ln}). \end{aligned} \quad (45)$$

**Proof.** The v-covariant differentiation of equation (38) with respect to  $n$  and using (38), we have

$$\{(b_{lm} - b_{ml})_{|n} - 2a_{|n}(b_{lm} + b_{ml})\} C_{jk}^i = 0. \quad (46)$$

If  $C_{jk}^i = 0$ , implies that  $F^n$  is essentially Riemannian. If the space is not Riemannian, we have

$$(b_{lm} - b_{ml})_{|n} = 2a_n(b_{lm} - b_{ml}). \tag{47}$$

Adding the expressions obtained by cyclic change of (47) with respect to indices  $l, m$  and  $n$ .

We have theorem (10). ■

**Theorem 11** *If the space is not Riemannian, then the recurrence tensor field  $b_{lm}$  of  $S^v$ -recurrent Finsler space of second order satisfies the relation,*

$$\begin{aligned} & 2a_{|n}(u_{lm} - u_{ml}) + 2a_{|l}(u_{mn} - u_{nm}) + 2a_{|m}(u_{nl} - u_{ln}) \\ &= (u_{lm} - u_{ml})_{|n} + (u_{mn} - u_{nm})_{|l} + (u_{nl} - u_{ln})_{|m}. \end{aligned} \tag{48}$$

**Proof.** Commutating (35) with respect to indices  $l, m$ , we have

$$S_{jkh|l|m}^i - S_{jkh|m|l}^i = 2(u_{lm} - u_{ml}) S_{jkh}^i. \tag{49}$$

From commutation formula (6), it gives

$$S_{jkh}^r S_{rlm}^i - S_{rkh}^i S_{jlm}^r - S_{jrh}^i S_{klm}^r - S_{jkr}^i S_{hlm}^r = 2(u_{lm} - u_{ml}) S_{jkh}^i. \tag{50}$$

The v-covariant differentiation of (50) with respect to  $n$ , we have

$$\{2a_{|n}(u_{lm} - u_{ml}) - (u_{lm} - u_{ml})_{|n}\} S_{jkh}^i = 0. \tag{51}$$

If  $S_{jkh}^i = 0$ , that is  $C_{jk}^i = 0$ , shows that  $F^n$  is essentially Riemannian. therefore assume that the space is not Riemannian, we have

$$2a_{|n}(u_{lm} - u_{ml}) = (u_{lm} - u_{ml})_{|n}. \tag{52}$$

Adding the expression obtained by cyclic change of (52) with respect to indices  $l, m$  and  $n$ .

We have theorem (11). ■

**Acknowledgment.** The authors would like to thank Prof. Dr. R. B. Misra for many useful suggestions and comments.

## References

- [1] M. Matsumoto, On h-isotropic and  $C^h$ -recurrent Finsler spaces, *J. Math. Kyoto Univ.*, 11 (1971) 1-9.
- [2] H. Rund, The differential geometry of Finsler spaces, *Springer verlag*, 1959.
- [3] A. Deicke, Über die Finsler Räume mit  $A_i=0$ , *Arch. Math.*, 4(1951) 45-51.

**Received: November 16, 2007**