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# Congruences for Odd Perfect Numbers Modulo Some Powers of 2

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**Abstract**

We present new congruences for odd perfect numbers involving convolution sums.

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## 1 Introduction

Our notation is classical (see e. g. [11]). For a positive integer $n$ we denote by $\sigma_k(n)$ the sum of all $k$-th powers of the positive divisors of $n$ and $\sigma(n)$ denotes $\sigma_1(n)$.
We say that $n$ is perfect if one has

$$2\,n = \sigma(n).$$

The main result of Ewell's paper [6] is the following. If $n$ is an odd perfect number then:

$$E_w = \sum_{k=1}^{(n-1)/2} \sigma(2k-1)\;\sigma(2n-(2k-1)) \equiv 1 \;(mod\;2). \tag{1}$$

The proof is intricate. Recently [7] we (in collaboration with rahavandrainy) discovered a simple proof of this result. It is a consequence of an easy counting argument and some formulae from Touchard [26] involving the weighted convolution sums

$$S_r(n) = \sum_{k=1}^{n-1} k^r \,\sigma(k)\; \sigma(n-k). \tag{2}$$

The result (1) was improved in [7] to the following congruence.

If $n$ is an odd perfect number, then:

$$E_w \equiv -2n + 1 \;(mod\; 32). \tag{3}$$

It turns out that this congruence is only one of several congruences modulo small powers of 2 that can be obtained immediately from some well known identities in which some more general convolution sums of divisors functions equals an explicit rational combination of the $\sigma_k(n)$ for small positive values of $k$ (indeed for odd $k \leq 13$). A typical example of such an identity, appear in the abstract of [12]. For all positive integers $n$ we have:

$$24 \sum_{k=1}^{n-1} \sigma(k)\; \sigma(3n-3k) \; = \sigma_3(3n) + (1-6n)\sigma(3n) + 9\sigma_3(n) + (1-18n)\sigma(n).$$

These identities are (or came from) essentially:

(a) Classical identities obtained at the end of the 19th century by Catalan, Liouville, Glaisher, Halphen, etc... See [3, 5, 8, 9].

(b) Classical identities obtained at begin and middle of the 20th century by (mainly) Ramanujan and Lehmer. See [25, 21, 22, 23].

(c) A systematic presentation of identities and congruences derived from them, done by Lahiri in 1946, 1947. He used himself some of them for later work. See [15, 16, 17, 18, 19, 20].

(d) Modular identities obtained recently by Melfi. See [24].

Recently, almost all of these identities, and some new ones, were proved by elementary means in [12, 14, 13, 4]. It is worth mentioning that the Bibliography of [12] is very interesting and comprehensive. Furthermore, some of Lahiri's identities were used recently (in [2]) to get congruences for coefficients of some quotients of classical Eisenstein series.

With these identities available, in order to obtain our congruences (see Lemma 2.2 that is our main result) it suffices to compute the values of the $\sigma_k(n) \pmod{64}$ for an odd perfect number $n$ and for all odd numbers $k$ such that $1 \leq k \leq 13$, since for any odd $k > 15$ we get the same congruence for $k$ than for $k \pmod{16}$.

For $k \geq 5$ the congruences are new, while the case $k = 3$ was done in [7].

Let $n > 0$ be an odd integer. Observe that it is not difficult to prove that:

$$\sigma(n) \equiv 2n \pmod 4 \Longleftrightarrow n \text{ has the form } n = p^{4k+1}r^2$$

where $p \equiv 1 \pmod 4$ is prime and $\gcd(p, r) = 1$.

So the main information available about the form of odd perfects numbers (if they exist) comes from the above (trivial) congruence modulo 4 involving the sum of divisors function $\sigma$.

An example of application of these congruences is the following:

$$B_{1,3} \equiv (n-1)/4 \ (mod\ 2), \tag{4}$$

where $n$ is an odd perfect number and the sum below runs over positive integers $a, k$ :

$$B_{1,3} = \sum_{4a< n,\ 4ak< n} a\,\sigma_3(n-4ak).$$

This result is obtained immediately from the 2 formulas in Theorem 2 of [14] by using our Lemma 2.2 of section 2, see also [10, p. 128].

The computation of these congruences shall be carried out in the proof of Lemma 2.2, while in our four Theorems (see Section 4) we shall collect the congruences obtained using that lemma. It will be also convenient to have available the value of $\tau(n) \pmod{64}$, where $\tau$ is Ramanujan's function.

## 2 Computing $\sigma_k(n) \pmod{64}$

Here we collect some known congruences:

**Lemma 2.1**
*Let $n \geq 1$ be a positive integer. Then*

*a)* $7\sigma_5(n) \equiv 10(3n-1)\sigma_3(n) - (24n^2 - 12n + 1)\sigma(n) \pmod{64}$.

*b)* $n\sigma_7(n) \equiv 7n\sigma_5(n) - (6n^2 - n)\sigma_3(n) - n\sigma(n) \pmod{64}$.

*c)* $11\sigma_9(n) \equiv 20(3n-2)\sigma_7(n) - 3(24n^2 - 28n + 7)\sigma_5(n) - 10\sigma_3(n) + 2(6n-1)\sigma(n) \pmod{64}$.

*d)* $\tau(n) \equiv 10(5n^4 - 7n^3)\sigma_3(n) - 7(12n^5 - 20n^4 + 5n^3)\sigma(n) \pmod{64}$.

*e)* $1800\tau(n) \equiv -273\sigma_{11}(n) + 691(6\sigma_7(n) - 3\sigma_3(n)) \pmod{64}$.

*f)* $\sigma_{13}(n) \equiv 11\sigma_9(n) + 22\sigma_7(n) - 32\sigma_3(n) \pmod{64}$.

**Proof**
For a) see Table C(1), equation (5.2) in page 202 of [15]. For b) see Table C(1), equation (7.8) in page 203 of [15]. For c) see Table C(1), equation (9.9) in page 205 of [15]. For d) see Table C(2), equation (3.6) in page 40 of [16]. For e) see Table C(2), equation (11.5) in page 46 of [16]. For f) see Table C(1), equation (13.9) in page 205 of [15].

Now, we present our main lemma:

**Lemma 2.2**
*Let $n$ be an odd perfect number. Then*

*a) For all odd $k \geq 1$ we have*

$$\sigma_k(n) \equiv (k+1)n - (k-1) \pmod{64}.$$

*b)* $\tau(n) \equiv 12n - 10 \equiv \sigma_{11}(n) \pmod{64}$.

**Proof**
The result is trivial for $k = 1$ and has already been proved in [7, Corollary 3.6 ] for $k = 3$.

Observe that trivially

$$n^3 \equiv 3n^2 - 3n + 1 \pmod{64}, \quad 16n^2 \equiv 32n - 16 \pmod{64} \qquad (5)$$

since $n \equiv 1 \pmod 4$.

Assume that $k = 5$. From Lemma 2.1 a) we get

$$\sigma_5(n) \equiv 48n^3 + 48n^2 + 22n + 12 \pmod{64}$$

that using (5) reduces to $192n^2 - 122n + 60 \pmod{64}$, i.e. to $6n - 4 \pmod{64}$.

Observe that we cannot get in the same manner the value of $\sigma_3(n) \pmod{64}$ since in Lahiri's list $C(1)$ in p. 202 of [15], the "better" congruence for $\sigma_3(n)$ is congruence (3.9):

$$(2n^3 - 3n^2)\sigma_3(n) \equiv (3n^4 - 5n^3 + n^2)\sigma(n) \pmod{2^5.3^2}.$$

This allows one only to get a congruence modulo 32. The full congruence modulo 64 requires the result in [7, Corollary 3.6 ] mentioned above.

The rest of the proof (i.e., the proof for odd $k$ with $7 \leq k \leq 13$) is similar.

Observe that in order to get $\sigma_{11}(n) \pmod{64}$ we must first of all get $\tau(n)$ $\pmod{64}$ from Lemma 2.1 (d). This gives $\tau(n) \equiv 12n - 10 \pmod{64}$, so that we get $\sigma_{11}(n) \equiv 12n - 10 \pmod{64}$ from Lemma 2.1 e).

It remains just to observe that the values of $\sigma_k(n) \pmod{64}$ for odd $k$ repeats cyclically when $k > 15$ since the order of an inversible element $d$ in $\mathbb{Z}/64\mathbb{Z}$ is in $\{1, 2, 4, 8, 16\}$ so that we get $d^{16} \equiv 1 \pmod{64}$ for all such $d$'s. For example

$$\sigma_{17}(n) \equiv \sum_{d|n} d^{17} \equiv \sum_{d|n} d \equiv \sigma_1(n) \equiv \sigma(n) \pmod{64}.$$

It remains only the special case $k = 15$. In this case it is easy to see that

$$\sigma_{15}(n) \equiv \sum_{d|n} d^{15} \equiv \sum_{d|n} \frac{1}{d} \equiv \frac{\sigma(n)}{n} \equiv 2 \equiv 16n - 14 \equiv \sigma_{-1}(n) \pmod{64}.$$

This finishes the proof of the lemma.

## 3 Some classical formulae and lemmata

The following lemma contains some classical formulae we shall use to get our first congruences:

**Lemma 3.1**

*Let* $n > 0$ *be a positive integer. Set* $S_0^*(n) = \sum_{k=1,\, k\ odd}^{n-1} \sigma(k)\ \sigma(n-k)$. *Then*

*a)* $12S_0(n) = 5\sigma_3(n) - (6n-1)\sigma(n)$.

*b)* $S_0^*(2\,n) = (1/8)\,(\sigma_3(2\,n) - \sigma_3(n))$.

**Proof**
Formula a) first appeared in [8] and appears also in [5, p. 300] ; it is formula (3.10) in [12] where the complete history of the formula is described.

Formula *b*) is from Glaisher, [5, p. 300] while the case $n$ odd is from Liouville [5, p. 287] and appears also as formula (4.6) in [12]. We already used it in [7].

We recall some simple facts:

**Lemma 3.2**

*a) If* $k \equiv 3 \pmod 4$ *then* $\sigma(k) \equiv 0 \pmod 4$.

*b) If* $n \equiv 1 \pmod 4$ *then* $2n^2 \equiv 4n - 2 \pmod{32}$.

*c) If* $n$ *is an odd perfect number then* 3 *does not divide* $n-2$. *So that* $3 \nmid n \iff n \equiv 1$ (*mod* 3).

*d) For all positive integers* $k$ *one has* $\sigma_k(n) \equiv \sigma(n) \pmod 2$.

We collect some formulas in Apostol's book [1]:

**Lemma 3.3**

*Let* $n > 0$ *be a positive integer. Then*

*a)* $\sigma_7(n) - \sigma_3(n) = 120\sum_{k=1}^{n-1}\sigma_3(k)\ \sigma_3(n-k)$.

*b)* $10\sigma_3(n) + 11\sigma_9(n) - 21\sigma_5(n) = 5040\sum_{k=1}^{n-1}\sigma_3(k)\ \sigma_5(n-k)$.

*c)* $756\tau(n) = 65\sigma_{11}(n) + 691\sigma_5(n) - 252\cdot 691\sum_{k=1}^{n-1}\sigma_5(k)\ \sigma_5(n-k)$.

*d)* $65520(\sigma_{11}(n)-\tau(n)) = 691(\tau(n+1)+24\tau(n)+\sum_{k=1}^{n-1} c(k)\tau(n-k))$,

*where in d) we denote by $c(k)$ the coefficient of $x^k$ in the fourier expansion of $j(\tau) = 12^3 J(\tau)$ where $J(\tau)$ is Klein's modular function, see [1, p. 15, p. 74].*

**Proof**

Formula a) first appeared in [9] and appears also in [1, p. 140, exercise 8] and in [15, formula (7.1) in p. 198] ; it is formula (3.17) in [12] where the complete history of the formula is described. Formula b) first appeared in [9] and appears also in [1, p. 140, exercise 9] and in [15, formula (9.1) in p. 199] ; it is formula (3.27) in [12] where the complete history of the formula is described. Formula c) appears in [1, p. 140, exercise 10]. Formula d) that appears in [1, p. 93, exercise 10] is due to Lehmer [21].

# 4 Congruences for odd perfect numbers

First of all we present congruences related to $S_0(2n)$ :

**Theorem 4.1**

*Let $n$ be an odd perfect number. Then we have*

*a)*

$$C_1 = \sum_{k=1}^{\frac{n-1}{2}} \sigma(k)\ \sigma(n-k) \equiv (-3)\ \frac{n-1}{4}\ (mod\ 8)$$

*so that*

$$C_{1a} = \sum_{k=1,\ k\equiv 0,1 \pmod 4}^{\frac{n-1}{2}} \sigma(k)\ \sigma(n-k) \equiv \frac{n-1}{4}\ (mod\ 4)$$

*b)*

$$C_2 = \sum_{k=1}^{\frac{n-1}{2}} \sigma(2k)\ \sigma(2n-2k) \equiv (-1)\ \frac{n-1}{4}\ (mod\ 8)$$

*c)*

$$C_3 = \sum_{k=1}^{\frac{n-1}{4}} \sigma(4k-3)\ \sigma(2n-(4k-3)) \equiv (-1)\ (2n-1)\ (mod\ 16)$$

**Proof**

From Lemma 2.2 we get $\sigma_3(n) \equiv 4n-2 \pmod{64}$ so that formula a) in Lemma 3.1 gives

$$12S_0(n) \equiv -12n^2 + 22n - 10 \pmod{64},$$

that using Lemma 3.2 b) reduces after multiplication by 11 (mod 32) to

$$2S_0(n) \equiv -2n^2 - 7n - 9 \equiv 11 - 11n \equiv 5n - 5 \pmod{32}.$$

Since $C_1 = S_0(n)/2$ we get the first result of a); the second result of a) follows from Lemma 3.2 a).

Observe that $C_2 = (S_0(2n) - S_0^*(2n))/2$. Working similarly to above we get

$$2S_0(2n) \equiv 7n + 29 \pmod{32}$$

and observing that $S_0^*(2n) = \sigma_3(n)$ by Lemma 3.1 b) we obtain using again $\sigma_3(n) \equiv 4n-2 \pmod{64}$ :

$$4C_2 \equiv 1 - n \pmod{32}$$

i.e. the result b).

Since $k \equiv 3 \pmod 4$ implies $2n - k \equiv 3 \pmod 4$ we have by Lemma 3.2 a):

$$E_w \equiv \sum_{k \equiv 1 \pmod 4}^{n-4} \sigma(k)\,\sigma(2n-k) \equiv C_3 \pmod{16}$$

so that we get

$$C_3 \equiv E_w \equiv 1 - 2n \pmod{16}.$$

from (3) thereby finishing the proof of the theorem.

Secondly, we present congruences derived from some formulas in Apostol's book [1]:

**Theorem 4.2**

*Let n be an odd perfect number. Then we have* *a)*

$$C_4 = \sum_{k=1}^{\frac{n-1}{2}} \sigma_3(k)\ \sigma_3(n-k) \equiv\ (-1)\frac{n-1}{4}\ (mod\ 4)$$

*b)*

$$C_5 = \sum_{k=1}^{n-1} \sigma_3(k)\ \sigma_5(n-k) \equiv\ \frac{n-1}{2}\ (mod\ 4)$$

*c)*

$$C_6 = \sum_{k=1}^{\frac{n-1}{2}} \sigma_5(k)\ \sigma_5(n-k) \equiv 3\ \frac{n-1}{4}\ (mod\ 8)$$

*d)*

$$C_7 = \tau(n+1) + \sum_{k=1}^{\frac{n-1}{2}} c(2k-1)\ \tau(n-(2k-1)) \equiv\ 224n + 240\ (mod\ 512)$$

**Proof**

From Lemma 2.2 a) applied to formula a) of Lemma 3.3 we get congruence a). From Lemma 2.2 a) applied to formula b) of Lemma 3.3 we get congruence b). From Lemma 2.2 applied to formula c) of Lemma 3.3 we get congruence c). Observe that $c(even) \equiv 0 \pmod{2^{11}}$, see [1, Theorem 4.12, p. 90]. So that from Lemma 2.2 applied to formula d) of Lemma 3.3 we get congruence d).

Thirdly, we present congruences studied in Huard et al. see [12]; these include congruences deduced from Melfi's formulas, see [24]. The proof of the following theorem is similar to the proof of the preceding two theorems, i.e., it is based on applying Lemma 2.2 to some appropiate formula together with some simple computations. Therefore, we shall just indicate from which formula in [12] we deduced each congruence in next theorem.

**Theorem 4.3**

*Let n be an odd perfect number. Then we have*

1) $C_8 = S_1(n) = \sum_{k=1}^{n-1} k\,\sigma(k)\ \sigma(n-k) \equiv\ \frac{n-1}{4}\ (mod\ 8)$

2) $C_9 = \sum_{k=1}^{n-1} \sigma(k)\ \sigma_3(n-k) \equiv\ 0\ (mod\ 4)$

3) $C_{10} = \sum_{k=1}^{n-1} k\ \sigma(k)\ \sigma_3(n-k) \equiv (-1)\ \frac{n-1}{4}\ (mod\ 4)$

*4)* $C_{11} = \sum_{k=1}^{n-1} k\ \sigma_3(k)\ \sigma(n-k) \equiv\ \dfrac{n-1}{4}\ (mod\ 8)$

*5)* $C_{12} = \sum_{k=1}^{n-1} k(n-k)\ \sigma(k)\ \sigma(n-k) \equiv\ 2n-2\ (mod\ 16)$

*6)* $C_{13} = S_2(n) = \sum_{k=1}^{n-1} k^2\ \sigma(k)\ \sigma(n-k) \equiv\ (-3)\dfrac{n-1}{4}\ (mod\ 8)$

*7)* $C_{14} = 2C_4 = \sum_{k=1}^{n-1}\ \sigma_3(k)\ \sigma_3(n-k) \equiv\ (-1)\dfrac{n-1}{2}\ (mod\ 8)$

*8)* $C_{15} = \sum_{k=1}^{n-1} \sigma(k)\ \sigma_5(n-k) \equiv\ 0\ (mod\ 8)$

*9)* $C_{16} = \sum_{k=1}^{n-1} k\ \sigma_3(k)\ \sigma_3(n-k) \equiv (-1)\ \dfrac{n-1}{4}\ (mod\ 4)$

*10)* $C_{17} = \sum_{k=1}^{n-1} k(n-k)^2\ \sigma(k)\ \sigma(n-k) \equiv\ n-1\ (mod\ 8)$

*11)* $C_{18} = S_3(n) = \sum_{k=1}^{n-1} k^3\ \sigma(k)\ \sigma(n-k) \equiv\ (-3)\dfrac{n-1}{4}\ (mod\ 8)$

*12)* $C_{19} = \sum_{k=1}^{n-1} k^2\ \sigma_3(k)\ \sigma(n-k) \equiv\ (-3)\dfrac{n-1}{4}\ (mod\ 8)$

*13)* $C_{20} = \sum_{k=1}^{n-1} k^2\ \sigma(k)\ \sigma_3(n-k) \equiv\ (-1)\dfrac{n-1}{4}\ (mod\ 4)$

*14)* $C_{21} = \sum_{k=1}^{n-1} k\ \sigma(k)\ \sigma_5(n-k) \equiv\ (-1)\dfrac{n-1}{4}\ (mod\ 8)$

*15)* $C_{22} = \sum_{k=1}^{n-1} k\ \sigma_5(k)\ \sigma(n-k) \equiv\ \dfrac{n-1}{4}\ (mod\ 8)$

*16)* $C_{23} = \sum_{k=1}^{n-1} k(n-k)\ \sigma(k)\ \sigma_3(n-k) \equiv\ 0\ (mod\ 16)$

*17)* $C_{24} = C_5 = \sum_{k=1}^{n-1} \sigma_3(k)\ \sigma_5(n-k) \equiv\ \dfrac{n-1}{2}\ (mod\ 4)$

*18)* $C_{25} = \sum_{k=1}^{n-1} \sigma(k)\ \sigma_7(n-k) \equiv\ 0\ (mod\ 2) \equiv 2C_1\ (mod\ 2)$

*19)* $C_{26} = \sum_{k=1}^{n-1} \sigma_5(k)\ \sigma_7(n-k) \equiv\ 0\ (mod\ 2) \equiv 2C_1\ (mod\ 2)$

*20)* $C_{27} = \sum_{k=1}^{n-1} \sigma_3(k)\ \sigma_9(n-k) \equiv\ \dfrac{n-1}{2}\ (mod\ 4)$

*21)* $C_{28} = \sum_{k=1}^{n-1} \sigma(k)\ \sigma_{11}(n-k) \equiv\ 0\ (mod\ 4)$

*22)* $C_{29} = \sum_{k=1}^{n-1} \sigma(k)\ \sigma(2n-2k) \equiv\ n-1\ (mod\ 8)$

*23)* $n \equiv 1\ (mod\ 3) \Rightarrow C_{30} = \sum_{k=1}^{n-1} \sigma(k)\ \sigma(3n-3k) \equiv\ \dfrac{n-1}{4}\ (mod\ 8)$

*24)* $C_{31} = \sum_{k=1}^{n-1} \sigma(k)\ \sigma(4n-4k) \equiv\ 0\ (mod\ 4)$

*25)* $C_{32} = \sum_{k=1}^{2n-1} \sigma(2k)\ \sigma(4n-2k) \equiv\ (-1)\dfrac{n-5}{2}\ (mod\ 8)$

*26)* $C_{33} = \sum_{k=1}^{2n} \sigma(2k-1)\ \sigma(4n-(2k-1)) \equiv\ 0\ (mod\ 8)$

*27)* $C_{34} = \sum_{k=1}^{n} \sigma(2k-1)\ \sigma(2n-(2k-1)) = \sigma_3(n)\ \equiv\ 4n-2\ (mod\ 64)$

*28)* $C_{35} = \sum_{k=1}^{n-1} \sigma_3(k)\ \sigma(2n-2k) \equiv\ \dfrac{n-1}{2}\ (mod\ 4)$

*29)* $C_{36} = \sum_{k=1}^{n-1} \sigma(k)\ \sigma_3(2n-2k) \equiv\ 0\ (mod\ 4)$

*30)* $C_{37} = \sum_{k=1}^{\frac{n-3}{2}} \sigma(2k+1)\ \sigma_3(\frac{n-1}{2}-k) \equiv\ (-1)\dfrac{n-1}{4}\ (mod\ 4)$

*31)* $C_{38} = \sum_{k=0}^{n-1} \sigma(2k+1)\ \sigma(2n-(2k+1)) = \sigma_5(n)\ \equiv\ 6n-4\ (mod\ 64)$

*32)* $n \equiv 1\ (mod\ 3) \Rightarrow C_{39} = \sum_{k=1}^{n-1} \sigma(3k)\ \sigma(3n-3k) \equiv\ 2n-2\ (mod\ 16)$

*33)* $n \equiv 1\ (mod\ 3) \Rightarrow C_{40} = \sum_{1\le k<\frac{2n}{3}} \sigma(3k)\ \sigma(2n-3k) \equiv\ (-1)\dfrac{n-1}{4}\ (mod\ 8)$

*34)* $n \equiv 1\ (mod\ 3) \Rightarrow$

$$C_{41} = \sum_{k=0}^{\frac{2n-2}{3}} \sigma(3k+1)\ \sigma(2n-(3k+1)) = \sigma_3(n)\ \equiv\ 4n-2\ (mod\ 64)$$

*35)* $n \equiv 1\ (mod\ 3) \Rightarrow C_{42} = \sum_{k=0}^{n-1} \sigma(3k+1)\ \sigma(3n-(3k+1)) \equiv\ 12n-6\ (mod\ 64)$

*36)* $C_{43} = \sum_{k=0}^{n-1} \sigma(4k+1)\ \sigma(4n-(4k+1)) \equiv\ 0\ (mod\ 4)$

*37)*

$$C_{44} = \sum_{k=1}^{n-1} \sigma(4k)\ \sigma(4n-4k) + \sum_{k=0}^{n-1} \sigma(4k+2)\ \sigma(4n-(4k+2)) \equiv\ (-1)\frac{n-5}{2}\ (mod\ 8)$$

*38)*

$$C_{45} = \sum_{k=1}^{\frac{n-1}{4}} \sigma(4k)\ \sigma(n-4k) + \sum_{k=0}^{\frac{n-5}{4}} \sigma(4k+2)\ \sigma(n-(4k+2)) \equiv\ (-3)\frac{n-1}{4}\ (mod\ 8)$$

*39)*

$$C_{46} = \sum_{k=1}^{\frac{3n-3}{4}} \sigma(4k)\ \sigma(3n-4k) + \sum_{k=0}^{\frac{3n-3}{4}} \sigma(4k+1)\ \sigma(3n-(4k+1)) \equiv\ n-3\ (mod\ 8)$$

*40)* $n \equiv 1\ (mod\ 3) \Rightarrow$

$$C_{47} = 3(\sum_{k=1}^{n-1} \sigma(3n-3k)\ \sigma_3(k)) + \sum_{k=1}^{n-1} \sigma(k)\ \sigma_3(3n-3k) \equiv\ \frac{n-1}{2}\ (mod\ 4)$$

*41)*

$$C_{48} = \sum_{k=1}^{n-1} \sigma(k)\ \sigma_5(2n-2k) \equiv\ 0\ (mod\ 8)$$

*42)*

$$C_{49} = 2(\sum_{k=1}^{n-1} \sigma_3(k)\ \sigma_3(2n-2k)) + 3(\sum_{k=1}^{n-1} \sigma_5(k)\ \sigma(2n-2k)) \equiv\ (-1)\frac{n-1}{2}\ (mod\ 8)$$

**Proof**

Congruence 1) came from formula (3.11) in [12]. We abbreviate this fact by writing only [1), 3.11]. Similar for the rest:

[2), 3.12], [3), 3.13], [4), 3.14], [5), 3.15], [6), 3.16], [7), 3.17], [8), 3.18], [9), 3.19], [10), 3.20], [11), 3.21], [12), 3.22] , [13), 3.23], [14), 3.24], [15), 3.25], [16), 3.26], [17), 3.27], [18), 3.28], [19), 3.29], [20), 3.30], [21), 3.31], [22), 4.4].

The following references includes call to theorems:

[23), Theorem 3], [24), Theorem 4], [25), Theorem 5], [26), Corollary 1], [27), 4.6, see also Lemma 3.1 b)], [28), Theorem 6, formula 1], [29), Theorem 6, formula 2], [30), 4.7], [31), Corollary 3], [32), Theorem 7, formula 1], [33), Theorem 7, formula 2], [34), 5.5], [35), Theorem 8, formula 2], [36), Theorem 9, formula 1], [37),Theorem 9, formula 2], [38), Theorem 9, formula 5], [39), Theorem 9, formula 10], [40), Theorem 14], [41), Theorem 15, formula 1], [42), Theorem 15, formula 2].

Observe also (see Lemma 3.2 d)) that the congruences (mod 2) above may be written using only the $\sigma$ function.

Fourthly, we present congruences from recent papers of Williams, Huard, Cheng, see [14, 13, 4, 27].

**Theorem 4.4**
*Let $n$ be an odd perfect number. Then we have*

*a)*

$$C_{50} = \sum_{k<\frac{n}{4}} \sigma(k)\ \sigma_3(n-4k) \equiv\ \frac{n-1}{4}\ (mod\ 2)$$

*b)*

$$C_{51} = \sum_{k<\frac{n}{2}} \sigma(k)\ \sigma_5(2n-4k) - \sum_{k<n} \sigma_3(k)\ \sigma_3(2n-2k) \equiv\ \frac{n-1}{4}\ (mod\ 2)$$

*c)*

$$C_{52} = \sum_{k<n} \sigma_3(k)\ \sigma_3(2n-2k) \equiv\ 0\ (mod\ 2) \equiv C_{29}\ (mod\ 2)$$

*d)* $n \equiv 1\ (mod\ 3) \Rightarrow$

$$C_{53} = \sum_{k<\frac{2n}{9}} \sigma(k)\ \sigma(2n-9k) \equiv\ (-1)\frac{n-1}{4}\ (mod\ 8)$$

*e)* $n \equiv 1\ (mod\ 3) \Rightarrow$

$$C_{54} = \sum_{k<n} \sigma(k)\ \sigma(9n-9k) \equiv\ 3\frac{n-1}{2}\ (mod\ 16)$$

*f)*

$$C_{55} = \sum_{k<2n} \sigma(4k)\ \sigma(8n-4k) \equiv\ \frac{3n+1}{2}\ (mod\ 16)$$

*g)*

$$C_{56} = \sum_{k\leq 2n} \sigma(4k-2)\ \sigma(8n-(4k-2)) \equiv\ 16\ (mod\ 64)$$

*h)*

$$C_{57} = \sum_{k<\frac{n+1}{2}} \sigma(4k)\ \sigma(2n-4k) \equiv\ \frac{3n+5}{4}\ (mod\ 8)$$

*i)*

$$C_{58} = \sum_{k \leq 2n} \sigma(4k-3)\ \sigma(8n-(4k-3)) \equiv\ 0\ (mod\ 64)$$

*j)*

$$C_{59} = \sum_{k \leq 2n} \sigma_3(4k-2)\ \sigma(8n-(4k-2)) \equiv\ 0\ (mod\ 64)$$

**Proof**

Congruence a) came from formula (2.11) in [14]. We abbreviate this fact by writing only [a), 2.11, [14]]. Similar for the rest: [b), Theorem 8, formula 1, [13]], here note that there is a misprint in the formula: the term $\frac{1}{16}(-1)^n\sigma_3(n)$ is incorrectly written two times; [c), Theorem 8, formula 2, [13]]; [d), Theorem 1.10, formula 1, [27]]; [e), Theorem 1.10, formula 2, [27]]; [f), Theorem 3, formula 4.1, [4]]; [g), Theorem 3, formula 4.2, [4]]; [h), Theorem 3, formula 4.3, [4]]; [i), Theorem 3, formula 4.4, [4]]; and [j), Theorem 3, formula 4.5, [4]]. Observe also (see Lemma 3.2 d)) that the congruences (mod 2) above may be written using only the $\sigma$ function.

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