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Semi-Selfsimilar Processes on R and I_0

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Abstract

Introducing non-negative integer-valued semi-selfsimilar processes we compare and contrast them with their continuous counterpart and characterize Lévy processes in this class in terms of an $AR(1)$ scheme.

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1 Introduction

Semi-selfsimilar (SSS) processes are stochastic processes that are invariant in distribution under suitable scaling of time and space. That is; a process $\{X(t), t \geq 0\}$ is SSS if for some $a > 0$ there exists an $H > 0$ such that,

$$\{X(at)\} \stackrel{d}{=} \{a^H X(t)\}. \tag{1}$$

Here a is called the epoch and H the exponent of the SSS process and we write $\{X(t)\}$ is (a, H)-SSS. Theorem.4.1 in [3] states that a Lévy process $\{X(t)\}$ is SSS *iff* the distribution of $X(1)$ is semi-stable. A distribution on $\mathbf{R}$ is semi-stable(a, b) if its characteristic function (CF) $f(u)$ satisfies $f(u) \neq 0$ and $\{f(u)\}^a = f(bu)$, for all $u \in \mathbf{R}$, some $b \in (0, 1) \cup (1, \infty)$ and $b = a^{1/\alpha}$ for a unique $\alpha \in (0, 2]$. Semi-stable laws on $\mathbf{I}_0 = \{0, 1, 2, \dots\}$ were described with examples in [4] and [1]. In this note we extend the notion of semi-selfsimilarity to $\mathbf{I}_0$-valued processes and discuss some of its properties.

Of late there has been an increasing interest in describing and discussing the integer valued analogues of semi-stable laws and processes. See, eg. [1], [4] and [2]. Clearly, equation (1) does not (cannot) include the integer-valued case. In the next section we describe $\mathbf{I}_0$-valued SSS processes, compare and contrast it with its continuous counterpart and characterize $\mathbf{I}_0$-valued SSS Lévy processes in terms of an $\mathbf{I}_0$-valued $AR(1)$ series. This characterization is the $\mathbf{I}_0$-valued analogue of a result in [5]. Here the notions of semi-stability and semi-selfsimilarity are considered in the strict sense only.

2 Results

The description in (1) cannot hold good for an integer-valued process, as the state space of the process on the RHS will be different. Hence to formulate $\mathbf{I}_0$-valued SSS processes we use the idea of binomial thinning operation (in [6]) "$\otimes$" on an $\mathbf{I}_0$-valued *r.v.* X defined as: $b \otimes X = \sum_{i=1}^{X} Z_i$ for some *i.i.d* Bernoulli *r.v.*s $\{Z_i\}$ independent of X with $P\{Z_i = 1\} = b$. It may be noted that if $P(s)$ is the probability generating function (PGF) of X then that of $b \otimes X$ is $P(1 - b + bs)$.

Definition 2.1. *An $\mathbf{I}_0$-valued r.v X is semi-stable(a, b) if its PGF $P(s)$ satisfies $P(s) \neq 0$ and $\{P(s)\}^a = P(1 - b + bs)$, for all $s \in [0, 1]$, some $b \in (0, 1)$ and $b = a^{1/\alpha}$ for a unique $\alpha \in (0, 1]$.*

Definition 2.2. *An $\mathbf{I}_0$-valued process $\{X(t), t \geq 0\}$ is SSS if for some $a \in (0, 1)$ there exists an $H > 0$ such that*

$$\{X(at)\} \stackrel{d}{=} \{a^H \otimes X(t)\}.$$

Also, as in the continuous case we call a the epoch and H the exponent of the SSS process and we write $\{X(t)\}$ is (a, H)-SSS.

Notice that if $X(t)$ is an $\mathbf{I}_0$-valued Lévy process and $P(s)$ is the PGF of $X(1)$, then that of $X(t)$ is $P(s)^t$. Hence in terms of PGFs a Lévy process is SSS if for some $a \in (0, 1)$ and $H > 0$

$$\{P(s)\}^{at} = \{P(1 - a^H + a^H s)\}^t, \tag{2}$$

for all $s \in [0, 1]$. Now we give the $\mathbf{I}_0$-valued analogue of theorem.4.1 in [3].

Theorem 2.3. *An $\mathbf{I}_0$-valued Lévy process $\{X(t), t \geq 0\}$ is $(a, \frac{1}{\alpha})$-SSS iff the distribution of $X(1)$ is $\mathbf{I}_0$-valued semi-stable(a, b), $\alpha \in (0, 1]$.*

Proof. If $X(t)$ is $\mathbf{I}_0$-valued Lévy and $X(1)$ is $\mathbf{I}_0$-valued semi-stable(a, b), then for each $t \geq 0$ and for all $s \in [0, 1]$,

$$\{P(s)\}^{at} = \{P(1 - b + bs)\}^t, \text{ and } b = a^{1/\alpha}.$$

That is, $X(at) \stackrel{d}{=} a^{1/\alpha} \otimes X(t)$ and hence $\{X(t)\}$ is $\mathbf{I}_0$-valued $(a, \frac{1}{\alpha})$-SSS. Conversely, if a Lévy process $\{X(t)\}$ is $\mathbf{I}_0$-valued $(a, \frac{1}{\alpha})$-SSS then

$$X(at) \stackrel{d}{=} a^{1/\alpha} \otimes X(t)$$

and hence in terms of PGFs at $t = 1$,

$$\{P(s)\}^a = \{P(1 - a^{1/\alpha} + a^{1/\alpha} s)\}$$

implying $X(1)$ is $\mathbf{I}_0$-valued semi-stable$(a, a^{1/\alpha})$. □

Theorem 2.4. *A Lévy process $\{X(t), t \geq 0\}$ is $(\frac{1}{a}, \frac{1}{\alpha})$-SSS if the distribution of $X(1)$ is semi-stable(a, b), $\alpha \in (0, 1]$.*

Proof. The CF $f(u)$ of a semi-stable(a, b) law can also be equivalently written as $\{f(\frac{u}{b})\}^a = f(u)$. The Lévy process version of this, since $b = a^{1/\alpha}$, is:

$$\frac{1}{b} X(t) \stackrel{d}{=} X(\frac{t}{a}) \text{ or } X(\frac{t}{a}) \stackrel{d}{=} (\frac{1}{a})^{1/\alpha} X(t).$$

Hence, a semi-stable(a, b) law gives rise to a $(\frac{1}{a}, \frac{1}{\alpha})$-SSS Lévy process also. □

Remark 2.5. *Thus the SSS Lévy process corresponding to a semi-stable(a, b) law can have two epochs, a and $\frac{1}{a}$. However, in the $\mathbf{I}_0$-valued setup the epoch can only be a, since we need $a < 1$, $a^{1/\alpha}$ being a Bernoulli probability.*

We now characterize $\mathbf{I}_0$-valued SSS Lévy processes in terms of an $\mathbf{I}_0$-valued $AR(1)$ model with marginals $\{X_n\}$ described by a sequence of $\mathbf{I}_0$-valued *r.vs* $\{X_n, n \geq 0\}$, and an innovation sequence $\{\epsilon_n\}$ of *i.i.d r.vs* satisfying;

$$X_n = b \otimes X_{n-1} + \epsilon_n, \text{ for some } b \in (0, 1). \tag{3}$$

Theorem 2.6. *Let $\{Z(t), t \geq 0\}$ be an $\mathbf{I}_0$-valued Lévy process and in (3) let $X_0 \stackrel{d}{=} Z(1)$ and $\epsilon_n \stackrel{d}{=} b \otimes Z(b^{-\alpha} - 1)$, for all n, some $b \in (0, 1)$ and $\alpha \in (0, 1]$. Then (3) is stationary with $\mathbf{I}_0$-valued semi-stable(b^α, b) marginals if $\{Z(t)\}$ is $(b^\alpha, \frac{1}{\alpha})$-SSS. Conversely, the marginals of (3) are $\mathbf{I}_0$-valued semi-stable(b^α, b) and $\{Z(t)\}$ is $(b^\alpha, \frac{1}{\alpha})$-SSS if (3) is stationary.*

Proof. Let the PGF of $Z(1)$ is $P(s)$, then $\{P(1 - b + bs)\}^{b^{-\alpha} - 1}$ is the PGF of $b \otimes Z(b^{-\alpha} - 1)$. If $\{Z(t)\}$ is $(b^\alpha, \frac{1}{\alpha})$-SSS then,

$$Z(b^\alpha t) \stackrel{d}{=} b \otimes Z(t) \text{ or } \{P(s)\}^{b^\alpha} = P(1 - b + bs),$$

and $Z(1)$ is semi-stable(b^α, b). Now under the given assumptions at $n = 1$ in (3), the PGF of X_1 is

$$P_1(s) = P(1 - b + bs)\{P(1 - b + bs)\}^{b^{-\alpha} - 1} = \{P(1 - b + bs)\}^{b^{-\alpha}} = P(s),$$

which is same as that of X_0. Hence on iteration (3) is stationary with $\mathbf{I}_0$-valued semi-stable(b^α, b) marginals.

Conversely, let (3) is stationary and $P(s)$ is the PGF of X_0. Then at $n = 1$,

$$P(s) = P(1 - b + bs)\{P(1 - b + bs)\}^{b^{-\alpha}-1} = \{P(1 - b + bs)\}^{b^{-\alpha}}.$$

Hence the marginals and $Z(1)$ are semi-stable(b^α, b) and consequently $\{Z(t)\}$ is $(b^\alpha, \frac{1}{\alpha})$-SSS. □

Remark 2.7. *If definition.2.2 holds for any $a > 0$ then $\{X(t)\}$ is selfsimilar (SS) and in the $\mathbf{I}_0$-valued setup we have: (a) a Lévy process $\{X(t)\}$ is SS iff $X(1)$ is stable (b) (3) is stationary with stable marginals if $\{Z(t)\}$ is $\frac{1}{\alpha}$-SS and (c) $\{Z(t)\}$ is SS and the marginals are stable if (3) is stationary.*

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