International Mathematical Forum, 3, 2008, no. 30, 1457 - 1467

Sensitivity Analysis of CCR-Efficient Units in the Presence of the Indicator with Limited Source

G. R. Jahanshahloo[a], **F. Hosseinzadeh Lotfi**[a1], **Hadi Bagherzadeh Valami**[a]

[a]Dept. of Mathematics, Science & Research Campus
Islamic Azad University, Tehran, Iran

Abstract

Data envelopment analysis (DEA), is a technique to evaluate the ability of the decision making units (DMUs) by using the mathematical programming inspired by some input and output homogeneous indicators. One of the topics of interest in DEA is the sensitivity and the stability analysis of an efficient DMU, when the data variations of inputs and outputs are considered. Presence of the indicators with limited sources effects the sensitivity of the DMUs. Same indicators exist as fixed amount in a community and the DMUs can own them with their ability and if a DMU loses the same amount of the indicator, the rest of the DMUs find the ability to own some of them without even changing their capacity of other indicators. This paper develop a sensitivity analysis for the efficient DMUs, when there is variation in an input or output indicator with a limited source.

Mathematics Subject Classification: Operations Research, No. 90

Keywords: DEA, Cost efficiency, Triangular fuzzy number

1 Introduction

Since the pioneering work of Charnes et al. [1], data envelopment analysis (DEA) has been extensively used for evaluating the performance of many activities. DEA evaluates the relative efficiency of a set of homogeneous decision making units (DMUs) by using a ratio of the weighted sum of outputs to the weighted sum of inputs. It generalizes the usual efficiency measurement from a single-input, single-output ratio to a multiple-input, multiple-output ratio. Specifically, it determines a set of weights in a way that the efficiency of a target DMU (DMU_o) relative to the other DMUs is maximized.

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[1] Corresponding author, hosseinzadeh_lotfi@yahoo.com

In some of the systems, availability of some of the input or output indicators in a community may be limited, and the DMUs can own them with their ability. For example, in a community there is limited investment to invest in banks and banks by considering their ability to attract customers can own some of this investment as an output indicator. In another example, the amount of sick people in a community in need of hospitalization in a hospital are limited. However, hospitals by considering their quality of care and their healing expense can own some of these patients as output indicators. In examples above, if a DMU losses an amount of this output, the rest of the DMUs find the ability to own some of them without even increasing thier inputs.

This indicator can exist as an input indicator of the DMUs. For example, the amount of the buses in a bus-firm, that has different lines are limited and each line can only take some of the buses as input indicator. Also about input indicator with limited sources, it must be considered that if a DMU wants more of the same input, this amount must be supported by decreasing the input in the other DMUs.

The main contribution of this paper is to develop a sensitivity analysis for evaluating the stability amount of an efficient DMU against variation in an input or output indicator with a limited source.

The paper is structured as follows: the next section provides an introduction to CCR model and the efficient DMUs in DEA. Section 3 presents a sensitivity analysis for an efficient DMU relative to an output with alimited source. Also the approach is illustrated by solving an example. The conclusion is provided in section 4.

2 CCR-efficiency

DEA is a non-parametric multiple input/output efficiency technique that measures the relative efficiency of DMUs using a linear programming based model. It is non-parametric because it requires no assumption on the shape or parameters of the underlying production function. Due to this property, DEA has gained increasing popularity in the last two decades.

Consider n DMUs, each consuming varying amounts of m inputs in the production of s outputs. The $m \times n$ matrix of inputs is denoted by X and the $s \times n$ matrix of outputs by Y. Furthermore, x_{ij} denotes the amount consumed of the i-th input by the j-th DMU and y_{rj} denotes the amount produced of its r-th output. Finally, X_j and Y_j denote, respectively, the vector of inputs and outputs for the j-th DMU. The input-oriented linear programming problem formulation for the Charnes, Cooper, and Rhodes (CCR) [1] model for evaluation of DMU_o, $o \in \{1, ..., n\}$, (the multiplier side) is as follows:

$$
\begin{array}{lll}
\theta_o^* = Max & \displaystyle\sum_{r=1}^{s} u_r y_{ro} & \\
s.t. & \displaystyle\sum_{i=1}^{m} v_i x_{io} = 1 & \\
 & \displaystyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 & j = 1, \dots, n \\
 & v_i \geq \varepsilon & i = 1, ..., m \\
 & u_r \geq \varepsilon & r = 1, ..., s
\end{array} \tag{1}
$$

where ε is a non-Archimedean infinitesimal.

It can be proven that $0 < \theta^* \leq 1$ and DMU_o is the efficient in the CCR model if $\theta_o^* = 1$. Otherwise, the DMU_o is inefficient[1]. Therefore, from (1), DMU_o is efficient, if there are $v_i \geq \varepsilon$, $i = 1, ..., m$ and $u_r \geq \varepsilon$, $r = 1, ..., s$, such that

$$\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} = 0,$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, ..., n,\ j \neq o. \tag{2}$$

Without loss of generality it can be supposed that $\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1$, because if v_i and u_r satisfy in (2), then $\bar{v}_i = t v_i$ and $\bar{u}_r = t u_r$ also satisfy, where

$$t = \frac{1}{\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i}.$$

3 Sensitivity analysis

In this section, we purpose to measure the stability amount of the efficient DMU relative to decrease one of its outputs with a limited source against increasing the output in other DMUs, such that the new outputs can be produced by the same inputs. It must be considered that increasing each output in an efficient DMU does not effect its efficient status.

Suppose that DMU_o is an efficient DMU and the first output is a limited source case. In sensitivity analysis during, the changing amount of y_{1j}, $j = 1, ..., n$, is denoted by α_j, such that $o \leq \alpha_o \leq y_{1o}$ and $o \leq \alpha_j \leq p_j$, $j \neq o$, where p_j is the almost increasing y_{1j} such that DMU_j is able to produce $Y'_j = (y'_{1j}, y_{2j}, ..., y_{sj})$ by the same inputs X_j, where

$$y'_{1o} = y_{1o} - \alpha_o$$

$$y'_{1j} = y_{1j} + \alpha_j, j \neq o$$

and $\sum_{j \neq o} \alpha_j = \alpha_o$. $p_j \geq y_{10}$ means that α_j can be increased without any limitation.

Suppose that the attraction contribution of α_o by DMU_j is specified and is denoted by w_j where

$$W = (w_1, ..., w_n), w_o = 0, w_j \geq 0, j \neq o, \sum_{j \neq o} w_j = 1$$

therefore, we have $\alpha_j = w_j \alpha_o$. The stability interval of DMU_o is as $[0, \alpha_o(W)]$, where $\alpha_o(W)$ has the most amount α_o such that DMU_o with input-output vector (X_o, Y'_o) holds over as an efficient DMU among other DMUs with input-output vector (X_j, Y'_j). From (2), we have

$$\begin{array}{lll}
\alpha_o(W) = Max & \alpha_o & \\
s.t. & \sum_{r=1}^{s} u_r y_{ro} - u_1 \alpha_o - \sum_{i=1}^{m} v_i x_{io} = 0, & \\
 & \sum_{r=1}^{s} u_r y_{rj} + u_1 w_j \alpha_o - \sum_{i=1}^{m} v_i x_{ij} \leq 0, & j \neq o, \\
 & \sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1, & \\
 & 0 \leq \alpha_o \leq y_{1o}, & \\
 & w_j \alpha_o \leq p_j, & j \neq o, \\
 & v_i \geq \varepsilon, & i = 1, ..., m, \\
 & u_r \geq \varepsilon, & r = 1, ..., s.
\end{array} \quad (3)$$

The above model is a nonlinear programming problem which transform to linear form by $\bar{\alpha}_o = u_1 \alpha_o$, as follows

$$
\begin{array}{lll}
Max & \bar{\alpha}_o & \\
s.t. & \displaystyle\sum_{r=1}^{s} u_r y_{ro} - \bar{\alpha}_o - \sum_{i=1}^{m} v_i x_{io} = 0, & \\
& \displaystyle\sum_{r=1}^{s} u_r y_{rj} + w_j \bar{\alpha}_o - \sum_{i=1}^{m} v_i x_{ij} \le 0, & j \neq o, \\
& \displaystyle\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1, & \\
& 0 \le \bar{\alpha}_o \le y_{1o} u_1, & \\
& w_j \bar{\alpha}_o \le p_j u_1, & j \neq o, \\
& v_i \ge \varepsilon, & i = 1, ..., m, \\
& u_r \ge \varepsilon, & r = 1, ..., s
\end{array}
\tag{4}
$$

therefore, for the optimum solution $\bar{\alpha}_o^*$ and u_1^* of (4), we have $\alpha_o(W) = \frac{\bar{\alpha}_o^*}{u_1^*}$.

Now, suppose that the contribution of each DMU of α_o is unknown. Hence, $\alpha_o(W)$ is a function of W. The upper bound of the stability interval, α_o^*, is the most decreasing y_{1o} which DMU_o certainly holds over the efficient DMU for each contribution W. That is

$$
\alpha_o^* = \min\{\alpha_o(W) \mid W = (w_1, ..., w_n), w_o = 0, w_j \ge 0, j \neq o, \sum_{j \neq o} w_j = 1.\} \tag{5}
$$

The amount of α_o^* is obtained by solving above $\min\max$ nonlinear programming problem. In following, we provide a method for calculating α_o^*. The dual problem of (4), to known W, is as follows

$$\begin{array}{lll}
\alpha_o(W) = \frac{1}{u_1^*(W)} Min & z - \epsilon(\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-) & \\
s.t. & \sum_{j=1}^{n} y_{1j}\lambda_j + z - y_{1o}\beta - \sum_{j\neq o} p_j\mu_j - s_1^+ = 0, & \\
 & \sum_{j=1}^{n} y_{rj}\lambda_j + z - s_r^+ = 0, & r = 2, ..., s, \\
 & \sum_{j=1}^{n} x_{ij}\lambda_j - z + s_i^- = 0, & i = 1, ..., m, \\
 & -\lambda_o + \beta + \sum_{j\neq o} w_j\lambda_j + \sum_{j\neq o} w_j\mu_j \geq 1, & \\
 & \lambda_j \geq 0, & j \neq o, \\
 & \mu_j \geq 0, & j \neq o, \\
 & \beta \geq 0, & \\
 & z, \lambda_o \quad free &
\end{array} \tag{6}$$

where, the vector W is known as $W = (w_1, ..., w_n), w_o = 0, w_j \geq 0, j \neq o, \sum_{j\neq o} w_j = 1$ and $u_1^*(W)$ is optimal solution of (4) corresponding to W. Therefore, the problem (5) is formulated as follows

$$
\begin{array}{lll}
\alpha_o^* = \frac{1}{u_1^*(W)} Min & z - \epsilon(\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-) & \\
s.t. & \sum_{j=1}^{n} y_{1j}\lambda_j + z - y_{1o}\beta - \sum_{j\neq o} p_j\mu_j - s_1^+ = 0, & \\
 & \sum_{j=1}^{n} y_{rj}\lambda_j + z - s_r^+ = 0, & r = 2, ..., s, \\
 & \sum_{j=1}^{n} x_{ij}\lambda_j - z + s_i^- = 0, & i = 1, ..., m, \\
 & -\lambda_o + \beta + \sum_{j\neq o} w_j\lambda_j + \sum_{j\neq o} w_j\mu_j \geq 1, & \\
 & \sum_{j\neq o} w_j = 1, & \\
 & w_o = 0, w_j \geq 0, & j \neq o, \\
 & \lambda_j \geq 0, & j \neq o, \\
 & \mu_j \geq 0, & j \neq o, \\
 & \beta \geq 0, & \\
 & z, \lambda_o \quad free. &
\end{array}
\tag{7}
$$

Using following theorem, the optimal solution of (7) is obtained by solving n linear programming problems.

Theorem 1. $\alpha_o^* = z^k$, for some k, where

$$
\begin{array}{lll}
z^k = \frac{1}{u_1^*} Min & z - \epsilon(\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-) & k = 1, ..., n, k \neq o \\
s.t. & \sum_{j=1}^{n} y_{1j}\lambda_j + z - y_{1o}\beta - \sum_{j\neq o} p_j\mu_j - s_1^+ = 0, & \\
 & \sum_{j=1}^{n} y_{rj}\lambda_j + z - s_r^+ = 0, & r = 2, ..., s, \\
 & \sum_{j=1}^{n} x_{ij}\lambda_j - z + s_i^- = 0, & i = 1, ..., m, \\
 & -\lambda_o + \beta + \lambda_k + \mu_k \geq 1, & \\
 & \lambda_j \geq 0, & j \neq o, \\
 & \mu_j \geq 0, & j \neq o, \\
 & \beta \geq 0, & \\
 & z, \lambda_o \quad free &
\end{array}
\tag{8}
$$

where u_1^* is optimal solution $\alpha_o(W^*)$ in (3) such that W^* is optimal solution of (5) corresponding to α_o^*.

Proof. Clearly, each solution of (8) is a solution for (7), for $w_j = 0$, $j \neq k$, and $w_k = 1$. Therefore, for all k, $z^k \leq \alpha_o^*$.

Suppose $(z^*, \lambda_1^*, ..., \lambda_n^*, \mu_1^*, \mu_n^*, w_1^*, ..., w_n^*, \beta^*, s_1^{+*}, s_s^{+*}, s_1^{-*}, ..., s_m^{-*})$ is an optimal solution of (7), we show that it satisfies in (8), for some k. It is sufficient to show it satisfies in the constraint $-\lambda_o + \beta + \lambda_k + \mu_k \geq 1$ for some k. By contradiction, suppose

$$-\lambda_o^* + \beta^* + \lambda_j^* + \mu_j^* < 1$$

for any j $(j \neq o)$. Since, $0 \leq w_j \leq 1$, we have

$$-\lambda_o^* + \beta^* + w_j^*\lambda_j^* + w_j^*\mu_j^* < 1$$

therefore

$$-\lambda_o^* + \beta^* + \sum_{j\neq o} w_j^*\lambda_j^* + \sum_{j\neq o} w_j^*\mu_j^* < 1.$$

Which contracts the feasible solution assumption of this solution in (7). Then, there exists a k which this solution satisfies in (6) and therefore $z^k \leq \alpha_o^*$ and the proof is completed.□

Example. Table 1 shows the summary input and output data of 7 DMUs that DMU_j use a single input x_{1j}, whose value is normalized to 1, to produce two outputs (y_{1j}, y_{2j}). The Production Possibility Set(PPS) in outputs space (see [1]) is portrayed in Fig. 1.

	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7
x_{1j}	1	1	1	1	1	1	1
y_{1j}	9	3	6	8	6	4	3
y_{2j}	1	2	3	3	4	4	5

Table 1. Data of seven DMUs.

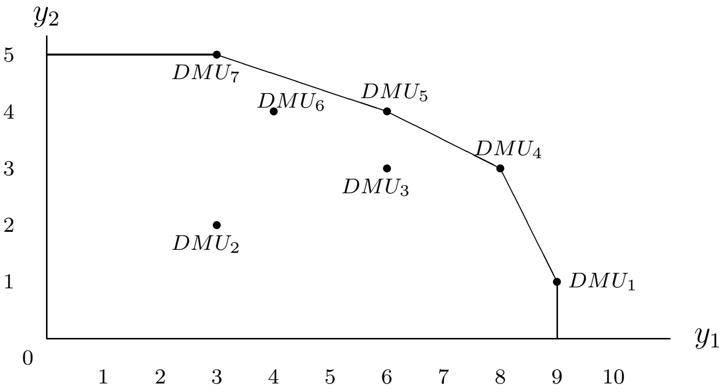

Fig. 1. The Production Possibility Set(PPS) in outputs space

If a DMU is efficient, it locates on the boundary of PPS (see [1]). Therefore DMUs 1,4,5 and 7 are efficient.

Suppose the source of the first output is limited and $p_j \geq 9$, for all j. Table 2 shows the summary result of the stability interval $[0, \alpha_o^*]$ and their optimal vector W^* corresponding to four efficient DMUs.

If the source of first output is not limited, the stability interval of DMU_o is as $[0, \delta_o^*]$, where δ_o^* has the most amount α_o in (9) such that DMU_o with input-output vector (X_o, Y_o') holds over as an efficient DMU among other DMUs with input-output vector (X_j, Y_j).

$$
\begin{aligned}
\delta_o^* = Max \quad & \alpha_o \\
s.t. \quad & \sum_{r=1}^{s} u_r y_{ro} - u_1 \alpha_o - \sum_{i=1}^{m} v_i x_{io} = 0, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, && j \neq o, \\
& \sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1, \\
& 0 \leq \alpha_o \leq y_{1o}, \\
& v_i \geq \varepsilon, && i = 1, ..., m, \\
& u_r \geq \varepsilon, && r = 1, ..., s.
\end{aligned}
\tag{9}
$$

Similarly, (9) is transformed to a linear programming as (4). The third row of Table 2 shows the stability interval of the efficient DMUs when the source of first output is not limited.

	DMU_1	DMU_4	DMU_5	DMU_7
$[0, \alpha_o^*]$	[0,0.4996]	[0,0.5998]	[0,0.3340]	[0,2.9945]
W^*	(0,0,0,1,0,0,0)	(0,0,0,0,1,0,0)	(0,0,0,1,0,0,0)	(0,0,0,0,1,0,0)
$[0, \delta_o]$	[0,1.000]	[0,1.0000]	[0,0.5000]	[0,3.0037]

Table 3. The stability intervals of the efficient DMUs.

4 Conclusion

The main contribution of this paper is to provide a sensitivity analysis approach when one of the input or the output indicators are with limited sources. the approach is presented for single output with a limited source case and it is similarly extendable for single input with a limited source case. Finally, this approach is explained with an example.

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Received: April 23, 2008