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Subclasses of Convex Functions with Respect to other Points

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Abstract

Let $\mathcal{S}$ be the class of functions f which are analytic and univalent in the open unit disc $D = \{z : |z| < 1\}$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and a_n a complex number. Let $\mathcal{T}$ denote the class consisting of functions f of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ where a_n is a non negative real number. In [6], Sitin and Janteng introduced 3 subclasses of $\mathcal{T}$; $C_sT(\alpha, \beta, \sigma, k)$, $C_cT(\alpha, \beta, \sigma, k)$ and $C_{sc}T(\alpha, \beta, \sigma, k)$, consisting of analytic functions with negative coefficients and are respectively convex with respect to symmetric points, convex with respect to conjugate points and convex with respect to symmetric conjugate points. Here, α and β are to satisfy certain constraints. This paper extends the result in [6] to other properties namely growth and extreme points.

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1 Introduction

Let $\mathcal{S}$ be the class of functions f which are analytic and univalent in the open unit disc $D = \{z : |z| < 1\}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and a_n a complex number. Let $S^\star$ be the subclass of $\mathcal{S}$ consisting of functions starlike in D. It is well known that $f \in S^\star$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in D.$$

Let $S_s^\star$ be the subclass of $\mathcal{S}$ consisting of functions given by (1) satisfying

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0,\quad z\in D.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi in [4]. The class has also been considered in Robertson [3], Stankiewicz [7], Wu [9] and Owa et al. [2]. El-Ashwah and Thomas in [1], introduced two other classes namely the class $S_c^\star$ consisting of functions starlike with respect to conjugate points and $S_{sc}^\star$ consisting of functions starlike with respect to symmetric conjugate points.

In [8], Sudharsan et al. introduced $S_s^\star(\alpha,\beta)$ of functions f analytic and univalent in D given by (1) and satisfying the condition

$$\left|\frac{zf'(z)}{f(z)-f(-z)}-1\right|<\beta\left|\frac{\alpha zf'(z)}{f(z)-f(-z)}+1\right|$$

for some $0\leq\alpha\leq 1, 0<\beta\leq 1$ and $z\in D$.

However, for this paper, we consider a subclass of $\mathcal{T}$ where $\mathcal{T}$ denotes the class consisting of functions f of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \tag{2}$$

where a_n is a non negative real number.

For $f\in\mathcal{T}$, Sitin and Janteng in [6], defined the classes $C_sT(\alpha,\beta,\sigma,k), C_cT(\alpha,\beta,\sigma,k)$ and $C_{sc}T(\alpha,\beta,\sigma,k)$ with α and β satisfying the conditions $0\leq\alpha<1,\ 0<\beta<1,\ \frac{1}{2}<2\sigma<k\leq 1$ and $0\leq\frac{2(k(1-\beta)+2\sigma\beta)}{1+\alpha\beta}<1$.

Definition 1.1 *A function $f\in C_sT(\alpha,\beta,\sigma,k)$ is said to be convex with respect to symmetric points if and only if it satisfies*

$$\left|\frac{(zf'(z))'}{(f(z)-f(-z))'}-k\right|<\beta\left|\frac{\alpha(zf'(z))'}{(f(z)-f(-z))'}-(2\sigma-k)\right|$$

for $z\in D$.

Definition 1.2 *A function $f\in C_cT(\alpha,\beta,\sigma,k)$ is said to be convex with respect to conjugate points if and only if it satisfies*

$$\left|\frac{(zf'(z))'}{\left(f(z)+\overline{f(\bar{z})}\right)'}-k\right|<\beta\left|\frac{\alpha(zf'(z))'}{\left(f(z)+\overline{f(\bar{z})}\right)'}-(2\sigma-k)\right|$$

for $z\in D$.

Definition 1.3 *A function $f \in C_{sc}T(\alpha, \beta, \sigma, k)$ is said to be convex with respect to symmetric conjugate points if and only if it satisfies*

$$\left| \frac{(zf'(z))'}{\left(f(z) - \overline{f(-\bar{z})}\right)'} - k \right| < \beta \left| \frac{\alpha(zf'(z))'}{\left(f(z) - \overline{f(-\bar{z})}\right)'} - (2\sigma - k) \right|$$

for $z \in D$.

At this point, we would like to note that the above conditions imposed on α and β, are necessary to ensure these classes form subclasses of $\mathcal{S}$.

2 Preliminaries

We first state preliminary results due by Sitin and Janteng in [6], required for proving our main results. For $C_sT(\alpha, \beta, \sigma, k)$, we have the following:

Theorem 2.1 *$f \in C_sT(\alpha, \beta, \sigma, k)$ if and only if*

$$\sum_{n=2}^{\infty} \left(\frac{(1+\beta\alpha)n^2}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} + \frac{n\beta(k-2\sigma)(1-(-1)^n) - nk(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} \right) a_n \leq 1.$$

Corollary 2.1 *If $f \in C_sT(\alpha, \beta, \sigma, k)$ then*

$$a_n \leq \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n^2 + n\beta(k-2\sigma)(1-(-1)^n) - nk(1-(-1)^n)}\ , \quad n \geq 2.$$

Next, similar coefficient properties for functions which belong to $C_cT(\alpha, \beta, \sigma, k)$ and $C_{sc}T(\alpha, \beta, \sigma, k)$ are obtained.

Theorem 2.2 *$f \in C_cT(\alpha, \beta, \sigma, k)$ if and only if*

$$\sum_{n=2}^{\infty} \left(\frac{(1+\beta\alpha)n^2}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} + \frac{2n(\beta(k-2\sigma)-k)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} \right) a_n \leq 1.$$

Corollary 2.2 *If $f \in C_cT(\alpha, \beta, \sigma, k)$ then*

$$a_n \leq \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n^2 + 2n(\beta(k-2\sigma)-k)}\ , \quad n \geq 2.$$

Theorem 2.3 *$f \in C_{sc}T(\alpha, \beta, \sigma, k)$ if and only if*

$$\sum_{n=2}^{\infty} \left(\frac{(1+\beta\alpha)n^2}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} + \frac{n\beta(k-2\sigma)(1-(-1)^n) - nk(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} \right) a_n \leq 1.$$

Corollary 2.3 *If $f \in C_{sc}T(\alpha, \beta, \sigma, k)$ then*

$$a_n \leq \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n^2 + n\beta(k-2\sigma)(1-(-1)^n) - nk(1-(-1)^n)}\ , \quad n \geq 2.$$

3 Results

In this section, we give results concerning the growth and extreme points of the 3 main classes.

Theorem 3.1 *Let the functions f be defined by (2) and belongs to the class $C_sT(\alpha,\beta,\sigma,k)$. Then for $\{z : 0 < |z| = r < 1\}$,*

$$r-\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)}\,r^2 \leq |f(z)| \leq r+\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)}\,r^2.$$

Proof.

First, it is obvious that

$$\frac{4(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\sum_{n=2}^{\infty}a_n$$

$$\leq \sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n^2}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{n\beta(k-2\sigma)(1-(-1)^n)-nk(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)a_n,$$

and as $f \in C_sT(\alpha,\beta,\sigma,k)$, using the inequality in Theorem 2.1 yields

$$\sum_{n=2}^{\infty}a_n \leq \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)}. \tag{3}$$

From (2) with $|z| = r(r < 1)$, we have

$$|f(z)| \leq r+\sum_{n=2}^{\infty}a_nr^n \leq r+\sum_{n=2}^{\infty}a_nr^2$$

and

$$|f(z)| \geq r-\sum_{n=2}^{\infty}a_nr^n \geq r-\sum_{n=2}^{\infty}a_nr^2.$$

Finally, using (3) in the above inequalities, gives the result in Theorem 3.1.

We note that result in Theorem 3.1 is sharp for the following function,

$$f_2(z) = z-\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)}\,z^2$$

at $z = \pm r$.

Next, similar growth results for functions which belong to $C_cT(\alpha,\beta,\sigma,k)$ and $C_{sc}T(\alpha,\beta,\sigma,k)$ are obtained. Similar method of proving is used for Theorem 3.2 and Theorem 3.3.

Theorem 3.2 *Let the functions f be defined by (2) and belongs to the class $C_cT(\alpha,\beta,\sigma,k)$. Then for $\{z : 0 < |z| = r < 1\}$,*

$$r-\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)+4(\beta(k-2\sigma)-k)}\,r^2 \le |f(z)| \le r+\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)+4(\beta(k-2\sigma)-k)}\,r^2.$$

Proof.

The result follows through from

$$\frac{4(1+\beta\alpha)+4(\beta(k-2\sigma)-k)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\sum_{n=2}^{\infty}a_n$$

$$\le \sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n^2}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{2n(\beta(k-2\sigma)-k)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)a_n,$$

and upon using the inequality in Theorem 2.2, which yields

$$\sum_{n=2}^{\infty}a_n \le \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)+4(\beta(k-2\sigma)-k)}.$$

The result in Theorem 3.2 is sharp for the following function,

$$f_2(z)=z-\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)+4(\beta(k-2\sigma)-k)}\,z^2$$

at $z=\pm r$.

For completeness, we state the following result with regards to the class $C_{sc}T(\alpha,\beta,\sigma,k)$.

Theorem 3.3 *Let the functions f be defined by (2) and belongs to the class $C_{sc}T(\alpha,\beta,\sigma,k)$. Then for $\{z : 0 < |z| = r < 1\}$,*

$$r-\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)}\,r^2 \le |f(z)| \le r+\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)}\,r^2.$$

The result in Theorem 3.3 is sharp for the following function,

$$f_2(z)=z-\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{4(1+\beta\alpha)}\,z^2$$

at $z=\pm r$.

In view of Theorem 2.1, the class $C_sT(\alpha,\beta,\sigma,k)$ is closed under convex linear combinations. Here, we determine the extreme points for $C_sT(\alpha,\beta,\sigma,k)$.

Theorem 3.4 *Let* $f_1(z) = z$ *and* $f_n(z) = z - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n^2+n\beta(k-2\sigma)(1-(-1)^n)-nk(1-(-1)^n)}\ z^n$ *for* $n \geq 2$. *Then* $f \in C_sT(\alpha, \beta, \sigma, k)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z)$ *where* $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty}\lambda_n = 1$.

Proof.

Adopting the same technique used by Silverman in [5], we first assume

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z) = z - \sum_{n=2}^{\infty}\lambda_n \left\{\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n^2+n\beta(k-2\sigma)(1-(-1)^n)-nk(1-(-1)^n)}\right\} z^n.$$

Next, since

$$\sum_{n=2}^{\infty}\lambda_n \left\{\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n^2+n\beta(k-2\sigma)(1-(-1)^n)-nk(1-(-1)^n)}\right\}\cdot\left\{\frac{(1+\beta\alpha)n^2+n\beta(k-2\sigma)(1-(-1)^n)-nk(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right\}$$

$$= \sum_{n=2}^{\infty}\lambda_n$$

$$= 1 - \lambda_1$$

$$\leq 1,$$

therefore by Theorem 2.1, $f \in C_sT(\alpha, \beta, \sigma, k)$.

Conversely, suppose $f \in C_sT(\alpha, \beta, \sigma, k)$. Since

$$a_n \leq \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n^2+n\beta(k-2\sigma)(1-(-1)^n)-nk(1-(-1)^n)}\ , \quad n \geq 2,$$

we may set $\lambda_n = \left\{\frac{(1+\beta\alpha)n^2+n\beta(k-2\sigma)(1-(-1)^n)-nk(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right\} a_n$, $(n \geq 2)$ and $\lambda_1 = 1 - \sum_{n=2}^{\infty}\lambda_n$.

Then

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n f_n(z) &= \lambda_1 f_1(z) + \sum_{n=2}^{\infty}\lambda_n f_n(z)\\ &= z - \sum_{n=2}^{\infty}\lambda_n z + \sum_{n=2}^{\infty}\lambda_n z - \sum_{n=2}^{\infty}a_n z^n\\ &= f(z).\end{aligned}$$

Hence, we complete the proof of Theorem 3.4.

Finally, we give similar extreme points for functions which belong to $C_cT(\alpha, \beta, \sigma, k)$ and $C_{sc}T(\alpha, \beta, \sigma, k)$. Method of proving Theorem 3.5 and Theorem 3.6 is similar as that of Theorem 3.4.

Theorem 3.5 *Let* $f_1(z) = z$ *and* $f_n(z) = z - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n^2+2n(\beta(k-2\sigma)-k)}\ z^n$ *for* $n \geq 2$.. *Then* $f \in C_cT(\alpha, \beta, \sigma, k)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z)$ *where* $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty}\lambda_n = 1$.

Theorem 3.6 *Let* $f_1(z) = z$ *and* $f_n(z) = z - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n^2+n\beta(k-2\sigma)(1-(-1)^n)-nk(1-(-1)^n)}\, z^n$ *for* $n \geq 2$. *Then* $f \in C_{sc}T(\alpha,\beta,\sigma,k)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z)$ *where* $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty}\lambda_n = 1$.

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