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# A Note on Continuity and Continuous Representability of Interval Orders

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**Abstract**

We introduce the notion of *weak continuity* relative to an interval order $\precsim$ on a topological space $(X, \tau)$. Then we show that a weakly continuous interval order $\precsim$ on a second countable topological space $(X, \tau)$ is represented by a pair $(u, v)$ of continuous real-valued functions (in the sense that, for all $x, y \in X$, $x \precsim y$ if and only if $u(x) \leq v(y)$). In this way we generalize the famous continuous utility representation theorem of Debreu according to which a total preorder $\precsim$ on a second countable topological space $(X, \tau)$ admits a continuous utility representation.

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# 1 Introduction

An interval order $\precsim$ on a set $X$ is a binary relation on $X$ which is *reflexive* and in addition verifies the following condition for all $x, y, z, w \in X$: $[(x \precsim z)$ *and* $(y \precsim w) \Rightarrow (x \precsim w)$ *or* $(y \precsim z)]$. Fishburn [11] defined two total preorders $\precsim^*$ and $\precsim^{**}$ which are naturally associated to an interval order $\precsim$. Conditions implying the existence of a pair $(u, v)$ of continuous real-valued functions representing an interval order $\precsim$ on a topological space $(X, \tau)$ (in the sense that, for all $x, y \in X$, $x \precsim y$ if and only if $u(x) \leq v(y)$) typically require continuity not only of the interval order $\precsim$ but also of the total preorders $\precsim^*$ and $\precsim^{**}$ (see Bosi et al. [1, Corollary 4.3]) and Bosi et al. [2, Theorem 1]). Nevertheless, a condition of this kind is not necessary for the existence of a continuous representation (see e.g. Bosi et al. [2, Example 1]). To the best of my knowledge, the only attempt in the direction of finding a characterization of a continuous representation for an interval order on a topological space is found in Bosi and Isler [6, Theorem 3.3], who used the concept of a *scale* in a topological space.

In this paper we make a further step in the direction of characterizing the existence of a continuous representation without additional requirements. To this aim, we introduce the notion of a *weakly continuous* interval order on a topological space. We show that any weakly continuous interval order on a second countable topological space is continuously representable. This result may be viewed as a slight generalization of the famous continuous utility representation theorem of Debreu according to which a total preorder $\precsim$ on a second countable topological space $(X, \tau)$ admits a continuous utility representation.

## 2 Notation and preliminaries

An *interval order* $\precsim$ on an arbitrary nonempty set $X$ is a binary relation on $X$ which is *reflexive* and in addition verifies the following condition for all $x, y, z, w \in X$:

$$(x \precsim z) \text{ and } (y \precsim w) \Rightarrow (x \precsim w) \text{ or } (y \precsim z).$$

An interval order $\precsim$ on a set $X$ is *total* (i.e., for all $x, y \in X$ we have that either $x \precsim y$ or $y \precsim x$) and not necessarily transitive (see e.g. Oloriz et al. [13]). The *irreflexive part* of an interval order $\precsim$ will be denoted by $\prec$ (i.e., for all $x, y \in X$, $x \prec y$ if and only if $(x \precsim y)$ and not$(y \precsim x)$).

Fishburn [11] proved that if $\precsim$ is an interval order on a set $X$, then the following two binary relations $\precsim^*$ and $\precsim^{**}$ on $X$ are both total preorders:

$$x \precsim^* y \Leftrightarrow (z \precsim x \Rightarrow z \precsim y) \text{ for all } z \in X,$$

$$x \precsim^{**} y \Leftrightarrow (y \precsim z \Rightarrow x \precsim z) \text{ for all } z \in X.$$

Obviously, if $\precsim$ is a *total preorder* $\precsim$ (i.e., $\precsim$ is reflexive, transitive and total), then $\precsim$ is an interval order. In this case, we have that $\precsim = \precsim^* = \precsim^{**}$. The irreflexive parts of $\precsim^*$ and $\precsim^{**}$ will be denoted by $\prec^*$ and $\prec^{**}$.

If $\precsim$ is an interval order on a set $X$, then denote by $L_\prec(x)$ ($U_\prec(x)$) the *strict lower* (*upper*) *section* of any element $x \in X$ (i.e., for every $x \in X$, $L_\prec(x) = \{y \in X : y \prec x\}$ and $U_\prec(x) = \{y \in X : x \prec y\}$). Further define, for every $x \in X$, $L_\precsim(x) = \{y \in X : y \precsim x\}$ and $U_\precsim(x) = \{y \in X : x \precsim y\}$.

We recall that a real-valued function $u$ on $X$ is said to be a *utility function* for a total preorder $\precsim$ on a set $X$ if, for all $x, y \in X$,

$$x \precsim y \Leftrightarrow u(x) \leq u(y).$$

A pair $(u, v)$ of real-valued functions on $X$ is said to *represent* an interval order $\precsim$ on $X$ if, for all $x, y \in X$,

$$x \precsim y \Leftrightarrow u(x) \leq v(y).$$

We say that a pair $(u, v)$ of real-valued functions on $X$ *almost represents* an interval order $\precsim$ on $X$ if, for all $x, y \in X$,

$$(x \precsim y \Rightarrow u(x) \leq v(y)) \text{ and } (x \prec y \Rightarrow v(x) \leq u(y)).$$

An interval order $\precsim$ on a topological space $(X, \tau)$ is said to be *upper (lower) semicontinuous* if $L_{\prec}(x)$ $(U_{\prec}(x))$ is an open subset of $X$ for every $x \in X$. If $\precsim$ is both upper and lower semicontinuous, then it is said to be *continuous* .

Let us now introduce the main definition in this paper.

**Definition 2.1** We say that an interval order $\precsim$ on a topological space $(X, \tau)$ is *weakly continuous* if for every $x, y \in X$ such that $x \prec y$ there exists a pair $(u_{xy}, v_{xy})$ of continuous real-valued functions on $(X, \tau)$ satisfying the following conditions:
(i) $(u_{xy}, v_{xy})$ almost represents $\precsim$;
(ii) $v_{xy}(x) < u_{xy}(y)$.

The concept of weak continuity described in Definition 2.1 is reminiscent of the concept of *weak continuity* of a preorder on a topological space. From Herden and Pallack [12, Definition 2.3], a (not necessarily total) preorder $\precsim$ on a topological space $(X, \tau)$ is said to be *weakly continuous* (see also Bosi and Herden [5]) if for every $x, y \in X$ such that $x \prec y$ there exists a continuous increasing real-valued function $u_{xy}$ on $(X, \tau)$ such that $u_{xy}(x) < u_{xy}(y)$.

The immediate proof of the following proposition is left to the reader.

**Proposition 2.2** *Let $\precsim$ be an interval order on a topological space $(X, \tau)$. If there exists a pair $(u, v)$ of continuous real-valued functions on $(X, \tau)$ representing $\precsim$, then $\precsim$ is weakly continuous.*

Let us now prove a more interesting condition implying weak continuity. Indeed, the following proposition in some sense motivates the definition of weak continuity itself, at least when we deal with second countable topological spaces.

**Proposition 2.3** *Let $\precsim$ be a continuous interval order on a second countable topological space $(X, \tau)$. If the total preorders $\precsim^*$ and $\precsim^{**}$ associated to $\precsim$ are also continuous, then $\precsim$ is weakly continuous.*

*Proof.* If $\precsim$ is a continuous interval order on a second countable topological space $(X, \tau)$ and the total preorders $\precsim^*$ and $\precsim^{**}$ associated to $\precsim$ are both continuous, then first of all there exists a pair $(u', v')$ of real-valued functions on $X$ representing $\precsim$ (see Bridges [7, Proposition 2.3]). Hence we have that there exists a pair $(u, v)$ of continuous real-valued functions on $(X, \tau)$ representing $\precsim$ (see Bosi et al. [2, Theorem 1] and Bosi et al. [1, Corollary 4.3]). The conclusion follows from Proposition 2.2. □

# 3 Continuous representability

We recall that a topology $\tau$ on a set $X$ is a *hereditarily Lindelöf topology* if for every subset $A$ of $X$ and every open covering $\mathcal{C}$ of $A$ there exists some countable subcovering $\mathcal{C}' \subset \mathcal{C}$ of $A$.

The following Theorem is analogous to Theorem 2.15 in Herden and Pallack [12] (see also Bosi et al. [4, Theorem 3.1]).

**Theorem 3.1** *Let $\precsim$ be an interval order on a topological space $(X, \tau)$ and assume that the product topology $\tau \times \tau$ on $X \times X$ is hereditarily Lindelöf. Then the following conditions are equivalent:*

(i) *There exists a pair $(u, v)$ of continuous real-valued functions on $(X, \tau)$ representing $\precsim$;*

(ii) *$\precsim$ is weakly continuous.*

*Proof.* (i) ⇒ (ii). See Proposition 2.2. (ii) ⇒ (i). Since the interval order $\precsim$ on the topological space $(X, \tau)$ is weakly continuous, we have that for every pair $(x, y) \in X \times X$ such that $x \prec y$ there exists a a pair $(u_{xy}, v_{xy})$ of continuous real-valued functions on $(X, \tau)$ which almost represents $\precsim$ and in addition are such that $v_{xy}(x) < u_{xy}(y)$ (see Definition 2.1). Clearly, it is not restrictive to assume that $u_{xy}$ and $v_{xy}$ both take values in $[0, 1]$. Define for every pair $(x, y) \in X \times X$ such that $x \prec y$

$$A_{xy}(x) := v_{xy}^{-1}([0, \frac{u_{xy}(x) + v_{xy}(y)}{2}[), \ B_{xy}(y) := u_{xy}^{-1}(]\frac{u_{xy}(x) + v_{xy}(y)}{2}, 1]).$$

Then the family $C := \{A_{xy}(x) \times B_{xy}(y)\}_{(x,y) \in X \times X, x \prec y}$ is an open cover of $\prec$. Since the product topology $\tau \times \tau$ on $X \times X$ is hereditarily Lindelöf, there exists a countable subfamily $C'$ of $C$ which also covers $\prec$, and therefore there is a countable family $\{(u_n, v_n)\}_{, n \in \mathbb{N} \setminus \{0\}}$ of pairs of continuous functions on $(X, \tau)$ with values in $[0, 1]$ such that $(u_n, v_n)$ almost represents $\precsim$ for every $n \in \mathbb{N} \setminus \{0\}$ and in addition for every $(x, y) \in X \times X$ with $x \prec y$ there exists some $n \in \mathbb{N} \setminus \{0\}$ such that $v_n(x) < u_n(y)$. Hence, if we finally define $u := \sum_{n=1}^{\infty} 2^{-n} u_n$ and $v := \sum_{n=1}^{\infty} 2^{-n} v_n$, we have that the pair $(u, v)$ of continuous real-valued functions on $(X, \tau)$ represents the interval order $\precsim$. □

Since whenever a topology $\tau$ on a set $X$ is second countable the product topology $\tau \times \tau$ on $X \times X$ is hereditarily Lindelöf, we immediately obtain the following corollary of Theorem 3.1.

**Corollary 3.2** *Let $\precsim$ be an interval order on a second countable topological space $(X, \tau)$. Then there exists a pair $(u, v)$ of continuous real-valued functions on $(X, \tau)$ representing $\precsim$ if and only if $\precsim$ is weakly continuous.*

Corollary 3.2 may be viewed as a generalization of the famous Debreu continuous utility representation Theorem (see Debreu [10]).

**Theorem 3.3 (Debreu continuous utility representation Theorem)** *Let $\precsim$ be a continuous total preorder on a second countable topological space $(X, \tau)$. Then there exists a continuous utility function $u$ for $\precsim$.*

*Proof.* If $\precsim$ is a continuous total preorder on a topological space $(X, \tau)$, then from considerations above we have that $\precsim = \precsim^* = \precsim^{**}$. Therefore, from Herden and Pallack [12, Lemma 2.2], we have that $\precsim$ is weakly continuous in the sense of Definition 2.1 and therefore, since we have assumed that $\tau$ is a second countable topology on $X$, we may apply Corollary 3.2. In particular, observe that in the proof of Theorem 3.1 we may assume that for every pair $(x, y) \in X \times X$ such that $x \prec y$ there exists a continuous increasing real-valued function $u_{xy}$ on $(X, \tau)$ such that $u_{xy}(x) < u_{xy}(y)$ (we may let $u_{xy} = v_{xy}$ for every $x, y \in X$ such that $x \prec y$). □

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