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Qualitative Analysis of the Overdamped Pendulum by Flows in the Circle

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Abstract

This work contains the qualitative and bifurcation analysis of an overdamped pendulum, which is a nonlinear system, initially describing some historical and theoretical events that marked the analysis of dynamic systems, later it will be presented with some examples the scope of the qualitative analysis for non-linear systems by means of flows in the lines, the importance of the bifurcation analysis and the variation of the parameters inherent in the system are also described, as the theoretical part is explained the flow analysis in the circle is explained. Consecutively, the overdamped pendulum model and each one of the considerations taken into account for its analysis are exposed. The results of the qualitative analysis, flows in the line and in the circle are

shown for different values of the parameters involved, the bifurcation diagram of the system is also exposed, and then the conclusions are concluded.

Keywords: Bifurcation, fixed points, overdamped pendulum

1 Introduction

It is of great importance for all fields of science to have mathematical models that describe the behavior of a physical system and that it does so in detail. With the aim set out above, different studies have emerged that combine especially the areas of mathematics and physics; The study of dynamic systems, which have a purely deterministic approach, focuses on the solution and explanation of the mathematical models that arise from a physical model.

The analysis of formal dynamic systems begins with the study of linear systems, since the behavior of these systems can be easily predicted analytically, however few physical systems are really linear, so the study should focus on Non-linear systems, a traditional way of solving these systems is linearized around a point, although this procedure is not inappropriate, biases the knowledge of the behavior of the non-linear model. Even so, there are easy application techniques to solve nonlinear systems in a simple way, these techniques contain a geometric analysis in which from the given system the trajectories are drawn and information is extracted from the solutions, this geometric reasoning allows to trace the trajectories in the phase plane without solving the system [3].

This (qualitative) analysis on which this document focuses, aims to show how, from observing the first-order differential equation plotted on the plane, this can be trajectories that contain, from the observation of critical points. In addition, it will be explained how the variation of the parameters involved in the model makes stable and unstable critical points change position or even stability (bifurcation). This analysis is made from the technique known as flows in the lines and can be simplified when you have periodic fixed points to flows in the circle, the latter will be the central idea of the document.

1.1 Nonlinear models

The analysis that works for linear systems is based on the principle of superposition, however, as mentioned above, most of the models that exist in nature have a non-linear structure which complicates the analysis a bit because the principle of overlap does not work anymore and it begins to require

longer-range mathematical tools [4]. One way to solve non-linear systems is to linearize around a nominal operating point and start working from there, and although this practice is correct, it has the disadvantage that only the local behavior of the system can be predicted. A clear example of this linearization procedure is when the simple pendulum [7] [9], Fig. 1, is modeled by a second order system (PS) and linearized (by Taylor series of first degree) considering oscillations of small amplitude (small angle θ), and in this way you can have clarity in the behavior of the pendulum with a more simple expression, however, you will not have knowledge for large oscillations [11].

$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0 \rightarrow \ddot{\theta} + \frac{g}{L}\theta = 0 \tag{1}$$

It is usual to perform the process described above, but in many non-linear

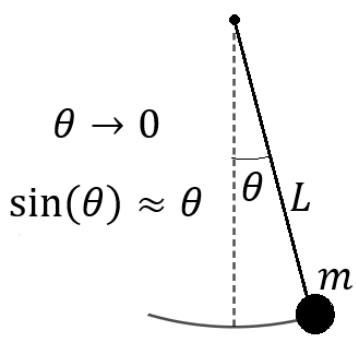

Figure 1: Simple pendulum.

models it is not possible to predict their behavior by linearizations [4] [11], the custom of performing these procedures is that the linear models are easy to solve because the problem it can be divided into smaller parts, and each of these parts is even easier to solve than the original problem and later these responses can be combined to obtain the general solution of the problem. The difficulty of non-linear systems, as has been indicated, lies in the fact that it is not possible to apply the principle of superposition [3].

1.2 Qualitative analysis

The qualitative analysis is a way of solving non-linear systems such as those mentioned in (1), the solutions of this type of systems can be visualized as trajectories that flow through a phase space of dimension n with coordinates $(x_1, x_2, \ldots, x_n)$. However, the purpose of this paper is to focus on solving a first-order system:

$$\dot{x} = f\left(x\right), \tag{2}$$

where $x(t)$ is a function of real value of time t and $f(x)$ is a smooth function of real values of x. The methodology to analyze this system is the observation of graphics, or said in a more formal way; a differential equation is interpreted as a vector field [2]. For example [3], for the system:

$$\dot{x} = \sin\left(x\right), \tag{3}$$

whose analytical solution for the initial value $x = x_0$, $t = 0$ is

$$t = \ln\left|\frac{cscx_0 + cotx_0}{cscx + cotx}\right|.$$

Even after having the analytical solution to the equation, it is difficult to know its behavior in time. Through qualitative analysis, we have t is time, x is the position of an imaginary particle moving along a real line, and $\dot{x}$ as the velocity of that particle. It will be then that the differential Eq.(3) represents a vector field in the line, which indicates the velocity vector $\dot{x}$ for each position x.

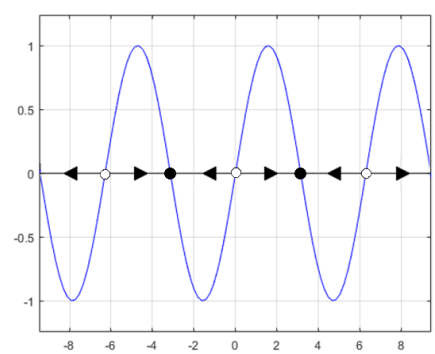

Figure 2: Flows in the system line.

2 Flows in the circle

The flow in the circle is a type of analysis to simplify the evolution of the trajectories of a system with periodic solution [5,12], as proposed in (3), Fig. 2, as can be seen the critical points are repeated every $2\pi k$ units, therefore, a system of type (3) can be written as:

$$\dot{x} = f\left(x\right) \;\rightarrow\; \dot{\theta} = f\left(\theta\right), \tag{4}$$

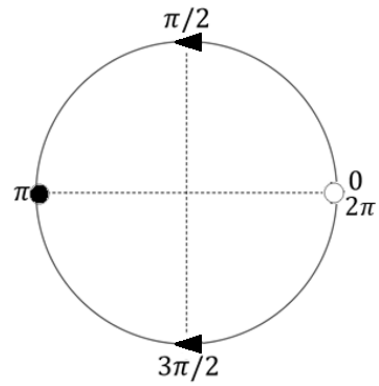

Figure 3: Flows in the system circle.

and the graph of the system of figure 2, equivalent to flows in the circle is:

As can be seen in Figure 3, the flows in the circle compress the periodic fixed points to just two; the unstable ones that are repeated every $2\pi k$ and the stable ones every πk, with $k = 1, 2 \ldots$.[5] The four previous figures can be

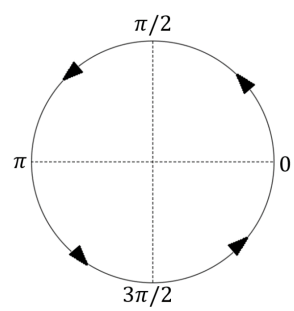

Figure 4: Flows in the system circle.

seen as a translation of the results obtained with the flows in the line, however, only the two periodic critical points of interest are observed, with more clarity it can be concluded that when $\mu > 1$ the value of the torque constant Γ will be greater than the torque produced by the mass mgL, that is to say that the pendulum will not have an equilibrium, when $\mu = 1$ the pendulum will have a critical point in $\pi/2$, in addition any initial position where the pendulum is positioned (initial angle θ_0) torque will take you to balance at the critical point mentioned. When $0 < \mu < 1$ the constant torque will be less than that exerted by the mass and there will be two equilibrium points, a stable one located between 0 and $\pi/2$, and an unstable point located between $\pi/2$ and π. When the constant torque is equal to zero then $\mu = 0$, and the critical points will be located in 0 the stable and in π the unstable, the latter behaving is

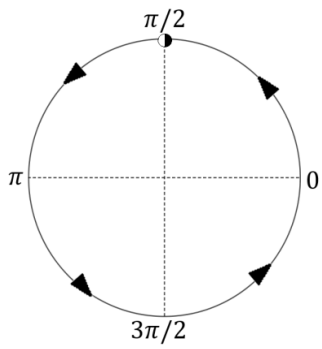

Figure 5: Flows in the system circle.

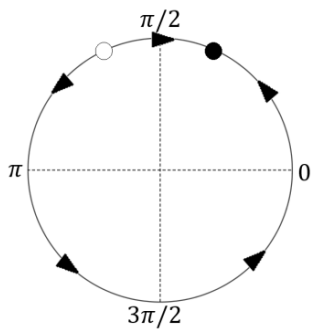

Figure 6: Flows in the system circle.

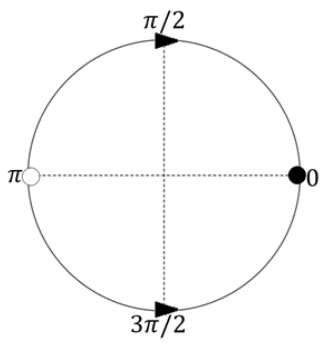

Figure 7: Flows in the system circle.

identical to the simple pendulum [6]. Finally, the bifurcation diagram is shown in which the stability of the system for different values of the parameter μ is completely shown.

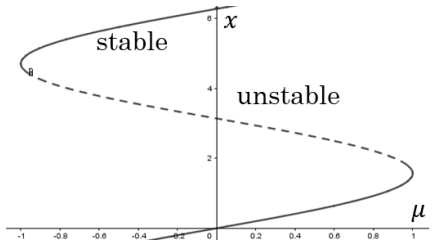

Figure 8: Bifurcation diagram.

3 Conclusion

The qualitative analysis is of great interest in the theory of dynamic systems using the observation of flows can predict the behavior of a non-linear system easily. The case of the flows in the circle is a more effective tool for the case of systems with periodic solutions, because in the case of the overdamped pendulum it is possible to observe through the proposed graphs the behavior of the system in different environmental conditions. With all the considerations for the overdamped pendulum system it is admissible to think that in the stable critical points obtained the pendulum stops instantaneously, however, this is an idealization that is explained when the buffer parameter b (molasses of the system) it is too large and therefore inertia is neglected, this condition of "instantaneous" equilibrium has its physical meaning in the sense that the change in speed is very small.

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