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Biparametric Bifurcation Analysis Applied to a Fishing Exploitation Model

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Abstract

The present work exposes the biparametric bifurcation of the dynamic model referring to a reserve of fishing exploitation, initially the concepts referring to the theory of dynamic systems are exposed, such as its history, types of models and stability of the system. Then the qualitative analysis and the relationship with the bifurcation are exposed, later the fishery exploitation model is presented with the condensation of its parameters and the conversion to a dimensionless biparametric model, the relation between the dimensionless parameters is raised and the bifurcations of the system, then the relationship of the parameters with the system variable is represented and finally the conclusions are formulated.

Keywords: Biparametric bifurcation, fishing exploitation, dimensionless biparametric model

1 Introduction

Multiple disciplines of knowledge explore physical phenomena concerning their area of knowledge, it is of great importance to know their behavior in a certain time horizon. A classic way of describing a physical phenomenon and its change over time is from a dynamic system, which is a mathematical model composed of one or several differential equations, the analysis of these have a deterministic approach and allow to describe the evolution of the physical phenomenon through time [1].

The theory that studies these dynamic systems has its beginnings approximately in the XVII century, where Isaac Newton formulates that the way of knowing the future is to solve the differential equations that regulate the dynamic systems, and although he was one of the great precursors of the calculation, his affirmation would involve a discussion between philosophical positions such as determinism and probabilism, as well as covering a topic that has been of great concern: the knowledge of future events. Without taking into account the superstitions and traditions about the destiny of different peoples it is possible to see that there has always been a longing for such knowledge and that to a certain extent it is impossible to know, nevertheless, the assertion of Newton and his contributions in mathematics and physics gave room for knowing the future of some events [2].

From here different characters incurred in the modeling and solution of dynamic systems using differential equations. But there was a problem that caught the attention of the great mathematicians; the problem of the three bodies, which consisted in modeling through explicit equations the movement of the sun, the earth and the moon, and after several decades of failed attempts, in the nineteenth century Henri Poincaré, French mathematician, delivered remarkable contributions as was the "Map of Poincaré", application with which solved the problem of the three bodies, also could analyze deterministic systems that exhibit an aperiodic behavior dependent on the initial conditions, for which it was difficult to make long-term predictions. Since then, the theory of dynamic systems is developed and is extended to different fields of knowledge, mainly in mathematics, physics and engineering.

The analysis of dynamic systems expressly begins with the study of linear systems, however, most physical processes are non-linear, so the theory focuses on the latter, even so, a traditional way of solving these systems is not linear is linearized around a point of operation and although this procedure is not incorrect, it curtails the display of the behavior of the non-linear model. The solution of analytical form of a dynamic system covers numerous methods which have high difficulty and sometimes do not solve the problem, however, there are easy application techniques to solve nonlinear systems from a geometric analysis which provides more information than analytical techniques and allows to trace the trajectories of the system in an extensive timeline, this form of solution, qualitative, reveals the behavior of the system through an analysis of differential equations and a subsequent location of critical points of the system, and the stability of these. In the qualitative analysis, the behavior of the dynamic systems obeys its inherent parameters, and the change of these can create, destroy or change the stability of the critical points, this fact is known as bifurcation and it is of great importance to know the value of the parameter (critical value) for which this event occurs. This analysis is made from the technique known as flows in the lines and will be the basis of the solution for the fishing exploitation model of this document [3].

2 Dynamical systems and nonlinear model

Dynamic systems describe the behavior of a physical process using internal parameters. They are called systems because they are described by one or a set of differential equations and are dynamic because their parameters vary with respect to some variable that is usually time. The study of dynamic systems can be divided into particularly in 3 fields:

- Applied dynamics: Process modeling by means of state equations that relate past states to future states.
- Experimental dynamics: Laboratory experiments, computer simulations of dynamic models.
- Mathematics of the dynamics: It is focused on the qualitative analysis of the dynamic model.

In addition to fields, dynamic systems can be divided into continuous or discrete, autonomous or non-autonomous, linear or non-linear. The model worked on in this document is of an autonomous and non-linear continuous type.

2.1 Nonlinear models

The theory of dynamic systems always begins by explaining systems of linear type, however, in nature it is difficult to find physical processes that behave in this way, most are described by nonlinear models with behavior highly sensitive to initial conditions. Although there are many techniques to solve linear systems analytically, often the solution process is too cumbersome and sometimes does not reach a solution, so we resort to qualitative analysis and most often to numerical analysis [4].

2.2 Stability in dynamic systems

The way to visualize the behavior of the state variables of a dynamic system can be in the form of time series (state variable versus time), or in the form of phase space. The phase space of an n-dimensional system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is the space where all the possible states of a system are represented, each system parameter is represented as an axis of a multidimensional space and each point of the space represents each possible state of the variables of the system.

The phase space is described by a vector field F that governs the path of the variables of the system $x(t)$ in time, the path of these variables is called trajectory. As an example, the phase space of a dynamic system is shown, where singularities such as points and cycles can be observed that attract the trajectories that pass near them and others that repel them [1].

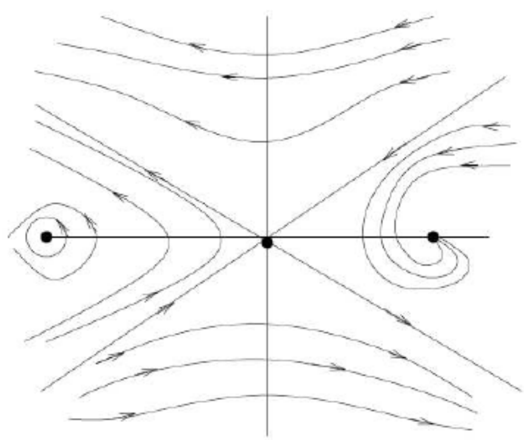

Figure 1: Phase space of a non-linear dynamic system.

2.3 Qualitative analysis and bifurcations

The qualitative analysis performed on a system $\mathbf{x}(\mathbf{t})$, in which a is a parameter of the system shows the nature of the fixed points (attractors, repulsors, etc.) analyzing only the value of x for which $f(x)$ is equal to zero (point fixed) as shown in Figure 2.

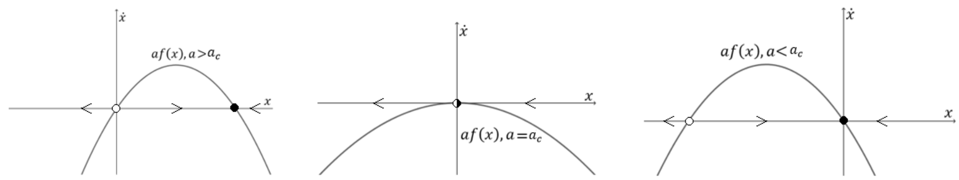

Figure 2: Qualitative analysis and bifurcations.

It can be observed in figure 2 that when a parameter is changed the fixed points also alter their position, in the value in which the parameter is equal to a critical value (a_c) such that the fixed points from being eliminated or change their stability, then it will be said that a bifurcation occurs there.

In the biparametric bifurcation, the parameters are redefined and condense into one, however, most of these analyzes only show the behavior of the system conditioned to the condensed values in a single parameter, preventing in this way having a more deep. In order to be able to observe more characteristics of the behavior of the system, we resort to the use of two parameters in the system, this practice also allows to visualize additional bifurcations that are not normally observed in a mono-parametric model.

2.4 Fishery exploitation model

Fisheries or fisheries are organized efforts to capture fish or other aquatic species through fishing. Currently, fisheries base their catches by 90% on maritime fish and 10% on fresh-aquaculture fish. Fisheries represent a large part of global trade and represent the livelihoods of millions of people and continue to be of cultural importance to many communities [2]. Because the stock of fish in a reserve is a variable that changes due to different conditions such as fishing and the size of the reserve, it is possible to model a dynamic system that represents fishing exploitation through the system proposed in (1).

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - f\left(N\right) \ , \quad f\left(N\right) = H\frac{N}{A+N} \tag{1}$$

The first term $rN(1-N/K)$ describes the logistic growth of the fish population in the absence of fishing or harvest, where r is the maximum growth rate and K is the capacity to contain fish in the system. The second term $f(N)$ gives the rate at which the fish is harvested as a function of population N. Here we must take into account that $f(N)$ is a monotonic increasing function of N. For the N populations that are much smaller than A, the harvest rate $f(N)$ is proportional to the N catch available, with the capture speed constant

H/A. For populations much larger than A, the harvest rate $f(N)$ is close to H. Therefore, H is the maximum rate at which the fish is harvested, even if an unlimited catch is available, while A is a measure of the fish population in which the fish is harvested, that is, $f(A) = H/2$. Before carrying out the bifurcation analysis, it is necessary to rewrite the system in a dimensionless way, as follows:

$$x = N/K$$
$$\frac{1}{rK}\frac{dx}{dt} = x\left(1-x\right) - \frac{H}{rK}\;\frac{x}{A/K\;+x}$$
$$\tau = rKt, \quad h = \frac{H}{rK}, \quad a = \frac{A}{K}$$

Here we must bear in mind that h, and A are dimensionless, finally the dimensionless model in (2).

$$\frac{dx}{d\tau} = x\left(1-x\right) - h\frac{x}{a+x} \tag{2}$$

The equilibrium points, when $dx/d\tau = 0$ are:

$$x_1 = 0, \quad x_{2,3} = \frac{1 - a \pm \sqrt{\left(a+1\right)^2 - 4h}}{2} \tag{3}$$

Here we have a saddle-node bifurcation in $4h = (a+1)^2$ with two solutions that appear for:

$$h < \frac{\left(a+1\right)^2}{4} \tag{4}$$

The solution $x_{2,3}$ has positive values if $a < 1$ and if $a > 1$ has negative values, which are not relevant to the problem. If $a \neq 1$, one of the solution branches of $x_{2,3}$ crosses the branch $x = 0$ and then there is a transcritical bifurcation in $h = a$.

From Eq.(2) when $dx/d\tau = 0$ and Eq.(4) the graphs that provide the information about the stability of the points dependent on the parameters of the system are observed [5].

The interpretation of the critical points obtained previously indicates that the stability in the population (availability) of fish will have a fixed point at 0, that is to say that the population is extinguished and continues in this way although the parameters are subsequently changed, if the parameter h it exceeds the relation $(a+1)^2/4$, that is, if the maximum harvest rate exceeds the rate of the population of harvest fish and regardless of the population, it

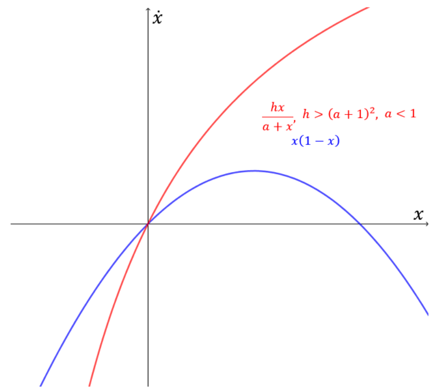

Figure 3: **Case I, fixed point, $\mathbf{x^* = 0}$**.

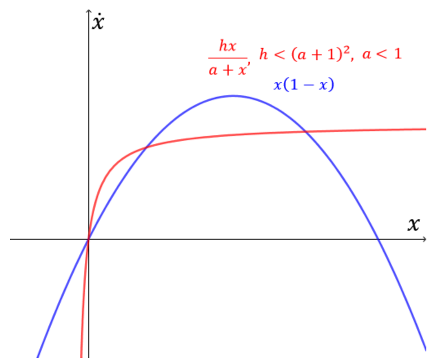

Figure 4: **Case II, three fixed points, $\mathbf{x_i^* \geq 0}$**.

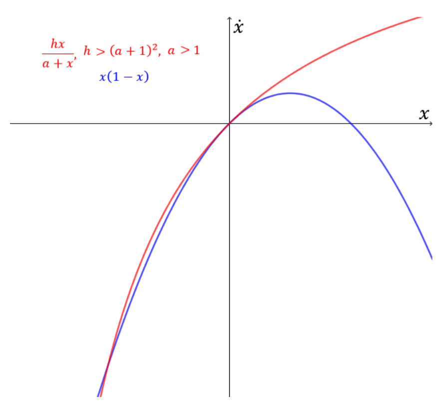

Figure 5: **Case III, three fixed points, $\mathbf{x_i^* \leq 0}$**.

will reach zero, something that is obvious and that is expressed in case I. On

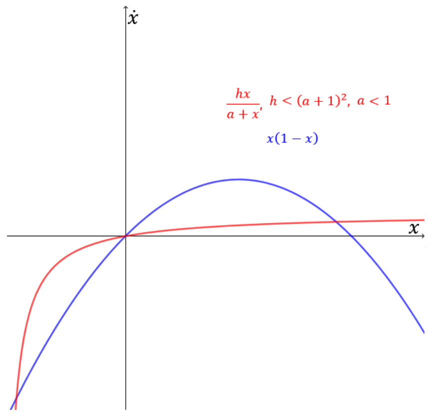

Figure 6: **Case IV, three fixed points, $\mathbf{x_1^* < 0}$, $\mathbf{x_2^* > 0}$, $\mathbf{x_3^* = 0}$.**

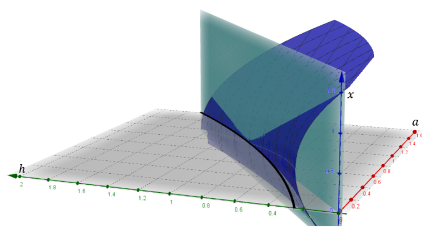

Figure 7: Three-dimensional scheme of the model.

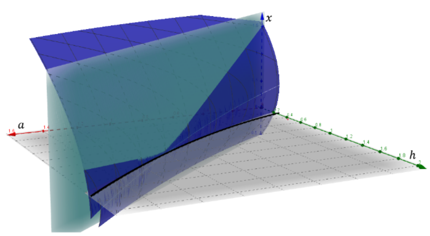

Figure 8: Three-dimensional scheme of the model.

the other hand, in case II there is an important event in which it is indicated that there will be a stable population greater than zero, if the parameter h does not exceed the rate of the population of harvest fish, while in case III you can see what happens the same as in case I, except that there is a fixed point with no physical sense for the problem since the population is negative. Finally, case 4 expresses that if the parameter h is less than the parameter a, a population of fish will always be available for any initial condition. The cases b, c, e, f show the bifurcations and evolution of the fixed points. In Figure 9 you can see the complete dynamics of the system.

3 Conclusion

The qualitative analysis of dynamic systems allows knowing their behavior thoroughly, simply by careful observation of the geometry of the inherent equations, thanks to this it is possible to see the conditions in which a bifurcation occurs. The qualitative analysis carried out on the fishing exploitation model made it possible to observe its dynamic behavior in depth. The consideration of two parameters for the bifurcation analysis provided the perception of three possible bifurcations in the system: Transcritical, saddle-node and fork, something that could not have been obtained if only a parameter was considered. With the results obtained it is possible to generate new models and standards for trade in fisheries, since with the data two situations can be avoided: that the population of fish is extinguished or that there is an overpopulation that causes a waste of the resource. In future investigations it is possible to use a multiparametric analysis that allows to deepen even more in the studies on dynamic systems, implementing on this or other models.

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