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Application of Finite Elements Considering Variation in Time on PDE

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Abstract

The model of the heat diffusion equation by various means shows the behavior of the variation of the temperature in a material (in principle uniform) due to the dynamics of this with respect to the time and the length of the material. This paper presents the development of the finite element method for the one-dimensional problem in order to apply it to the heat equation with a variant source over time, whose solution is a linear combination of linear test interpolation functions and a function that It depends on time. Finally, some operations of the application problem section are demonstrated to simplify the calculations of the model used in the results section and to attack a problem that includes the dynamics of the temperature variable with respect to time in a steel block of finite dimensions, such as a one-dimensional problem to be solved by means of the finite element method.

Keywords: Galerkin method, finite elements, heat equation, transfer by conviction

1 Introduction

The finite element method is a numerical method used to solve problems in the areas of engineering, mathematics and physics. The typical problems that have been solved in these areas of interest by means of the finite element method include structural analysis, heat transfer, fluid flow, mass transport and electromagnetic potential. For problems involving complex geometries, loads and non-isotropic properties of the materials, it is usually not possible to obtain an analytical solution. These mathematical expressions allow knowing the value of an unknown quantity at any point in the domain [1]. Computational techniques such as interpolation of linear bases and domain discretization, are known as the finite element method, which allows finding an approximation of an ordinary or partial differential equation in one, two or three dimensions. To solve the differential equations by means of the finite element method it is necessary to discretize the domain into uniformly spaced fragments depending on the domain, that is, linear elements in one dimension, triangles or quadrilaterals in two dimensions or tetrahedral elements of 5 or 8 nodes depending on the precision that is desired in three dimensions, consequently to the partition of the domain into fragments arises the need to find a set of linear equations that allow to build from the bases described above, a global function that characterizes the solution of the differential equation and this implies a directly proportional relationship between the precision and the quantity of linear coefficients. On the other hand, the finite element method is proposed as a tool, which is described in the following sections as an alternative to numerically solve ordinary and partial differential equations that model physics and engineering phenomena. The advantage of this method over others is the ability to adapt the finite elements in the form of the domain of interest even though there are discontinuities [2].

2 Application of finite elements

Consider the differential equation for a dimension that describes the flow of heat due to conduction, convection, or mass transfer of a fluid:

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+Q=\frac{\dot{m}c}{A}\frac{\partial T}{\partial x}+\frac{hP}{A}\left(T-T_{\infty}\right)+\rho c\frac{\partial T}{\partial t},$$

where, $\frac{\dot{m}c}{A}\frac{\partial T}{\partial x}$ refers to the transport of mass and the variable $\dot{m}$ is the rate of change of the mass flow; $\frac{hP}{A}\left(T - T_{\infty}\right)$ relates the interaction of the temperature of the medium with the study material [3]. Here, the dynamics of the temperature of the material, the interaction with medium (or means) and the source are modeled with $\rho c\frac{\partial T}{\partial t}$.

Once the differential equation modeling heat transfer with mass transfer is obtained, the equations will be formulated by means of the Galerkin residual representation to obtain the finite elements in a general way to arrive at the numerical solution of the partial differential equation. It is based on the following assumptions [4]:

- $Q = 0$
- Stable condition conditions, i.e., $\frac{\partial T}{\partial t} = 0$

The residual representation of Galerkin is given by:

$$R\left(T\right) = -\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\dot{m}c}{A}\frac{\partial T}{\partial x} + \frac{hP}{A}\left(T - T_{\infty}\right)$$

Then, we obtain

$$\int_0^L \left(-\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\dot{m}c}{A}\frac{\partial T}{\partial x} + \frac{hP}{A}\left(T - T_{\infty}\right)\right) N_i dx = 0$$

After integrating by parts, the following expression is obtained:

$$u = N_i$$

$$du = \frac{dN_i}{dx}dx$$

$$dv = -\frac{d}{dx}\left(\lambda\frac{\partial T}{\partial x}\right)dx$$

$$v = \ -\lambda\frac{\partial T}{\partial x}$$

$$\left(-\lambda\frac{\partial T}{\partial x}\right) N\Big|_{i0}^{L} - \int_0^L \left(-\lambda\frac{\partial T}{\partial x}\right)\left(\frac{dN_i}{dx}dx\right) + \int_0^L \left(\frac{\dot{m}c}{A}\frac{\partial T}{\partial x} + \frac{hP}{A}\left(T - T_{\infty}\right)\right) N_i dx = 0$$

$$\int_0^L \left(\lambda\frac{\partial T}{\partial x}\right)\left(\frac{dN_i}{dx}dx\right) + \int_0^L \left(\frac{\dot{m}c}{A}\frac{\partial T}{\partial x} + \frac{hP}{A}\left(T - T_{\infty}\right)\right) N_i dx = \left(\lambda\frac{\partial T}{\partial x}\right) N\Big|_{i0}^{L}$$

Now, if $\Omega \in [0, L]$ and are calculated for two elements, assume the answer as $T(x) = N_1 T_1 + N_2 T_2$, with $N_1 = 1 - \frac{x}{L}$ and $N_2 = \frac{x}{L}$, therefore, the term $\frac{\partial T}{\partial x}$ can be approximated as $\frac{dT}{dx} = \frac{T_2}{L} - \frac{T_1}{L}$.
After setting $N_i = N_1 = 1 - \frac{x}{L}$ and substituting all the approximations and solving the integrals, we have to:

$$\int_0^L \left(\lambda\left(\frac{T_2}{L} - \frac{T_1}{L}\right)\right)\left(\frac{-1}{L}\right)dx + \int_0^L \left(\frac{\dot{m}c}{A}\left(\frac{T_2}{L} - \frac{T_1}{L}\right)\right)\left(1 - \frac{x}{L}\right)dx +$$

$$\int_0^L \left(\frac{hP}{A}\left(\left(1 - \frac{x}{L}\right)T_1 + \left(\frac{x}{L}\right)T_2 - T_\infty\right)\right)\left(1 - \frac{x}{L}\right)dx$$

$$= \left(\lambda\left(\frac{T_2}{L} - \frac{T_1}{L}\right)\right)\left(1 - \frac{x}{L}\right)_0^L$$

Analyzing each one of the resulting terms of the previously mentioned integral one has to [5]:

$$\int_0^L \left(\lambda\left(\frac{T_2}{L} - \frac{T_1}{L}\right)\right)\left(\frac{-1}{L}\right)dx = \left(\lambda\left(\frac{T_2}{L} - \frac{T_1}{L}\right)\right)\left(\frac{-1}{L}\right)x\Big|_0^L = \lambda\left(\frac{T_1}{L} - \frac{T_2}{L}\right)$$

$$\int_0^L \left(\frac{\dot{m}c}{A}\left(\frac{T_2}{L} - \frac{T_1}{L}\right)\right)\left(1 - \frac{x}{L}\right)dx = \left(\frac{\dot{m}c}{A}\left(\frac{T_2}{L} - \frac{T_1}{L}\right)\right)\left(x - \frac{x^2}{2L}\right)\Big|_0^L$$

$$= \frac{\dot{m}c}{2A}(T_2 - T_1)$$

$$\int_0^L \left(\frac{hP}{A}\left(\left(1 - \frac{x}{L}\right)T_1 + \left(\frac{x}{L}\right)T_2 - T_\infty\right)\right)\left(1 - \frac{x}{L}\right)dx = \frac{hP(LT_1)}{3A} +$$

$$\frac{hP(LT_2)}{6A} - \frac{hP(LT_\infty)}{2A}$$

$$\left(\lambda\left(\frac{T_2}{L} - \frac{T_1}{L}\right)\right)\left(1 - \frac{x}{L}\right)_0^L = \lambda\left(\frac{T_1}{L} - \frac{T_2}{L}\right) = T_{x1}$$

$$\left(\frac{\lambda}{L} - \frac{\dot{m}c}{2A} + \frac{hP(L)}{3A}\right)T_1 + \left(-\frac{\lambda}{L} + \frac{\dot{m}c}{2A} + \frac{hP(L)}{6A}\right)T_2 = T_{x1} + \frac{hP(LT_\infty)}{2A}$$

For the second element, a procedure similar to the one done with the first element is followed, like this:

$$N_i = N_2 = \frac{x}{L}$$

$$\int_0^L \left(\lambda\left(\frac{T_2}{L}-\frac{T_1}{L}\right)\right)\left(\frac{1}{L}\right)dx+\int_0^L\left(\frac{\dot{m}c}{A}\left(\frac{T_2}{L}-\frac{T_1}{L}\right)\right)\left(\frac{x}{L}\right)dx+$$

$$\int_0^L\left(\frac{hP}{A}\left(\left(1-\frac{x}{L}\right)T_1+\left(\frac{x}{L}\right)T_2-T_\infty\right)\right)\left(\frac{x}{L}\right)dx=\left(\lambda\left(\frac{T_2}{L}-\frac{T_1}{L}\right)\right)\left(\frac{x}{L}\right)_0^L$$

$$\int_0^L \left(\lambda\left(\frac{T_2}{L}-\frac{T_1}{L}\right)\right)\left(\frac{1}{L}\right)dx=\lambda\left(\frac{T_2}{L}-\frac{T_1}{L}\right)\left(\frac{x}{L}\right)\begin{matrix}L\\0\end{matrix}=\lambda\left(\frac{T_2}{L}-\frac{T_1}{L}\right)$$

$$\int_0^L\left(\frac{\dot{m}c}{A}\left(\frac{T_2}{L}-\frac{T_1}{L}\right)\right)\left(\frac{x}{L}\right)dx=\frac{\dot{m}c}{A}\left(\frac{T_2}{L}-\frac{T_1}{L}\right)\left(\frac{x^2}{2L}\right)\begin{matrix}L\\0\end{matrix}=\frac{\dot{m}c}{2A}\left(T_2-T_1\right)$$

$$\int_0^L\left(\frac{hP}{A}\left(\left(1-\frac{x}{L}\right)T_1+\left(\frac{x}{L}\right)T_2-T_\infty\right)\right)\left(\frac{x}{L}\right)=\frac{hPLT_1}{6A}+\frac{hPLT_2}{3A}-\frac{hPLT_\infty}{2A}$$

$$\left(\lambda\left(\frac{T_2}{L}-\frac{T_1}{L}\right)\right)\left(\frac{x}{L}\right)_0^L=\lambda\left(\frac{T_2}{L}-\frac{T_1}{L}\right)=T_{x2}$$

$$\left(-\frac{\lambda}{L}-\frac{\dot{m}c}{2A}+\frac{hPL}{6A}\right)T_1+\left(\frac{\lambda}{L}+\frac{\dot{m}c}{2A}+\frac{hPL}{3A}\right)T_2=T_{x2}+\frac{hPLT_\infty}{2A}$$

The terms T_{x1} is the temperature defined in element 1 and the term T_{x2} is the temperature defined in element 2, matrix rewriting the results obtained previously, it must:

$$\left[\frac{\lambda}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{\dot{m}c}{2A}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}+\frac{hPL}{6A}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right]\begin{bmatrix}T_1\\ T_2\end{bmatrix}=\frac{hPLT_\infty}{2A}\begin{bmatrix}1\\ 1\end{bmatrix}+\begin{bmatrix}T_{x1}\\ T_{x2}\end{bmatrix}$$

Consider a particular type of problem with time dependence, where the values at the boundary are an oscillating source around a value. For example, the heating power per unit volume can vary with respect to time according to the equation $h(t)=h_0Re\left(e^{i\omega t}\right)=h_0\cos(\omega t)$, where h_0 is the amplitude and $Re\left(e^{i\omega t}\right)$ is the real component of the quantity $\left(e^{i\omega t}\right)$. The partial differential equation that models a problem with variation in time (periodicity) involves the time variable implicitly, and for the heat conduction equation we have $(f=-\lambda\nabla T)$. Now, the simplified model adopts the form

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+Q=\frac{\dot{m}c}{A}\frac{\partial T}{\partial x}+\frac{hP}{A}\left(T-T_\infty\right)+\rho c\frac{\partial T}{\partial t}$$

Rewritten from the point of view of the Laplacian operator, we have:

$$\frac{\partial}{\partial x}\left(-\lambda\frac{\partial T}{\partial x}\right)+\rho C\frac{\partial T}{\partial t}=\frac{hP}{A}\left(T-T_{\infty}\right)$$

We seek to find a solution that involves the complex amplitude as expressed in the solution (variant test function in time) [6]:

$$T=T_0\left(r\right)e^{i\omega t}=\left[T_{0r}\left(r\right)+iT_{0i}\left(r\right)\right]\left[cos(\omega t)+isen(\omega t)\right]$$

Therefore,

$$T_r=T_{0r}\cos\left(\omega t\right)-T_{oi}sen(\omega t)$$

$$T_i=T_{0r}sen(\omega t)+T_{oi}\cos\left(\omega t\right)$$

Their respective partial derivatives with respect to time are:

$$\frac{\partial T_r}{\partial t}=-\omega\left[T_{0r}sen(\omega t)+T_{oi}\cos\left(\omega t\right)\right]=-\omega T_i$$

$$\frac{\partial T_i}{\partial t}=\omega\left[T_{0r}\cos\left(\omega t\right)-T_{oi}sen(\omega t)\right]=\omega T_r,$$

where,

$$T=T_r+iT_i$$

Substituting $T=T_r+iT_i$ and $\frac{\partial T}{\partial t}=-\omega T_i+i\omega T_r$in the partial differential equation gives a system of two partial differential equations, one equation refers to the real terms and the other equation relates the imaginary part as shown below:

$$\nabla\cdot(-\lambda\nabla T_r)-\omega\rho C_pT_i=h_0$$

$$\nabla\cdot(-\lambda\nabla T_i)+\omega\rho C_pT_r=0$$

$$\frac{\partial}{\partial x}\cdot\left(-\lambda\frac{\partial T_r}{\partial x}\right)-\omega\rho C_pT_i=h_0$$

$$\frac{\partial}{\partial x}\cdot\left(-\lambda\frac{\partial T_i}{\partial x}\right)+\omega\rho C_pT_r=0$$

Finally, we get

$$T(x,t)=\sqrt{{T_r}^2+{T_i}^2}\cos\left(\omega t+tan^{-1}\left(\frac{T_i}{T_r}\right)\frac{180}{\pi}\right)$$

We assume that the properties of the materials are independent of the variable T (isotropic materials), the parameters used to calculate the numerical solution of the system of differential equations previously raised are: length of the steel bar $Lx = 0.5u$, insulation temperature $h_0 = 0$, specific chloric capacity C_p, angular frequency $\omega = 0.025r/s$, thermal conductivity $\lambda = 45$, material mass density ρ, with $\rho C_p = 3 \cdot 10^6$. Although the properties can vary with the spatial coordinate even with small deformations or discontinuities, for small temperature oscillations, the equation $T(x,t)$ is a valid approximation, therefore, the system of partial differential equations is linear [7].

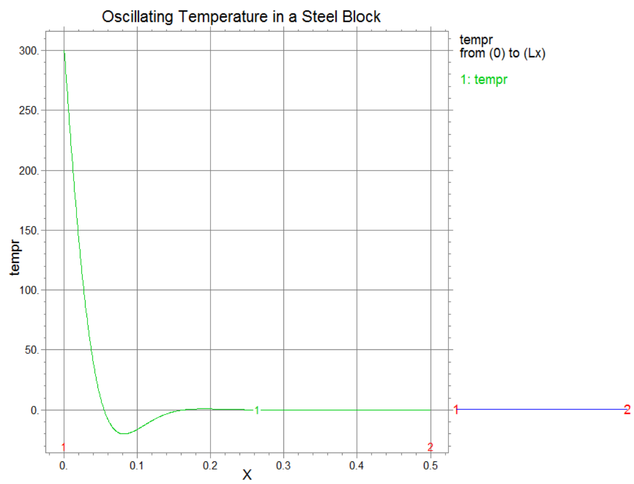

Figure 1: Real component Tr, numerical solution.

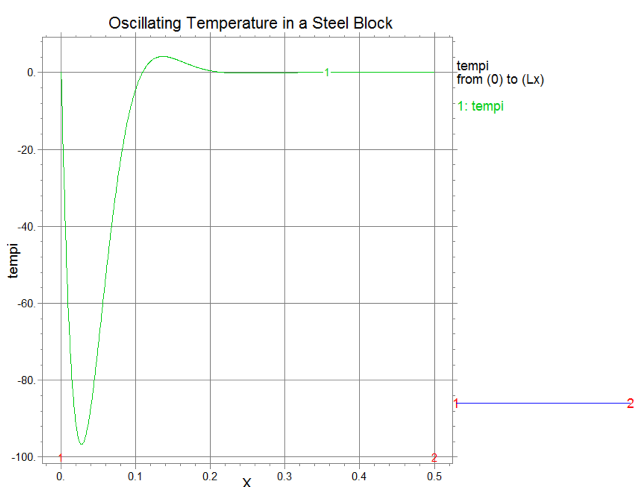

Figure 2: Real component Ti, numerical solution.

3 Numerical results

The problem arises with an isolated steel block, except on the front face, where the temperature oscillates around $300K$. There are two dependent variables, the real and imaginary components of the temperature T, for each one of the variables contour conditions are established. An oscillating temperature source is imposed on the left side as the real part of T (numerical solution figure 1). Therefore, oscillations at any point of the steel block (imaginary part of the temperature variable, graph of Figure 2) refer to variations with angular deviation due to the nature of the proposed solution, ie the part real and imaginary generate an angular deviation as a function of the location (domain), since the attenuation of the temperature wave is extremely strong, we need logarithmic diagrams to present the variation.

4 Conclusion

Temperature waves are a natural phenomenon that can be modeled as was done in this document. From the climatic point of view, the phenomenon occurs as the sun heats up more intensely at some times of the year compared to colder times, that is, the sun heats more intensely in summer than in winter and as a result of these changes a temperature wave is presented. Comparing this phenomenon with the obtained results it is possible to deduce that the temperature variations in the surface of the earth are attenuated as the wave enters more and more the materials that make up the layers of the earth as shown in the figure 1.

The results obtained in this document allow to know the value of the temperature in a steel block with isotropic properties at any point of the domain of this and at any time, being able to appreciate the attenuation of the temperature variable as a relation of complex variables and inversely proportional to the distance where you want to analyze the variable, that is, if the point of interest is further away from the source variant in time then this point will have a lower temperature associated with other points that are located closest to the source.

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