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Method of the Images in Problems of Electromagnetism

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Abstract

The present article shows the analysis of a technique of the electromagnetic theory used to simplify some problems that occurs in different fields and performs the study of the applications with the technical technique, which is known as the method of images.

Keywords: Method of the images, electromagnetism, potential

1 Introduction

The method of images is a fundamental concept of the theory of electromagnetism and is widely used to solve problems that generally arise in the areas of engineering, physics and mathematics. It consists of a certain limited region, the solution for the potential is unique once we know the boundary conditions. By placing additional charges outside the region of interest, we can sometimes reproduce the desired limit, condition, which allows us to easily write the solution [1].

The image method (or mirror image method) is a mathematical tool for solving differential equations, in which the domain of the function sought is extended by the addition of its mirror image with respect to a hyperplane of symmetry. As a result, certain boundary conditions are automatically satisfied by the presence of a mirror image, greatly facilitating the solution of the original problem. The domain of the function is not extended. The function is made to satisfy certain boundary conditions by placing singularities *outside* the domain of the function. The original singularities are within the domain of interest. The additional singularities (fictitious) are an artifact necessary to satisfy the prescribed border conditions, but still unsatisfied [2,3].

2 Modeling the problem - Electric dipole

For the study of the proposed method, the study of an electric dipole is initially proposed, as shown in Figure 1.

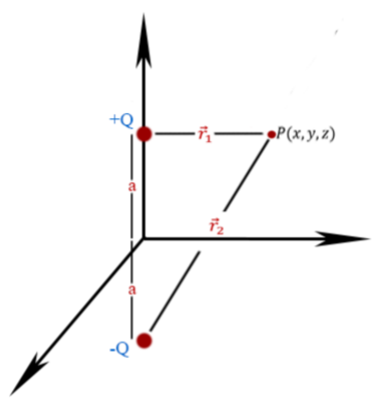

Figure 1: Electric dipole.

For the electric dipole shown in figure 1, the scalar potential at point $P(x, y, z)$ is given by the expression:

$$v(x, y, z) = \frac{k_e Q}{r_1} + \frac{k_e(-Q)}{r_2}$$

$$v\,(x.y.z) = k_e Q \left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

The distances (and magnitudes) of the charges to the point can be written as shown below.

$$\vec{r}_1 = x\hat{a}_x + y\hat{a}_y + (z-a)\,\hat{a}_z$$
$$r_1 = \sqrt{x^2 + y^2 + (z-a)^2}$$

$$\vec{r}_2 = x\hat{a}_x + y\hat{a}_y + (z+a)\,\hat{a}_z$$
$$r_2 = \sqrt{x^2 + y^2 + (z-a)^2}$$

2.1 Conditions

- The XY plane is an equipotential surface with potential equal to zero (Condition I). Mathematically it can be written as [4]: In the palne $XY \;\rightarrow\;\; z = 0,\; r_1 = r_2$ and $v\,(x,y,0) = 0$

  Field: $E\,(x,y,z) = -\nabla v = \frac{k_e Q}{r_1}\vec{r}_1 + \frac{k_e(-Q)}{r_2}\vec{r}_2$

  If $R = (x-x_0)\,\hat{a}_x + (y-y_0)\,\hat{a}_y + (z-z_0)\,\hat{a}_z$

$$\nabla\left(\frac{1}{r}\right) = -\frac{R}{R^3}$$

  In particular, in the XY plane, we have:

$$E\,(x,y,0) = \frac{k_e Q}{r_1^3}\left[x\hat{a}_x + y\hat{a}_y - a\hat{a}_z - \left(x\hat{a}_x + y\hat{a}_y + a\hat{a}_z\right)\right]$$

$$E\,(x,y,0) = -\frac{2ak_eQ}{(x^2+y^2+a^2)^{\frac{3}{2}}}$$

- In the XY plane, the field lines arrive perpendicularly (Condition II).

  The two conditions mentioned above in the electric dipole problem are sufficient to study the method of images in different electromagnetic problems. Next, the analysis of two common problems is analyzed analyzing them from their equivalent representation for the image method [5].

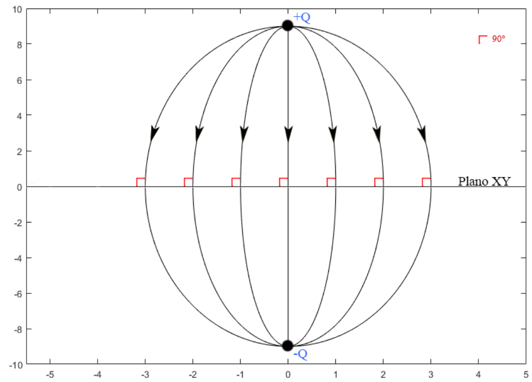

Figure 2: Condition of field lines.

2.2 Conductors in electrostatic equilibrium

$$E_{internal} = 0, \qquad v_{internal} = constant$$

The external field (E_{ext}) is normal (perpendicular) to the surface and of magnitude $\frac{\sigma}{\epsilon_0}$($\sigma$ Z is the surface density of charge)

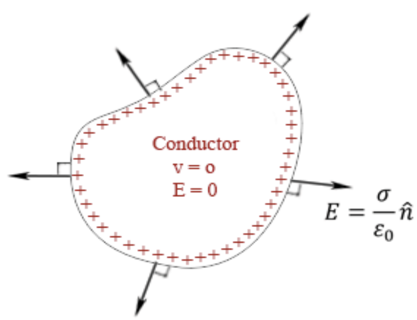

Figure 3: Conductors in electrostatic equilibrium.

The load is more concentrated in the areas of greater curvature and the field lines come out perpendicular to the surface, but never cross (deviate). The problem of the dipole, for the potential and the field in the plane, fulfills these conditions for the conductors [6].

3 equivalent problem

A problem equivalent to the previous one (conductor) can be a point load in front of a large conductor plane at zero potential, represented in figure 4.

which can be represented as illustrated in figure 5.

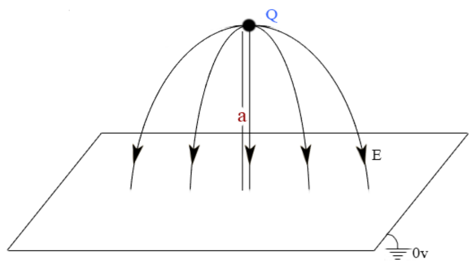

Figure 4: Load in front of the plane.

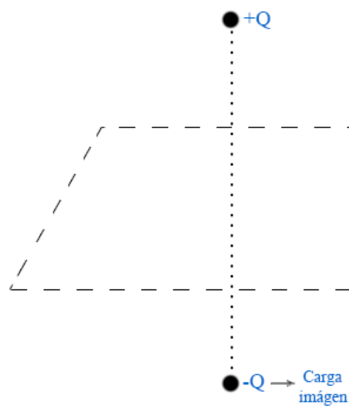

Figure 5: Equivalent.

The method of the images proposes that it is possible in certain cases to replace the conductor with one or more point charges, so that the conductive surfaces are replaced by equipotential surfaces. The calculations that are made are valid on the side of point loads, that is, outside the conductors [7].

The previous problem, then is to find, for the charge Q at $(0, 0, a)$ against the XY plane at potential 0, the potential v, the field E and the total charge induced in the plane, making use of the image method. For the load induced in the plane, you have to:

$$E_{plane} = \frac{-2ak_e Q\hat{a}_z}{\left[x^2+y^2+a^2\right]^{\frac{3}{2}}} = \frac{\sigma}{\varepsilon_0}\hat{a}_z,$$

and it is known as the field on the surface of a conductor. Therefore, starting from the previous expression, we can raise the expression for the surface charge density as:

$$\sigma\left(x, y\right) = \frac{-2ak_e\varepsilon_0 Q\hat{a}_z}{\left[x^2+y^2+a^2\right]^{\frac{3}{2}}} = \frac{dQ}{ds}$$

Therefore, the induced load is given by:

$$Q_{ind} = \int_s \sigma ds = -\frac{aQ}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dxdy}{(x^2 + y^2 + a^2)^{\frac{3}{2}}}$$

Now the plane is considered as a very large disk and the integral is performed in polar coordinates,

$$Q_{ind} = -\frac{aQ}{2\pi} \int_{-0}^{2\pi} \int_{0}^{\infty} \frac{rdrd\theta}{(r^2 + a^2)^{\frac{3}{2}}}$$

Then we get,

$$u = \sqrt{r^2 + a^2}$$

$$u^2 = r^2 + a^2 \qquad \rightarrow \qquad 2udu = 2rdr$$

$$si\ r \rightarrow 0, \qquad u \rightarrow a$$
$$si\ r \rightarrow \infty, \qquad u \rightarrow \infty$$

$$Q_{ind} = -aQ \int_a^{\infty} \frac{udu}{u^3} = aQ\ \frac{1}{u}\bigg|_a^{\infty} = aQ \left[0 - \frac{1}{a}\right]$$
$$Q_{ind} = -Q = Q_{image}$$

Finally, the potential for any point can be found by the equation that was determined in the study of the electric dipole,

$$v\,(x.y.z) = k_e Q \left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

4 Rectangular plane versus conductive sphere

For this case the equivalent problem of images is given by three radio spheres with charges opposite to the initial sphere as the configuration shown in the figure.

The expression for the electric field can be obtained by the Gaussian law.

$$\iint \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

For the sphere,

$$E\,(r) * 4\pi r^2 = \frac{4\pi R^2 \sigma}{\varepsilon_0}$$

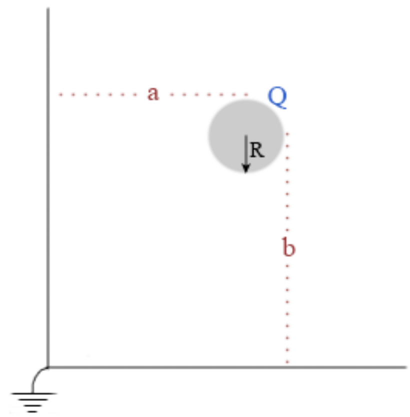

Figure 6: Conductive sphere in front of rectangular plane.

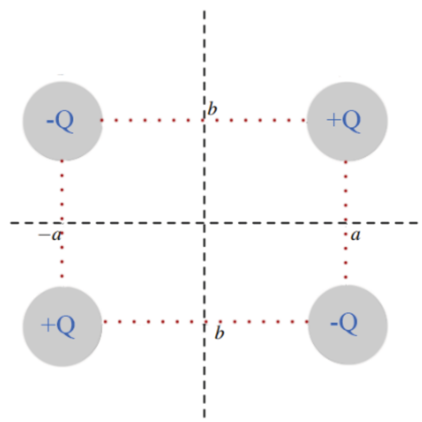

Figure 7: Equivalent representation.

$$\vec{E} = \frac{\sigma R^2}{\varepsilon_0 r^2}\hat{r}$$

The previous expression can be generalized even more when the sphere is not centered, as is the case of study, where there are 4 decentered spheres. Said expression is shown below.

$$\vec{E} = \frac{\sigma R^2(\vec{r} - \vec{r}_i)}{\varepsilon_0 \left|\left|\vec{r} - \vec{r}_i\right|\right|^3}$$

For the shown configuration the distances are shown in table 1. The total value of the field for any point r_i, is obtained using the principle of superposition.

i	$\vec{r}_i = (x_i, y_i)$
1	(a, b)
2	$(-a, b)$
3	$(-a, -b)$
4	$(a, -b)$

Table 1: Distances of spheres

$$\vec{E}(x, y) = \vec{E}(x_1, y_1) + \vec{E}(x_2, y_2) + \vec{E}(x_3, y_3) + \vec{E}(x_4, y_4)$$

$$\vec{E}(x, y) = \frac{\sigma R^2}{\varepsilon_0} \sum_{n}^{4} \frac{(-1)^{n-1}(\vec{r} - \vec{r}_i)}{||\vec{r} - \vec{r}_i||^3}$$

Finally, the field is determined for each of the sides of the rectangular plane, which correspond to the axes in the equivalent problem, the following expressions are obtained for each of the sides.

$$\vec{E}(0, y) = \frac{\sigma R^2}{\varepsilon_0} \left[\frac{(-2a, 0)}{\left(a^2 + (y - b)^2\right)^{\frac{3}{2}}} + \frac{(2a, 0)}{\left(a^2 + (y + b)^2\right)^{\frac{3}{2}}} \right]$$

$$\vec{E}(x, 0) = \frac{\sigma R^2}{\varepsilon_0} \left[\frac{(0, -2b)}{\left((x - a)^2 + b^2\right)^{\frac{3}{2}}} + \frac{(0, 2b)}{\left((x - a)^2 + b^2\right)^{\frac{3}{2}}} \right]$$

5 Conclusion

The method of the images is of great help when you want to solve certain problems that can be represented and meet the conditions that were raised for the electric dipole. There are many other imaging problems, each of which involves replacing a conductor with an imaginary charge (or charges) that mimics the electric field in some region (but not everywhere). With the technique that is studied it is much easier to perform the analysis of these problems.

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