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Optimum Control Using Finite Time Quadratic Linear Regulator

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Abstract

In the present paper a brief introduction to the theory of optimal control and its constructive form is exposed, in addition the analysis of a particular controller is made within this theory known as the linear quadratic regulator (LQR) of finite time. Finally, the application of the LQR of finite time to the second order model of a synchronous machine is shown.

Keywords: Control quadratic, optimal control, linear regulator

1 Introduction

The optimal control refers to the optimization of dynamic systems. Dynamic systems represent processes that evolve over time and that can be characterized by states $x(t)$ that allow us to predict the future behavior of the system. If the initial state is not known, it must be estimated based on the available measurement information. The estimation process is often based on optimization and uses the same optimization methods that are used for optimal control. In optimal control, these controls must be chosen appropriately to optimize (minimize or maximize) some objective function and respect some restrictions.

For optimal control you can think of an airplane where the state x consists of the current position and speed, and where the control is the power of the engine that the pilot of the aircraft can choose at any time. We could consider the movement of the plane in a time interval and the objective could be to minimize the energy consumption to fly from airport A to airport B, and one of the limitations would be that the plane should arrive at airport B in the final fixed time t_f.

The mathematical statement of the optimal control problem consists of:

- A description of the system to be controlled.
- A description of the limitations and possible alternatives of the system.
- A description of the task to be developed.
- A statement of the criteria for judging optimal performance.

If time is not a concern for the available control problem, we can use an infinite time LQR with less computational effort. But if the control problem needs a very fast response and precision, the finite time LQR is a necessity and carries with it the cost of loading the computational effort.

2 Dynamical systems - Optimal control

A dynamic system is one that evolves over time, but time can come in two variants: while physical time is continuous and forms the natural environment for most technical and biological systems, other dynamic systems can be better modeled in discrete time, such as digitally controlled sample data systems or games. We call a system a discrete time system every time the time in which the system evolves only takes values in a predefined time grid, it is generally assumed that they are integers. If we have a range of real numbers, as for physical time, we call it a continuous time system [2]. Most systems of interest in science and engineering are described in the form of differential equations that live in continuous time.

Now, we propose a system of the form

$$\dot{x}\left(t\right)=Ax\left(t\right)+Bu\left(t\right); \quad x\left(0\right)=0$$

$$y\left(t\right)=Cx\left(t\right),$$

where

$x(t)$: vector of states
$u\left(t\right)$: control vector
$x\left(0\right)$: initial state known

The control problem is reduced to finding or choosing the trajectories $u^{*}\left(t\right)$ with $0\leq t\leq t_f$, which optimizes the cost function [3]:

$$J=\phi\left(x\left(t_f\right),t_f\right)+\int_0^{t_f}L(x\left(t\right),u\left(t\right),t)dt$$

Functions $\phi\left(x\left(t_f\right),t_f\right)$ and $L(x\left(t\right),u\left(t\right),t)$ are scalar and generally nonlinear, ϕ guarantees that in the final time $t_f$the state $x(t)$ approaches an objective state and L ensures that no excessive efforts are made at $0\leq t\leq t_f$, to reach optimality.

There are different cost functions for the optimal control systems, and the use of each of them depends on the problem being addressed, within these are the cost function for the minimum time, for the minimum fuel problem, for the minimum energy problem of quadratic index, etc. For the problem that is dealt with within this document, the quadratic index function is used, used in LQR control problems.

2.1 Finite time LQR

The performance criterion, denoted J, is a measure of the quality of the performance of the system. Usually, we try to minimize or maximize the performance criteria by selecting the control input. For each $u(t)$ a system path $x(t)$ is associated. This quadratic performance criterion can be expressed in general form as [4]:

$$J=\phi\left(x\left(t_f\right),t_f\right)+\int_0^{t_f}L(x\left(t\right),u\left(t\right),t)dt,$$

where $L(x(t),u(t),t)$ is known as the function that penalizes the states and control actions in the intermediate states and $\phi\left(x\left(T\right),T\right)$ is the function that penalizes the final state. The control problem consists of choosing control trajectories $u^*(t)$, with $0 \leq t \leq T$, where T is the final time and can be fixed, in such a way that minimizes the cost function J. The functions $L(x(t), u(t), t)$ and $\phi(x(T), T)$, are real, scalar and generally non-linear [1].

$$J = \frac{1}{2} x^T(t_f) P x(t_f) + \frac{1}{2} \int_0^{t_f} \left[x^T(t) Q(t) x(t) + u^T(t) R(t) u(t) \right] dt$$

The matrices P and Q are symmetric and not defined negative, while R is symmetric and positive definite. A symmetric matrix $M \in R^{n \times n}$ is positive definite if $x^T M \geq 0 \; \forall x \neq 0; x \in R^n$, and not defined negative if $M \in R^{n \times n}$ is positive definite if $x^T M \geq 0 \; \forall x \neq 0; x \in R^n$. A symmetric matrix is positive defined (not defined negative) if and only if all its eigenvalues are positive (not negative).

To solve the problem of optimal control, the Lagrange multipliers are used and the Hamiltonian is defined.

$$H(x(t), u(t), \lambda(t)) = \frac{1}{2}(x^T(t) Q(t) x(t) + u^T(t) R(t) u(t) + \lambda^T(t)(A(t) x(t) + B(t) u(t))$$

The cost function can be written as:

$$J = \frac{1}{2} x^T(t_f) P x(t_f) + \frac{1}{2} \int_0^{t_f} \left[H(x(t), u(t), \lambda(t)) - \lambda^T(t) \cdot \dot{x}(t) \right] dt$$

Using the principle of variations [7], the necessary conditions to determine optimality are presented, which are:

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x(t)} = -Q(t) x^*(t) - A^T(t) \lambda^*(t)$$

$$\dot{x}(t) = \frac{\partial H}{\partial \lambda(t)} = A(t) x^*(t) + B(t) u^*(t)$$

$$0 = \frac{\partial H}{\partial u(t)} = R(t) u^*(t) + B^T(t) \lambda^*(t)$$

From these equations we obtain the expressions for the control law $u^*(t) = -R^{-1}(t) B^T(t) \lambda^*(t)$ and the dynamic continuous equation of Ricatti with contour condition $P(t)$.

$$-\dot{P} = A^T P + PA + PA - PBR^{-1}B^T P + Q; \qquad T_0 \leq t \leq t_f$$

Finding the values of p, is then to solve the previous system of differential equations, and known matrix P, we have that the law of control in closed loop is obtained by the expression:

$$u^*\left(t\right) = -R^{-1}B^TP = -K\left(t\right)x^*(t),$$

where $K(t)$ is known as the state feedback matrix, variable with time [5].

2.2 Second order model of the synchronous machine

To perform an optimal control analysis applied to real systems, the non-linear second order model of a synchronous machine represented by the system of equations is taken:

$$M\ddot{\delta} = P_m - D\dot{\delta} - \eta_1 E_q \sin\delta$$

$$\tau \dot{E}_q = -\eta_2 E_q + \eta_3 \cos\delta + E,$$

where δ is the angle of the rotor in radians, E_q is the generator voltage in $p.u$, D is the damping coefficient, E is the input voltage, p_m is the mechanical power, M is the rotor inertia coefficient, τ is a time constant and, η_1, η_2, η_3 are constant parameters related to internal characteristics of the generator. Table 1 presents the value of all the parameters that characterize the model of the machine [6].

P_m	0.815
E	1.22
η_1	1.22
η_2	2
η_3	1.7
τ	6.6
M	0.0147
D	0.0588

Table 1: Parameters of the synchronous machine.

Now a linearization of the previously proposed model around a point of operation is performed as shown below.

$$\dot{x}_1 = \Delta x_2$$
$$\dot{x}_2 = \frac{1}{M}(-\eta_1 \cos x_{10} * \Delta x_1 - D * \Delta x_2 + P_m),$$

where x_{10} and x_{20} are the operating point values of the state variables, which are $0.3840 rad/s$ and $0 rad/s$ respectively. The linear equations of the linearized model are:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{-\eta_1\cos x_{10}}{M} & -\frac{D}{M}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{P_m}{M}\end{bmatrix}$$

$$A = \begin{bmatrix}0 & 1\\ \frac{-\eta_1\cos x_{10}}{M} & -\frac{D}{M}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -76.95 & -4\end{bmatrix}$$

3 Analysis and results

The LQR model presented was applied to the proposed system of the synchronous machine and the following results were obtained:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -76.95 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$x\left(0\right) = \begin{bmatrix}0.384\\ 0\end{bmatrix}$$

Here, the objective function is

$$J = \frac{1}{2}x^T\left(t_f\right)\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}x\left(t_f\right) + \frac{1}{2}\int_0^{t_f}\left[x^T\left(t\right)\begin{bmatrix}q_1 & 0\\ 0 & q_2\end{bmatrix}x\left(t\right) + ru(t)^2\right]dt$$

$$P = \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix},\quad Q = \begin{bmatrix}q_1 & 0\\ 0 & q_2\end{bmatrix},\quad R\left(t\right) = r$$

For an optimal solution $P\left(t_f\right) \geq 0, \quad Q\left(t\right) \geq 0, \quad R\left(t\right) > 0$. From Ricatti's dynamic continuous equation we obtain

$$-\begin{bmatrix}\dot{p}_1 & \dot{p}_2\\ \dot{p}_2 & \dot{p}_3\end{bmatrix} = \begin{bmatrix}0 & 1\\ -76.9 & -4\end{bmatrix}\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix} + \begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}\begin{bmatrix}0 & 1\\ -76.9 & -4\end{bmatrix} -$$

$$\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\frac{1}{r}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix} + \begin{bmatrix}q_1 & 0\\ 0 & q_2\end{bmatrix}$$

The design parameters are:

$$Q = \begin{bmatrix}80 & 0\\ 0 & 1\end{bmatrix},\quad R = 10,\quad t_f = 10$$

We obtain

$$-\begin{bmatrix}\dot{p}_1 & \dot{p}_2\\ \dot{p}_2 & \dot{p}_3\end{bmatrix} = -\begin{bmatrix}-\frac{p_2^2}{10} - 75.95p_2 + 80 & p_1 + p_2\left(\frac{-p_3}{10} - 4\right) + p_3\\ -76.95p_1 + p_2\left(\frac{-p_3}{10} - 4\right) - 76.95p_3 & -75.95p_2 - \frac{p_3^2}{10} - 8p_3 + 1\end{bmatrix}$$

The system of differential equations is solved by using the $ODE45$ function presented by the Matlab software. For the gain K of the control signal you have to:

$$K = R^{-1}B^TP$$

$$K = \frac{1}{r}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix} = \frac{1}{r}\begin{bmatrix}p_2 & p_3\end{bmatrix}$$

In addition, the graphs of the input and control signals for the specified finite time are generated, which are shown below.

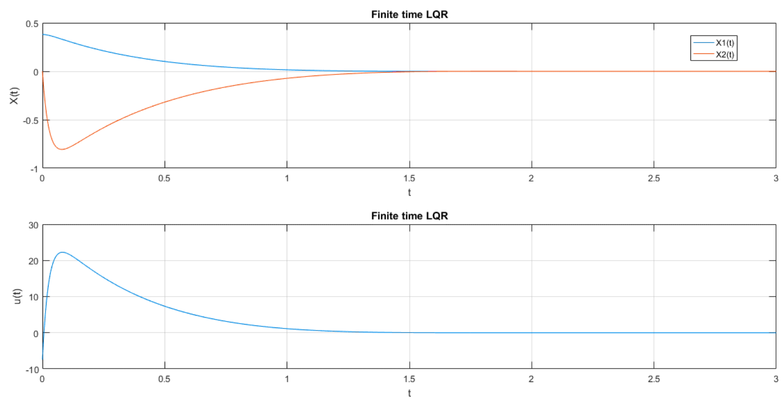

Figure 1: LQR Controller for finite time.

4 Conclusion

When optimizing processes or tasks that meet certain systems, optimal control offers a very good possibility of doing so and with very good results, depending on the application and area of study, various cost functions are presented that adapt to each task in specific. For the system shown in this document, with the help of the optimal control, the quadratic index cost function and the design parameters, a good response of the control system is obtained with a gain in the acceptable control signal.

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