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Poincaré-Bendixson Criterion for the Analysis of Periodic Orbits in the Behavior of Neurotransmitters

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Abstract

In this paper different criteria are presented to determine the existence or non-existence of periodic orbits for planar systems. In addition, an example of a model of the behavior of two types of neurons is shown, which can be characterized in a certain way by using the Poincaré-Bendixson criterion.

Keywords: Stability, Poincaré-Bendixon, periodic orbits, neurotransmitters

1 Introduction

The use of differential equations has allowed us to describe a large number of dynamic systems of physics, mathematics, mechanics, electromagnetism and medicine in a quantitative and qualitative way. However, there are still many questions to be resolved in this wide area as are differential equations [1].

In this work we focus on the verification of the existence or non-existence of periodic orbits by using different criteria, studying a very particular dynamic system which describes the oscillatory behavior present in two types of neurons, an exciter and an inhibitor. The dynamic systems for which the criteria for the existence of periodic orbits will be established, will be autonomous systems described by ordinary differential equations, mathematically, if Ω is an open and connected region of $\mathbb{R}^n$, and $f:\ \Omega \rightarrow \mathbb{R}^n$ is a continuous function, then

$$\dot{x} = f\left(x\right), \quad x \in\ \Omega. \tag{1}$$

Thus, we say that the system is an ordinary differential system. Also, if it exists and then the system (1) will be a problem of initial value [2].

Definition. *For the system (1), it is said that a point $p \in \Omega$ is a fixed point if $\phi_t(p) = p$, $\forall t \in \mathbb{R}$ is satisfied. Now, it is said that a solution $\phi_t(x)$ is periodic if $\phi_T(x) = x$ with $T > 0$.*

For the case of study in this document planar systems ($\mathbb{R}^2$) will be treated, since the existing theory is only valid for this type of systems, nevertheless some advances have been achieved for higher order systems. We will show some criteria later to characterize or define the non-existence of periodic solutions for a system in the plane. Then we will present the criteria to show that certain systems in the plane have presence of periodic orbits. Finally, we will present a model for which the existence of a periodic orbit can be verified once the value of the parameters involved is fixed [3].

2 Criteria to discard periodic orbits

Assuming that a system of the form (1) can be described by:

$$\dot{x} = -\nabla V(x), \tag{2}$$

where, $V : U \rightarrow \mathbb{R}$ with $U \subseteq \mathbb{R}^2$ is a function C^2 and $\nabla = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right)$

Theorem 1. *Periodic orbits are impossible in gradient systems $\dot{x} = -\nabla \mathrm{V}(x)$.*

Theorem 2. *(Bendixson-Dulac criterion). Let $\dot{x} = f(x)$ be a continuously differentiable vector field in a simply connected $D \subset \mathbb{R}^2$ domain, if there is a scalar function $g(x)$ of class C^1, such that $\nabla \cdot (g(x)\dot{x})$ is not identically zero and does not change the sign in D, then the system (1) does not have periodic orbits.*

2.1 Lyapunov functions

If it is possible to find an energy function that decreases along all trajectories, called the Lyapunov function, then closed orbits will be prohibited. mathematically for the system (1) with equilibrium point x^* and assuming that there is a function of Lyapunov $V(x)$ of class C^1 that complies with the following properties [4]:

- $V(x^*) = 0 \quad and \quad V(x) > 0, \ \ \forall\ x \neq x^*$
- $\dot{V}(x) < 0, \ \ \forall\ x \neq x^*,$

then x^* is an asymptotically stable global equilibrium point, that is, for any initial condition all trajectories converge to the fixed point, therefore there are no periodic orbits.

3 Criteria for the existence of periodic solutions

A work perhaps more useful than the previous one is to find if in a certain system there is presence of periodic orbits, which is achieved by using the following criteria.

Liénard type systems. A second order differential equation is of the Liénard type if it has the form:

$$\ddot{x} + f(x)\,\dot{x} + g(x)\,x = 0 \tag{3}$$

Eq.(3) can be written as a state space system, such as:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) - f(x)y \end{aligned} \tag{4}$$

where according to the type of functions $f(x)$ and $g(x)$, said system dynamics can be described as a limit cycle around the origin in the phase space, for this, the previous functions must satisfy the Linard theorem [5].

Theorem 3. *(Liénard). The system (4) has a single stable limit cycle around the origin if the functions $f(x)$ and $g(x)$ meet the following conditions.*

- $f(x)$ y $g(x)$ must be continuously differentiable for all values of x.
- $g(-x) = -g(x)$ for all x, that is, it must be an odd function.
- $g(x) > 0$ for all $x > 0$.
- $f(-x) = f(x)$ for all x.
- The odd function $F(x) = \int_0^x f(u)du$ has exactly one zero at $x = a$, being negative for $0 < x < a$, positive and not decreasing for $x > a$ and $F(x) \to \infty$ when $x \to \infty$.

Theorem 3. *(Poincaré-Bendixson Criterion). Let M be an open subset of $\mathbb{R}^2$ and $f \in C^1(M, \mathbb{R}^2)$. Let's fix a point $x \in M$, $\sigma \in \pm$ and suppose that $\omega_\sigma(x) \neq \emptyset$ is compact, connected and only has a number (finite at most) of fixed points. Then one of the following cases occurs:*

1. $\omega_\sigma(x)$ *it is a fixed orbit.*
2. $\omega_\sigma(x)$ *it's a regular periodic orbit.*
3. $\omega_\sigma(x)$ *is a set (finite as much) of fixed points $\{x_j\}_{j=1}^n$ and a single closed orbit $\gamma(x)$ such that $\omega_\pm \in \{x_j\}_{j=1}^n$.*

4 Numerical results

This section is dedicated to the demonstration of a periodic orbit present in activity of two types of neurons, an exciter and an inhibitor, which can be modeled by a second-order system whose general form is given by [6]:

$$\frac{dx_i}{dt} = -c_i x_i + \sum_{j=1}^{n} W_{ij} g_j(x_i), \text{ with } i = 1, 2, \ldots, n, \tag{5}$$

where g_j is the activation function of unit j, c_i is the relaxation coefficient, W_{ij} is the weight of the connection between unit j and unit i and x_i is the activity of the unit.

For the particular case of a two-neuron oscillator, we have the following model:

$$\begin{aligned} \dot{x}_1 &= -\tfrac{1}{\tau}x_1 + \tanh(\beta x_1) - \tanh(\beta x_2) \\ \dot{x}_2 &= -\tfrac{1}{\tau}x_2 + \tanh(\beta x_1) + \tanh(\beta x_2), \end{aligned} \tag{6}$$

where $\tau > 0$ is a characteristic time constant and $\beta > 0$ is the amplification gain. Now, to determine the existence of periodic orbits of the system (6), we start by calculating its fixed points, which are given by:

$$0 = -\frac{1}{\tau}x_1 + \tanh(\beta x_1) - \tanh(\beta x_2)$$
$$0 = -\frac{1}{\tau}x_2 + \tanh(\beta x_1) + \tanh(\beta x_2),$$

from which the following system of equations is obtained, called Nullclinales.

$$x_2 = -x_1 + 2\tau\tanh(\beta x_1)$$
$$x_1 = x_2 - 2\tau\tanh(\beta x_2),$$

for all possible values of τ and β the origin will be the only fixed point of the system, this can be verified by observing the figure 1 and 2, where the nullclinales for $(\tau, \beta) = (1, 0.5)$ and $(\tau, \beta) = (1, 2)$ and respectively are shown, and the only point of cut of these it is point $(0, 0)$. Now to determine the

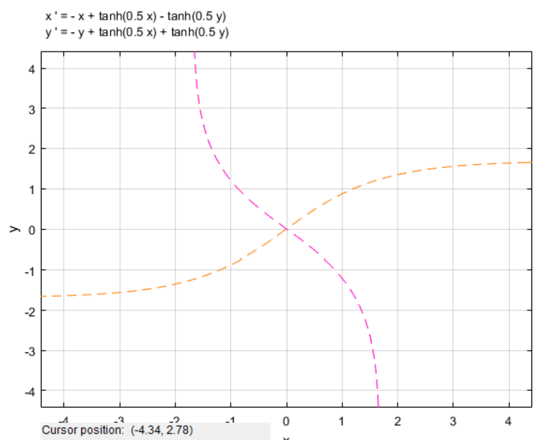

Figure 1: Nullclines of the system 6 with $\tau = 1$, $\beta = 0.5$.

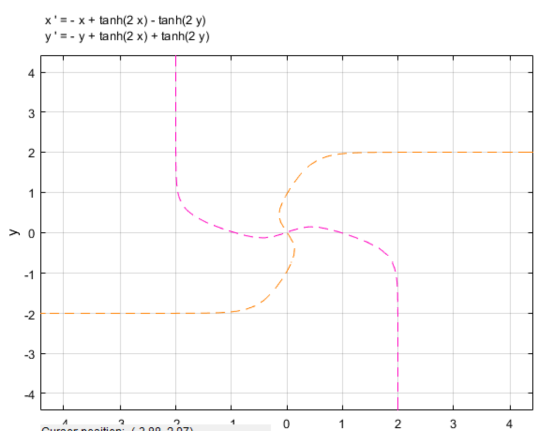

Figure 2: Nullclines of the system 6 with $\tau = 1$, $\beta = 2$.

nature of the fixed point found, we linearize around this, where we have:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau} + \frac{\beta}{\cosh^2(\beta x_{10})} & -\frac{\beta}{\cosh^2(\beta x_{20})} \\ \frac{\beta}{\cosh^2(\beta x_{10})} & -\frac{1}{\tau} + \frac{\beta}{\cosh^2(\beta x_{20})} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2, \end{bmatrix}$$

where $(x_{10}, x_{20}) = (0, 0)$, therefore the system around the fixed point will be:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau} + \beta & -\beta \\ \beta & -\frac{1}{\tau} + \beta \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Therefore, their eigenvalues are given by:

$$\lambda_{1,2} = \frac{\tau\beta - 1}{\tau} \pm i\beta$$

It can be noted that for $\tau\beta < 1$ we have that the fixed point $(0, 0)$ will be stable, while for $\tau\beta > 1$ the point will be an unstable focus. This change [7] in system dynamics can be seen as a type of bifurcation where the bifurcation parameter is $\tau\beta = \mu$.

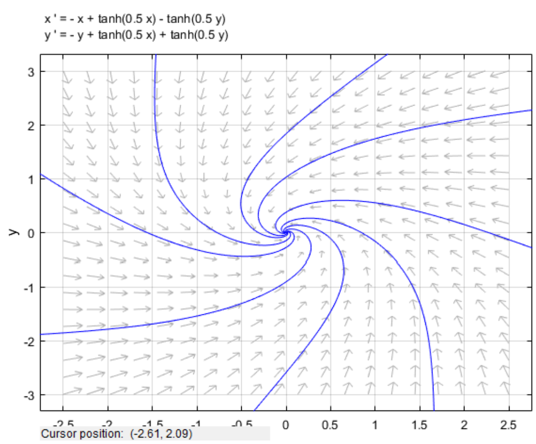

Figure 3: Stable equilibrium point.

5 Conclusion

The determination of periodic orbits for flat systems, can be developed in a "simple" way by applying the criteria mentioned in this work, however, for higher order systems this task of looking for the existence of periodic solutions is still a field study. It is important in some way to be able to characterize the orbits of the system, since they will allow to establish conclusions about solutions whose initial conditions are close to the trajectories identified. The phenomenon of bifurcation is very common in dynamic systems, since these will always depend on parameters that, due to physical conditions, will change with the evolution of time. Finally, although there is a vast theory about the existence of periodic orbits for the two-dimensional case, much remains to be investigated, such as being able to arrive at the solution to Hilbert's problem 16, which basically consists of being able to determine how many solutions isolated periodic may appear for the case of polynomial systems.

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