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New Exact Solutions for a Two-Mode KdV Equation with Higher-Order Nonlinearity

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Abstract

A generalized two-mode KdV equation with higher-order nonlinearity is considered. New periodic and soliton solutions are obtained for special case by using the improved tanh-coth method. Many other cases can be considered. From the generalized model, we can obtain new exact exact solutions for some particular models, which include the classical KdV equation with higher-order nonlinearity, the classical KdV equation as well as other important models used in several branch of sciences and engineering.

Keywords: Exact solutions; two-mode KdV equation; higher-order nonlinearity; Improved tanh-coth method

1 Introduction

It is well known that the origin of the Korteweg-de Vries equation (KdV) is related to the observations made by the naval engineer Scott-Russell in 1984 [1] when he was experimenting with a boat on a canal. Then, the investigations of Bussinesq and Rayleigh appeared around 1870 [2][3] with the aim to give explications on those observations. However, the preliminary investigation on the KdV equation ended with the work of Korteweg and de-Vries in 1895 [4]. Today is well known that the one dimensional propagation of an isolated wave mode in nonlinear dispersive system which exhibits several different modes is governed by the KdV equation which reads

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1}$$

where $u = u(x,t)$ denote the wave surface in a position x and time t. However, the wave phenomena that involve several wave modes simultaneously have attracted recently the attention of the investigators. A preliminary approximation on this direction was given by the so called two-mode KdV equation (TMKdV) which describes the propagation of two different wave modes in the same direction in a simultaneously way with the same dispersion relation [5][6][7][8],

$$\begin{cases} u_{tt} + (c_1 + c_2)u_{xt} + c_1c_2u_{xx} + \\ ((\alpha_1 + \alpha_2)\frac{\partial}{\partial t} + (\alpha_1c_2 + \alpha_2c_1)\frac{\partial}{\partial x})uu_x + \\ ((\beta_1 + \beta_2)\frac{\partial}{\partial t} + (\beta_1c_2 + \beta_2c_1)\frac{\partial}{\partial x})u_{xxx} = 0, \end{cases} \tag{2}$$

being the coefficients c_i, α_i, β_i, $i = 1,2$ constants and $\alpha_i \geq -1$, $\beta_i < 1$, $i = 1,2$, c_1 and c_2 the respective velocities, α_1, α_2 the nonlinearities, and β_1, β_2 the dispersion parameters. Studies relative to (2) and its solutions can be find in the references [5][6][7][8] and [9].

The following is a generalization of (2), which contains one term with higher-order nonlinearity

$$\begin{cases} U_{TT} + (c_1 + c_2)U_{XT} + c_1c_2U_{XX} + \\ ((\alpha_1 + \alpha_2)\frac{\partial}{\partial T} + (\alpha_1c_2 + \alpha_2c_1)\frac{\partial}{\partial X})U^pU_X + \\ ((\beta_1 + \beta_2)\frac{\partial}{\partial T} + (\beta_1c_2 + \beta_2c_1)\frac{\partial}{\partial X})U_{XXX} = 0, \end{cases} \tag{3}$$

Clearly, in the case of $p = 1$, the model (3) reduces to (2) studied recently in the reference [9] considering variable coefficients, where some important cases was derived, in particular (2). Other important case derived from (3) is

$$U_T + \sigma U^pU_X + \delta U_{XXX} = 0, \tag{4}$$

called a KdV equation with higher order nonlinearity, from which, we can obtain (1) and the following equations

$$\begin{cases} U_T + k_1U^2U_X + U_{XXX} = 0, \\ U_T + k_1U^pU_X + U_{XXX} = 0, \end{cases} \tag{5}$$

which have applications in several branch of sciences, particulary in fluid mechanics (see [10] and references therein). Exact solutions to (4) was constructed in the reference [11] in the case of variable coefficients.

Now, using the following transformations,

$$\begin{cases} x = (\beta_1 + \beta_2)^{-\frac{1}{2}}(X - c_0 T), \quad c_0 = \frac{c_1 + c_2}{2}, \quad t = (\beta_1 + \beta_2)^{-\frac{1}{2}} T, \\ \\ u = (\alpha_1 + \alpha_2)^{\frac{1}{p}} U, \end{cases} \tag{6}$$

and constrains

$$c_1 > c_2, \quad |\frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}| \leq 1, \quad |\frac{\beta_2 - \beta_1}{\beta_1 + \beta_2}| \leq 1 \tag{7}$$

then, (3) can be reduced to [5][6][7],

$$u_{tt} - \sigma^2 u_{xx} + (\frac{\partial}{\partial t} + A\,\sigma \frac{\partial}{\partial x}) u^p u_x + (\frac{\partial}{\partial t} + B\,\sigma \frac{\partial}{\partial x}) u_{xxx} = 0, \tag{8}$$

with

$$\sigma = \frac{c_1 - c_2}{2}, \quad A = \frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}, \quad B = \frac{\beta_2 - \beta_1}{\beta_1 + \beta_2}. \tag{9}$$

Our main goal in this paper, is to obtain exact traveling waves solutions for the two-mode KdV equations with higher order nonlinearity given by (8) for a special case, using the improved tanh-coth method.

2 Exact traveling wave solutions for Eq. (8)

There exists several computational methods to obtain exact solutions for nonlinear partial differential equations, however, we will use the improved tanh-coth method presented by the authors in [11], which is a very good generalization of others methods used in this direction. Following the steps of the method, first we consider the transformation

$$\begin{cases} u(x,t) = v(\xi), \\ \xi = x + \lambda t + \xi_0, \end{cases} \tag{10}$$

where, ξ_0 is an arbitrary constant and λ the wave speed. Using (10) then (8) reduces to

$$\big[\lambda^2 - \sigma^2\big] v'' + \big[\lambda + A\sigma\big] p v^{p-1} (v')^2 + \big[\lambda + A\sigma\big] v^p v'' + \big[\lambda + B\sigma\big] v'''' = 0, \tag{11}$$

where $v = v(\xi)$, $v' = v'(\xi) = \frac{dv}{d\xi}$. To apply the method, we need to make the transformation

$$v(\xi) = V(\xi)^{\frac{2}{p}}. \tag{12}$$

So that, using (12), (11) converts to

$$\begin{cases} (\lambda^2-\sigma^2)(4p^2-2p^3)V^2(V')^2+(\lambda+A\sigma)(4p^2+2p^3)V^4(V')^2+ \\ (\lambda+B\sigma)(16-48p+44p^2-12p^3)(V')^4+2(\lambda^2-\sigma^2)p^3V^3V''+ \\ 2(\lambda+A\sigma)p^3V^5V''+(\lambda+B\sigma)(48p-72p^2+24p^3)V(V')^2V''+ \\ (\lambda+B\sigma)(12p^2-6p^3)V^2(V'')^2+(\lambda+B\sigma)(16p^2-8p^3)V^2V'V'''+ \\ 2(\lambda+B\sigma)V^3V''''=0. \end{cases} \tag{13}$$

Second, we seek solutions to (13) using the expansion

$$V(\xi)=\sum_{i=0}^{M}a_i\phi(\xi)^i+\sum_{i=M+1}^{2M}a_i\phi(\xi)^{M-i}, \tag{14}$$

where M is a positive integer to be determinate later using the balance method, a_i constants to be determinate and $\phi(\xi)$ function which satisfy the Riccati equation

$$\phi'(\xi)=\alpha+\beta\phi(\xi)+\gamma\phi(\xi)^2, \tag{15}$$

whose solution reads [13]

$$\phi(\xi)=\frac{-\sqrt{\beta^2-4\alpha\gamma}\tanh[\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\xi]-\beta}{2\gamma},\ \ \beta^2-4\alpha\gamma\neq 0, \tag{16}$$

being α and β and γ constants.

We determine M using the balance method, so that, we substitute (14) into (13) and balancing V^5V'' with $VV''(V')^2$ we obtain $5M+M+2=M+M+2+2(M+1)$, so that, $M=1$. Then, (14) reduces to

$$V(\xi)=a_0+a_1\phi(\xi)+a_2\phi(\xi)^{-1}. \tag{17}$$

We substitute (17) into (13) taking into account (15). After simplifications, and equaling to zero the coefficients of the respective powers of $\phi(\xi)$ that appear (after multiplying by $p^4V^{\frac{4p-2}{p}}$) we obtain a very big algebraic system. The system is not very easy of solve in general form using the *Mathematica* software (my computer ran out of memory). However, taking into account (7) and (9) we can consider the particular case $A=1$, $B=1$. For this case, we can solve the system for the unknowns a_i, $(i=0,\ldots,2)$, α, β, γ and λ. We obtain the solution

$$\begin{cases} a_0=0, \quad \beta=0, \quad \alpha=\mp\frac{pa_2}{\sqrt{2}\sqrt{-2-3p-p^2}}, \\ \\ \gamma=\mp\frac{pa_1}{\sqrt{2}\sqrt{-2-3p-p^2}}, \quad \lambda=\sigma-\frac{8a_1a_2}{2+3p+p^2}. \end{cases} \tag{18}$$

Respect to this set of solutions (16) reduces to

$$\phi(\xi) = \frac{\sqrt{\frac{p^2 a_1 a_2}{p^2+3p+2}}\tanh[\frac{\sqrt{2}}{2}\sqrt{\frac{p^2 a_1 a_2}{p^2+3p+2}}\xi]}{\frac{2pa_1}{\sqrt{-p^2-3p-2}}}. \tag{19}$$

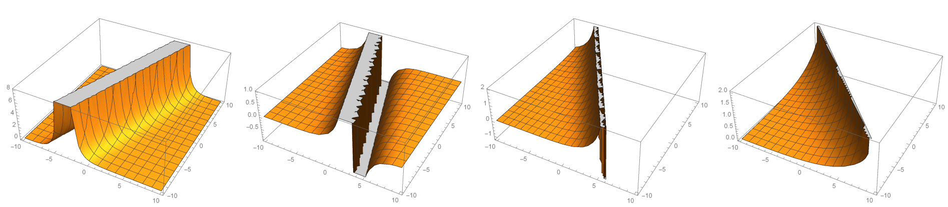
In accordance with (10),(12) and (19), we have that the solution to (8) take the form

$$u(x,t) = v(\xi) = V(\xi)^{\frac{2}{p}} = \left[a_1\phi(\xi) + a_2\phi(\xi)^{-1}\right]^{\frac{2}{p}}, \tag{20}$$

where $\phi(\xi)$ is given by (19), $\xi = x + (\sigma - \frac{8a_1a_2}{2+3p+p^2})t + \xi_0$ and a_1, a_2 arbitrary constants.

3 Results and Discussion

We have obtained exact traveling wave solutions for the two mode KdV equation (8), which contains a nonlinear term of higher-order. Clearly, solutions for (4) and (5) can be derived. On the other hand, reversing the transformations given by (6) we can obtain solutions for initial two-mode KdV equations with higher-order nonlinearity (2). Again, as in the reference [9], we can to use variable coefficients, so that we can derive particular cases such as (4), but now, with variable coefficients, and therefore, to obtain solutions for instance for the important model $u_t + k_1t^nuu_x + k_2t^mu_{xxx} = 0$ as a particular case. The system derived from applications of the improved tanh-coth method it is not easy to solve in a general form due to the memory of my computer is not sufficient, so that we have considered the particular case $A = 1$, $B = 1$. Clearly, this implies several values for α_1, α_2, β_1, β_2 according with (7). Many other values can by assigned to A and B with the aim to solve the system, however, for the sake of simplicity, we have considered only the previous. The figure 1 ($|\ u(x,t)\ |$), correspond to graphic of the solution (19) for several values of p, with $a_1 = 1$, $a_2 = 1$, $\sigma = 1$, $x \in [-10, 10]$ and $t \in [-10, 10]$:



(a) $p = 1$ (b) $p = 2$ (c) $p = 3$ (d) $p = 9$

Figure 1: $|u(x,t)|$.

Finally, varying the parameter ξ_0 and depending of the sign of $\beta^2 - 4\alpha\gamma$ we can obtain other type of solutions such as rational an periodic solutions, as can be seen in [13].

4 Conclusion

We have derived solitary wave solutions for the two-mode KdV equation with higher order nonlinearity (8) for the special case $A = 1$ $B = 1$. The solutions are of type solitons, periodic or rational solutions according with [13]. Clearly, solutions for (2) can be derived. Many other values for A and B can be considered, however due to availability of computational memory, each case must be considered individually. We have showed the graphs of some of the solutions for some values of p. Following the steps made in the reference [9], all calculations made in this work, can be considered with variable coefficients.

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