Building the Graphics Memory of the Stiffness Matrix of the Beam

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Abstract

This paper represents an essential departure from the conventional techniques to obtain the stiffness matrix of a beam. It has two main distinguishing features: 1) The formulation of the stiffness matrix can under certain conditions be superimposed using the superposition principle; 2) Application of Newton's Third Law (action-reaction) to build the graphics memory of the stiffness matrix of the beam. The beam model, as well as parameters, of the proposed problem is extensively investigated in order to get the most accurate results.

Keywords: Superposition Principles, Finite Elements, Structural Analysis, Displacement Method, Stiffness Matrix, Energy, Formulation, Global Coordinate System, Graphics Memory

1. Introduction

Finite-element technology and application of matrix methods are very powerful and impressive structural analysis tools and were created for articulated frameworks with truss, beams and frame elements. More advanced matrix methods, referred to as finite element analysis, modeling entire structures with one-, two-, and three-dimensional members were achieved later and can be used for articulated systems together with continuous systems such as a pressure vessel, plates, shells, and three-dimensional solids. In general, two properties are required to describe a beam: 1. Beams are commonly subjected to transverse loading, which will create bending in the beam. Beams are defined as a structural element whose cross – sectional dimensions is relatively smaller than its length.
For structural element that is considered as a beam, loads may be applied anywhere along the beam and the loading will create bending in the beam. Moreover, we can simplify the derivation of stiffness matrices. For example, if we evaluate the global displacement of each node. The term force may be used in its most fundamental sense and can refer for example, to a Moment, M producing a rotation, \( \theta \) as shown in. Differentiation of the strain energy with respect to the nodal degrees of freedom leads to the formulation of the frame’s stiffness matrix; the deflection of neutral axes is also related to the internal bending moment \( M(x) \), the transverse shear \( F_{i2} \) or \( F_{j8} \) and by computing the frame’s reaction forces. Consider a frame of length \( L \), as shown in Figure 4, and figure 5. The reaction forces and moments at the endpoints are also shown in the figure. The stiffness matrix for the beam element will be 4x4, the computation deformations, deflections, to present a formal unification of force and displacement methods using energy \([2]\), to determine internal forces and moments within structures. Structural analysis need input data such as loads, the beam’s geometry with a set of corresponding boundary and initial conditions. The model was created with measured data of variables selected on the basis of theoretical considerations of beam element \([3]\).

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The global displacements: \( U_{i2}, U_{i6} \) at node i and \( U_{j8}, U_{j12} \) at node j. In a similar way, the global forces may be related according to the global forces: \( F_{i2}, M_{i6} \) at node i and \( F_{j8}, M_{j12} \) at node j. The term force may be used in its most fundamental sense and can refer for example, to a Moment, M producing a rotation, \( \theta \) as shown in Figure 2.

2. The Method

2.1. Development of the study

Developing an application which involves the finite element method (procedure that can be applied to obtain solutions to a variety of problems in engineering) requires the same methodology and ingenuity as usual in engineering: one must clearly identify and analyze the application, gather data, evaluate costs, and check the statistical validity of results. Usually, the analysis phase will end up with an architecture composed of successive modules, built with a graphics memory in four acts; the stiffness matrix for a beam element with two degrees of freedom at each node: the vertical displacement and rotation: See Figure 3.
Building the graphics memory of the stiffness matrix of the beam

These steps help build a graphics memory the stiffness matrix for a beam element with two degrees of freedom at each node: the vertical displacement and rotation. To enliven and focus the exposition these will be organized as four acts of a play, properly supplemented with prologue, Newton's Third Law (Action/Reaction), two graphics with vertical displacement - two graphics with rotation angle, and three stiffness matrices illustrated graphically with beams. Here is the program [2].

2.2. Procedure

The procedure is referring to Figure 2. First, fixed node \( j \) and node \( i \) free. Note that \( u_{i2} \), represent the vertical displacement and \( u_{i6} = \theta_{i6} \) (local coordinates) represent the rotation at node \( i \), respectively. In the same manner, \( u_{j8} \) and \( u_{j12} = \theta_{j12} \) represent the lateral (vertical) displacement, and the rotation at node \( j \), respectively, when \( i \) is joint fixed end.

Differentiation of the strain energy with respect to the nodal degrees of freedom leads to the formulation of the beam’s stiffness matrix; the deflection of neutral axes is also related to the internal bending moment \( M_{(x)} \), the transverse shear \( F_{i2} = f_{i2} \) or \( F_{j8} = f_{j8} \) and by computing the beam’s reaction forces [2]. Consider a beam of length \( L \), as shown in Figure 2. The reaction forces and moments at the endpoints are also shown in the figures 3 and 4. The calculation of reactions at these artificially restrained ends. Differentiation of the strain energy with respect to the joint DOF leads to the formulation of the beam’s stiffness matrix. We begin minimizing the strain energy with respect to \( F_{i2} \) and \( F_{i6} = M_{i6} \) (moment at node \( i \)) and \( F_{j8} \) and \( F_{j12} = M_{j12} \) (moment at node \( j \)) to obtain the stiffness matrix [3], [7], [8], and [9]. Starting with the strain energy part of the total potential energy, we get.
\[ \frac{\partial U}{\partial F_k} = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial F_k} \, dx \] (1)

Now that you know what we mean by a beam element, proceed with derivation of stiffness matrix. In the following, we neglected the contribution of the flexion stresses to the strain energy.

2.2.1. Build the graphics memory of the stiffness matrix of the beam

Stiffness matrix of a beam: To Build a Graphics Memory in Four Acts. These steps help build a graphics memory the stiffness matrix for a beam element with two degrees of freedom at each node: the vertical displacement and rotation [7].

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**Act I:** Free node i and fixed node j – Vertical displacement \( u_{i2} \) at node i and fixed node j.

**Act II:** Fixed node i and Free node j – Vertical displacement \( u_{j8} \) at node j and fixed node i.

**Act III:** Free node i and fixed node j – Rotational displacement \( u_{i6} = \theta_{i6} \) at node i and fixed node j.

**Act IV:** Fixed node i and Free node j – Rotational displacement \( u_{j12} = \theta_{j12} \) at node j and fixed node i.

We now consider how the discussion of the preceding section may shed some light on the problem of implementing planned behavior by a pattern of motor commands (Action-reaction). A number of investigations have suggested that actions are planned in endpoint coordinates. To some extent, this is a rather obvious concept that does not require much experimental testing. It is clear, for example, that we can mentally formulate and execute a command such as “trace the shape of an arc with right hand”.

Without being concerned about the set of neurons that are involved in this behavior. However, once we have decided to trace an arc. In carrying out this task, our brain must face the challenges associated with (two degrees of freedom at each node) [1].
The purpose of this section is to show how a proper representation of dynamics and of coordinate transformations leads to a straightforward solution to the problems associated [2].

\[ E, G = \text{longitudinal elastic and shear moduli.} \]

\[ I = \text{second moment of inertia of cross-section with respect to neutral axis.} \]

\[ A = \text{shear resisting area.} \]

\[ \mathbf{u} = \begin{bmatrix} u_{i2} & u_{i6} & u_{j8} & u_{j12} \end{bmatrix}^T = \begin{bmatrix} u_{i2} & \theta_{i6} & u_{j8} & \theta_{j12} \end{bmatrix}^T \] (2)

Differentiation of the strain energy with respect to the nodal degrees of freedom leads to the formulation of the beam’s stiffness matrix; the deflection of neutral axes is also related to the internal bending moment \( M_i(x) \) [2] and [6], the transverse shear \( F_{i2} = f_{i2} \) or \( F_{j8} = f_{j8} \) and by computing the beam’s reaction forces. Consider a beam of length \( L \), as shown in Figure 5. The reaction forces and moments at the endpoints are also shown in the figures.

The calculation of reactions at these artificially restrained ends. Differentiation of the strain energy with respect to the joint DOF leads to the formulation of the beam’s stiffness matrix. We begin minimizing the strain energy with respect to \( F_{i2}, \) and \( F_{i6} = M_{i6} \) (moment at node \( i \)) and \( F_{j8}, \) and \( F_{j12} = M_{j12} \) (moment at node \( j \)) to obtain the stiffness matrix [7], [8], and [9]. Starting with the strain energy part of the total potential energy, we get, see equation (1)

Now that you know what we mean by a beam element, we will proceed with derivation of stiffness matrix. In the following, we neglected the contribution of the flexion stresses to the strain energy.

**Act I:** Free node \( i \) and fixed node \( j \) – Vertical displacement \( u_{i2} \) at node \( i \) and fixed node \( j \).
\[
\frac{\partial U}{\partial F_i} = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial F_i} \, dx
\]

(3)

Action:

\[ f_{i2} = \frac{12EI}{L^3} u_{i2} \]  \hspace{1cm} (4)

\[ M_{i6} = \frac{6EI}{L^2} u_{i2} \]  \hspace{1cm} (5)

Reaction:

\[ f_{j8} = -\frac{12EI}{L^3} u_{i2} \]  \hspace{1cm} (6)

\[ M_{j12} = \frac{6EI}{L^2} u_{j8} \]  \hspace{1cm} (7)

Vertical displacement matrix at node i.

\[
\begin{pmatrix}
    u_{i2} & u_{i6} & u_{j8} & u_{j12} \\
    \frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]  \hspace{1cm} (8)

Figure: 3 Vertical displacement at node i.
Act II: Fixed node i and Free node j – Vertical displacement $u_{j8}$ at node j and fixed node i.

$$\frac{\partial u}{\partial F_{j8}} = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial F_{j8}} \, dx$$  \hspace{1cm} (9)

Action:

$$f_{i2} = -\frac{12EIz}{L^3} u_{j8}$$  \hspace{1cm} (10)

$$M_{i6} = -\frac{6EIz}{L^2} u_{j8}$$  \hspace{1cm} (11)

Reaction:

$$f_{j8} = \frac{12EIz}{L^3} u_{j8}$$  \hspace{1cm} (12)

$$M_{j12} = -\frac{6EIz}{L^2} u_{j8}$$  \hspace{1cm} (13)

Vertical displacement matrix at node j.

$$\begin{bmatrix}
0 & 0 & 0 & 0 & u_{i2} \\
0 & 0 & 0 & 0 & u_{i6} \\
\frac{12EIz}{L^3} & -\frac{6EIz}{L^2} & \frac{12EIz}{L^3} & -\frac{6EIz}{L^2} & u_{j8} \\
0 & 0 & 0 & 0 & u_{j12}
\end{bmatrix}$$  \hspace{1cm} (14)
**Figure**: 4 Vertical displacement at node j.

**Act III**: Free node i and fixed node j – Rotational displacement $u_{i6} = \theta_{i6}$ at node i and fixed node j.

$$\frac{\partial U}{\partial M_{i6}} = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial M_{i6}} \, dx \tag{15}$$

**Action**:

$$f_{i2} = \frac{6EI_z}{L^2} u_{i6} \tag{16}$$

$$M_{i6} = \frac{4EI_z}{L} u_{i6} \tag{17}$$

**Reaction**:

$$f_{j8} = -\frac{6EI_z}{L^2} u_{i6} \tag{18}$$

$$M_{j12} = \frac{2EI_z}{L} u_{i6} \tag{19}$$

Rotational displacement matrix at node i.

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
6EI_z & 4EI_z & -6EI_z & 2EI_z \\
L^2 & L & -L^2 & L \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
u_{i2} \\
u_{i6} \\
u_{j8} \\
u_{j12}
\end{pmatrix}
= 
\begin{pmatrix}
u_{i2} \\
u_{i6} \\
u_{j8} \\
u_{j12}
\end{pmatrix}
\tag{20}
$$
**Figure: 5 Rotational displacement at node i.**

**Act IV:** Fixed node i and Free node j – Rotational displacement $u_{j12} = \theta_{j12}$ at node j and fixed node i.

\[
\begin{align*}
\frac{\partial U}{\partial M_{j12}} &= \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial M_{j12}} \, dx \\
\text{Action:} & \\
& f_{i2} = \frac{6EIz}{L^2} u_{j12} \\
M_{i6} &= \frac{2EIz}{L} u_{j12} \tag{22} \\
\text{Reaction:} & \\
& f_{j8} = -\frac{6EIz}{L^2} u_{j12} \\
M_{j12} &= \frac{4EIz}{L} u_{j12} \tag{23} \\
\text{Rotational displacement matrix at node j.} \\
\begin{bmatrix}
0 & 0 & u_{i2} & u_{i6} & u_{i8} & u_{j8} & u_{j12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\frac{6EIz}{L^2} \\
\frac{2EIz}{L} \\
\frac{6EIz}{L^2} \\
\frac{4EIz}{L} \\
\end{bmatrix}
\begin{bmatrix}
u_{i2} \\
u_{i6} \\
u_{j8} \\
u_{j12} \\
\end{bmatrix}
\tag{24}
\end{align*}
\]
2.2.2. Interlude

Interlude I is a 'step of the solutions' that can under certain conditions be superimposed using the superposition principle to analyze a element undergoing combined loading. Solutions for special cases exist for common structures such as beams [9]. For example, applying the principle of superposition of expressions ACT I, ACT II, ACT III, and ACT IV. Then, the stiffness matrix for a beam element with two degrees of freedom at each node the vertical displacement and rotation is:

\[
[K]_{(x,y)}^{(e)} = \begin{bmatrix}
    u_{i2} & u_{i6} & u_{j8} & u_{j12} \\
    \frac{12EIZ}{L^3} & \frac{6EIZ}{L^2} & \frac{6EIZ}{L^2} & \frac{12EIZ}{L^3} \\
    \frac{6EIZ}{L^2} & \frac{4EIZ}{L} & \frac{6EIZ}{L^2} & \frac{6EIZ}{L^3} \\
    \frac{6EIZ}{L^2} & \frac{2EIZ}{L} & \frac{6EIZ}{L^2} & \frac{4EIZ}{L^3} \\
\end{bmatrix}
\]

(27)
The stiffness matrix represent the contribution of each term to nodal degrees of freedom. Note that we need to represent Eq. (27) with respect to the global coordinate system \{X, Y\}. To perform this task, we must substitute for the local displacement in terms of the global displacements. These steps result in:

\[
[K]_{(e)}^{(X, Y)} = [K]_{(x, y)}^{(e)}
\]  

(28)

Where \([K]_{(X, Y)}^{(e)}\) is the stiffness matrix for a beam element expressed in the global coordinate system \{X, Y\}.

An explanation of some of the mathematical properties of stiffness matrix \([K]_{(X, Y)}^{(e)}\), relates the forces \(\{f\}\), applied at a set of coordinates on a structure to the displacements \(\{u\}\), at the same set of coordinates [9].

\[
\{f\} = [K]_{(x, y)}^{(e)} \{u\}
\]

(29)

As a simple beam element consist of two nodes. At each node, there are two DOF, a vertical displacement, and a rotation angle. The relationship between the local coordinates system \{x, y\} and the simple beam element the global displacements \(U_{i2}, U_{i6}\) at node \(i\) and \(U_{j8}, U_{j12}\) at node \(j\) are related to the local displacements \(u_{i2}, u_{i6}\) at node \(i\) and \(u_{j8}, u_{j12}\) at node \(j\) according to the equations. The element’s end conditions are given by the following nodal values:

For node \(i\): The vertical displacement at \(x = 0\):

\[
u_{i2} = U_{i2}
\]

(30)
The rotation at \( x = 0 \)

\[
u_{i6} = U_{i6}
\]  \hspace{1cm} (31)

For node \( j \): the vertical displacement at \( x = L \)

\[
u_{j8} = U_{j8}
\]  \hspace{1cm} (32)

The rotation at \( x = L \)

\[
u_{j12} = U_{j12}
\]  \hspace{1cm} (33)

Therefore, the global internal forces \( \{F\} \) and displacements \( \{U\} \) are related through the stiffness matrix

\[
\{F\} = [K]_{(X,Y)}^{(G)} \{U\}
\]  \hspace{1cm} (34)

3. Conclusions

1. The first column of the stiffness matrix corresponds to the forces that must be applied at the ends of the beam to produce the displacement \( u_1 \).
2. The first row of the stiffness matrix corresponds to the total force that must be applied in direction 1 due to the total displacements.
3. Known how to formulate stiffness matrix and load matrix for a beam element are derived.
4. At this point, know that stiffness matrix for beam element with two DOF at each node is derived.

References


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