From Black-Scholes to Hamilton-Jacobi

José Rodrigo González Granada
Universidad Tecnológica de Pereira-UTP, Colombia

Luis Fernando Plaza Gálvez
Universidad Central del Valle-UCEVA, Colombia

Olena Vasyunkina
Universidad Tecnológica de Pereira-UTP, Colombia

Abstract

In the present work the Stochastic Partial Differential Equations as the Black-Scholes which determines the valuation of goods and/or assets called financial options is studied, also we presents some relationships between the Black Scholes differential equation for European Call Options and Hamilton-Jacobi differential equation.

Keywords: Black - Scholes, Call Options, Differential Equation, Hamilton Jacobi, European Option

1 Introduction

1.0.1 Financial Option and the Black - Scholes model

The differential equation that models the value of a European Call type option called Black-Scholes is represented by

\[ rV = \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}, \]  
(1)
with initial condition: \( V(S, T) = \max\{S - E, 0\}, \quad S \in (0, \infty), \quad t = 0. \)

Where \( V = f(S, t) \) is the value of a call option, \( S \) represents the price of the underlying asset, \( r \) the risk-free rate, \( \sigma \) the volatility of the asset price, \( E \) the strike and \( T \) the maturation time of the option. The solution of this problem is classical when the volatility is a constant and can be obtained from the heat equation and appears related in (13), and when a volatility is no-constant in [8].

### 1.0.2 Dynamic Systems in Physics and Hamilton - Jacobi

In dynamic systems (classical mechanics and relativistic mechanics), the equations of motion can be found through the equation of Hamilton - Jacobi, which is a non-linear partial differential equation and can be linked to variational calculus.

The Hamilton-Jacobi first-order differential equation is represented by

\[
U_t + H(DU) = 0.
\]

Starting from heuristic principles and analyzing the expressions presented in (1) and (2), that corresponds to spatial variables, domain, initial condition, space, energy relations, characteristics the dynamic systems that model, as well as their solutions and power- presenting them in another form through a potential, there are some contributions and relationships common to the two models, that participate in different scenarios, such as those of finance and physical systems, and can be shown that Black - Scholes has immersed a potential and an energy. This work was a result beginning in [9], after having studied a suggestion raised in [11]. It is important to remark that such problems was studied in recent works in Econo-physics. In economics, the laws and concepts of physics do not apply directly to systems of economics but what is done is to model these systems with the aid of mathematical tools, concepts and methods of thermodynamics, quantum mechanics, statistical mechanics and quantum field theory [1].

### 2 Hamilton-Jacobi differential equation

#### 2.1 Basic concepts

Joseph L. Lagrange (1736 - 1813) contributed to the development of mechanics by means of the so-called Lagrange differential equations, which can describe the behavior of dynamic systems over time, taking into account the number of degrees of freedom of the mechanism and the coordinates that characterized the movement. Subsequently William R. Hamilton (1805 - 1865) used the principles of variational calculus to endow the theoretical mechanics of an
intrinsic conception that exceeded the developing of Lagrange. These works were complemented by G. Jacobi, S. Poisson and J. Lioville [4]. The Hamiltonian theory has as independents variables the generalized coordinates and associated moments. The Hamiltonian is a mathematical object related to the total energy of the system from which can be calculate the future evolution of the Hamiltonian system, known as a classic system. The previous studies led to the origin of the of Hamilton-Jacobi equation, which can be solved by different analytically methods, as it used in mechanical of systems (traveling particle, the harmonic oscillator, etc).

**Definition 2.1 (Hamilton-Jacobi equation).** It's an initial value problem, which is represented by

\[ U_t + H(DU) = 0, \]

valid in \( \mathbb{R}^n \times (0, +\infty) \), and the initial condition \( U = g \), with \( g \in \mathbb{R}^n \times (t = 0) \).

Here we have:

- \( U : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R} \). \( H \) is the Hamiltonian defined by:
- \( H : \mathbb{R}^n \to \mathbb{R} \), and the initial function \( g : \mathbb{R}^n \to \mathbb{R} \).

### 3 Solution of the Hamilton-Jacobi partial differential equation

Let \( L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \), a soft function called Lagrangian (difference between the kinetic energy and the potential energy), fixing 2 points \( x, y \in \mathbb{R}^n \), with \( t > 0 \), the functional is introduced as:

\[
I[w(\cdot)] = \int_0^t L(w(s), w(s))ds,
\]

where \( w(\cdot) \) are functions of the form \( w(\cdot) = \{w^1(\cdot), w^2(\cdot), ..., w^n(\cdot)\} \), belonging to the admissible class

\[
\{w(\cdot) \in C^2([0, t]; \mathbb{R}^n) \mid w(0) = y, w(t) = x\}.
\]

\( H \) and \( L \) are dual and convex functions, where the relationship between them is in the form [3, p. 122]:

\[
H = pq - L.
\]  

(3)

Where \( pq \) represents a potential form of the associated moment. A solution of the initial value problem (2) in variational terms, is given by

\[
U(x, t) = \inf \left\{ \int_0^t L(\dot{w}(s))ds + g(y) \mid w(0) = y, w(t) = x \right\},
\]

(4)
here $inf$ is taken over every function $w(\cdot) \in C^1$, with $w(t) = x$. It must also satisfies for $g : \mathbb{R}^n \to \mathbb{R}$ the lipschitz conditions of continuity:

$$Lip(g) := \sup_{x,y \in \mathbb{R}^n, x \neq y} \left\{ \frac{|g(x) - g(y)|}{|x - y|} \right\} < \infty.$$  \hspace{1cm} (5)

The expression (4) can be minimized (Hopf - Lax), if $x \in \mathbb{R}^n$, $t > 0$, in such way that the solution is given by

$$U(x,t) \leq \min_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) \right\}.$$  \hspace{1cm} (6)

The last expression is the weak solution of the initial value problem (2) for Hamilton - Jacobi partial differential equation [3].

4 Conditions of the Hamilton-Jacobi equation

Following are the main conditions that govern differential Hamilton-Jacobi equation, as well as its respective weak solution, taking into account the theory exposed in variational analysis.

1. The problem of Hamilton - Jacobi is a Cauchy problem (existence of an initial condition) and it is given for dynamic systems by the presence of energies in different shapes, the metric space where $U(x,t)$ in Hamilton-Jacobi equation, is a Banach space [3], [6]. The Lagrangian is invariant under transformations [10, p. 3] also the Hamiltonian $H$ must be at least of class $C^2$ [10, p. 16].

2. The Hamilton-Jacobi equation is an evolution equation.

5 Some relations between Hamilton-Jacobi and the Black-Scholes

The Black - Scholes differential equation, for the valuation of an option Call type, with its respective solution, presents the following relationships with the Hamilton-Jacobi equation, where its solution is a function of the form $V = f(S,t)$.

1. If we express the differential equation (1) of the form given in (2), we have

$$\frac{\partial V}{\partial t} + \left[ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \right] = 0.$$  \hspace{1cm} (7)
If \( n = 1 \) from (2), we deduced that

\[
H(DV) = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV.
\]

(8)

Where \( H(DV) \) is the Hamiltonian associated with the Black-Scholes differential equation. From (7) and (8) we have

\[
V_t + H(DV) = 0,
\]

(9)

in \( \mathbb{R} \times (0, +\infty) \), with the initial condition \( V = g, \) in \( (t = 0) \).

2. The initial condition for \( t = 0 \) in (2) is \( U = g \).
The initial condition for \( t = 0 \) in (1) is \( V(S,T) = \max\{S - E, 0\} \).

3. Taking into account (2) and (1), when \( S \) represents the price of an action or active in the Black-Scholes models, plays the role of the spatial variable in Hamilton-Jacobi equation.

4. The Black-Scholes relation is an evolution equation compared with (2).

5. The Black-Scholes model is a dynamic system, since its solution allows to find a function for the evaluation of options and this depends on the changes over time. Therefore, the behavior of options represent a physical field.

6. Being the Black-Scholes model a dynamic system, it can be affirmed which has an associated energy, and that can be seen reflected in the Hamiltonian, as well as in his Lagrangian by the fact that both functions are double convex. Starting from [10], and (3), we have that

\[
H = P_i \dot{x}^i - L(t, x^j, \dot{x}^j),
\]

(10)

where \( P_i \) is the generalized moment, and taking \( P_i \dot{x}^i = \Phi \), as a potential function, it can be concluded that \( L = \Phi - H \), therefore the Lagrangian for Black-Scholes has the form:

\[
L(S, \dot{S}, t) = \Phi - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV,
\]

(11)

with \( \Phi = P \dot{S} \).

Therefore, when applying (6), a weak solution to the problem of Black-Scholes and on the principle of variational calculation:

\[
V(S, t) \leq \min_{y \in V} \left\{ tL \left( \frac{S - y}{t} \right) + g(y) \right\},
\]

(12)
with $S, y \in V$. Taking into account that $\dot{S}$ is the variation of the price of the action with respect to time.

7. The Hamilton-Jacobi equation has interesting symmetries properties in the analysis of Lie theory, because of superposition, and it is a great tool to work in the Black-Scholes equation for the theory of operators [11, 7], and can be show that Black-Scholes has a differential operator immersed.

8. The solutions of the Black-Scholes differential equation are defined in a Banach space $V$ defined by the metric:

$$
\| V \| = D(V_1, V_2) = |V_1 - V_2|,
$$

where $V_1$ and $V_2$ are solutions of (1), for $t_1$ and $t_2$ respectively. The solution of (7) is given by the expression

$$
V(S, t) = SN(d_1) - E e^{-r(T-t)} N(d_2). \quad (13)
$$

Here $N(d), d_1$ y $d_2$, are given respectively by (14), (15) as follows:

$$
N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{w^2}{2}} dw, \quad (14)
$$

$$
d_1 = \frac{\ln \left( \frac{S}{E} \right) + \left[ r + \frac{\sigma^2}{2} \right] (T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t}. \quad (15)
$$

If $V_1$ is given for $t = t_1$, and $V_2$ is given for $t = t_2$, then

$$
V_1(S, t) = SN(d_{11}) - E e^{-r(T-t_1)} N(d_{12}), \quad (16)
$$

$$
V_2(S, t) = SN(d_{21}) - E e^{-r(T-t_2)} N(d_{22}). \quad (17)
$$

Where

$$
d_{11} = \frac{\ln \left( \frac{S}{E} \right) + [r + \frac{\sigma^2}{2}] (T-t_1)}{\sigma \sqrt{T-t_1}}, \quad d_{12} = d_{11} - \sigma \sqrt{T-t_1}, \quad (18)
$$

$$
d_{21} = \frac{\ln \left( \frac{S}{E} \right) + [r + \frac{\sigma^2}{2}] (T-t_2)}{\sigma \sqrt{T-t_2}}, \quad d_{22} = d_{21} - \sigma \sqrt{T-t_2}. \quad (19)
$$
By subtracting (16) and (17), we obtained:

\[ \Delta V = S[N(d_{11}) - N(d_{21})] + E e^{-rT} \left[ e^{-rt_2} N(d_{22}) - e^{-rt_1} N(d_{12}) \right]. \tag{20} \]

With \( \Delta V = V_1 - V_2 \). If

\[ N(d_{11}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_{11}} e^{-w^2/2} dw, \quad N(d_{21}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_{21}} e^{-w^2/2} dw. \tag{21} \]

Taking into account (21) we have:

\[ N(d_{11}) - N(d_{21}) = \frac{1}{\sqrt{2\pi}} \int_{d_{11}}^{d_{21}} e^{-w^2/2} dw. \tag{22} \]

Replacing (22) in (20):

\[ \Delta V = \frac{S}{\sqrt{2\pi}} \int_{d_{11}}^{d_{21}} e^{-w^2/2} dw + \frac{E e^{-rT}}{\sqrt{2\pi}} \left[ e^{-rt_2} \int_{-\infty}^{d_{22}} e^{-w^2/2} dw - e^{-rt_1} \int_{-\infty}^{d_{12}} e^{-w^2/2} dw \right]. \]

Taking absolute value to the previous expression, and applying the triangular inequality we can obtain:

\[ |\Delta V| \leq \frac{S}{\sqrt{2\pi}} \left| \int_{d_{11}}^{d_{21}} e^{-w^2/2} dw \right| + \frac{E e^{-rT}}{\sqrt{2\pi}} \left| e^{-rt_2} \int_{-\infty}^{d_{22}} e^{-w^2/2} dw - e^{-rt_1} \int_{-\infty}^{d_{12}} e^{-w^2/2} dw \right| \]

\[ + \frac{E e^{-rT}}{\sqrt{2\pi}} \left| e^{-rt_1} \int_{-\infty}^{d_{12}} e^{-w^2/2} dw \right|. \]

9. If we verified that initial condition of equation (1),

\[ V(S, T) = \max\{S - E, 0\}, \tag{23} \]

it’s lipschitz’s function, i.e if \( S_1 = x, S_2 = y \), with \( x \neq y \), then

\[ |V(x, t) - V(y, t)| \leq K|x - y|. \tag{24} \]

Initially, analyze the expression given in (23), with \( S > 0 \):
\[ V(S, T) = \begin{cases} S - E, & \text{si } S - E \geq 0, \rightarrow S \geq E \\ 0, & \text{si } S - E < 0, \rightarrow S < E \end{cases} \]

Applying the last expression for \( V(x, T) \) and \( V(y, T) \) we can get:

\[ |V(x, T) - V(y, T)| = \begin{cases} |x - y|, & \text{si } x - y \geq E \\ 0, & \text{si } x - y < E \end{cases} \]  \quad (25)

Assuming that \( |V(x, T) - V(y, T)| = |x - y| \), comparing with (24), and analyzing only the equality, we can conclude that the Lipschitz condition is given by \( K = 1 \), and for this case \( V(S, T) \) is a short function.

6 Conclusions

We proof how physics and finances markets have a link through the Black-Scholes model, making the latter a dynamic system by average of the relations found with the Hamilton - Jacobi equation.

According to the study, it was found that the Black Scholes model has a associated energy, which is reflected in its Hamiltonian, so in the future it can carry out studies that lead to finding the Lagrangian and the function potential present in the Black-Scholes model, as it appears in the equation (11), where \( P \) is the generalized moment.

References


Received: September 27, 2018; Published: October 23, 2018