Abstract

This article presents the approach from the point of view of the transformation of Legendre to obtain Hamiltonian systems from Lagrangian systems. The main objective is to show the sequence of obtaining these systems with the Legendre transformation. The problems outlined in [5] and [7] are explained with a qualitative approach, then a weak formulation for the dynamic system of the synchronous machine is shown by means of the Legendre transformation and the Lagrangian system formulation to obtain the Hamiltonian system, which describes the dynamics of an electric machine.
Keywords: Legendre transformation, Lagrangian system, electric machine

1 Introduction

1.1 Legendre transformation

The Legendre transformation allows to represent the dynamics of a system in terms of the first derivatives of the variables that model a system of interest. This transformation is widely used in mathematics, physics and engineering as a tool to move from Lagrangian to Hamiltonian dynamics, or as the connection of internal energy and enthalpy in mechanical systems [5]. The transformation of a pair of variables that models the system allows analyzing in another field with new variables known as conjugate variables. When you have a model of variables are analyzed in terms of momentum and \( 2n \) new variables that are called potentials. In general what we want to obtain by means of the representation of dynamic systems with the Lagrangian, Hamiltonian or Legendre formulation is the model of these systems in terms of their energies, that is, in a system where the variables that model it are the displacement, the speed, can be carried to a Hamiltonian system where the model is represented by the variables displacement and momentum. On the other hand, what Legendre’s formulation proposes is to convert a Lagrangian system into a Hamiltonian system in terms of the energy of that system as a conjugate pair of variables, that is, to obtain a model with the first derivatives.

The Legendre transformation can be applied under the following assumptions:

- The functions must be continuous and derivable in the domain. This assumption provides the idea of continuity, smoothness and non-singularity in a domain for the variables that model the dynamic system.

- If \( x \in \Omega \), the first derivative \( f'(x) \) is continuous in \( \Omega \) and can be calculated. That is, we know the function \( f(x) \) that models the system and the first derivatives can be calculated and are continuous in a range, there is no discontinuity in \( \Omega \) for both the function and the first derivative.

- The formulation in terms of energy are convex problems since the Legendre transformation is possible when there are no abrupt changes in the model [4].

Now, suppose that instead of using the variable \( x \), it is possible to change the variable in such a way that \( p(x) = f'(x) \). The Legendre transformation generates an equation in terms of \( p(x) \) (the first derivative), to obtain a new
function $g(p)$. The transformation is invertible since the information obtained in $g(p)$ is also used in $f(x)$. Given the model $x(p)$, it is possible to obtain as: $g(p) = p \cdot x(p) - f(x(p))$. Now, to obtain the sequence of a system we have:

- Verify the first assumption: $f(x)$ is continuous and derivable over $\Omega$.
- Calculate the first derivative $p(x) = f'(x)$ and invert $x(p) = p$.
- Define the new system with the Legendre transformation, i.e., $g(p) = p(x) \cdot x - f(x)$.

**Example.** In two variables, we propose a first order model to model the electrical circuit of Figure 1. To solve the problem previously raised could be solved by the laws of Kirchhoff, but it will not solve that way, it will formulate a Lagrangian system in terms of the energy of the system load.

$$\mathcal{L} = T_{\text{Kinetic energy}} - V_{\text{Potential energy}}$$

![Wave trap circuit](image)

Figure 1: Wave trap circuit.

$$T(\dot{q}) = \frac{L \cdot i^2}{2} = \frac{L \cdot \dot{q}^2}{2}; \quad V(q) = \frac{q^2}{2C} \quad \therefore$$

$$\mathcal{L} = T(\dot{q}) - V(q) = \frac{L \cdot \dot{q}^2}{2} - \frac{q^2}{2C}; \quad \mathcal{L} \text{ is a function of } q, \dot{q}$$

The transformation of Legendre allows finding a new system in terms of the first derivative of a variable, in this case with the first derivative $\dot{q}$ the variable change proposed in [1], [2], [5] will be made [7]. $\mathcal{L} \in t > 0$, the Lagrangian function is continuous and derivable for all $t > 0$. Therefore, we have

- Calculate $\frac{\partial L}{\dot{q}} = L \cdot \dot{q}$;
- Invert $\dot{q}(p)$; $\dot{q} = \frac{p}{L}$.
• Use the Legendre transformation
  \[ g(p) = p(q) \cdot \dot{q}(p) - \mathcal{L} \]
  \[ g(q, p) = p \cdot \frac{\dot{q}}{L} - \frac{L \cdot p^2}{2} + \frac{q^2}{2 \cdot C}; \]
  After simplifying we obtain:
  \[ g(q, p) = \frac{p^2}{2 \cdot L} + \frac{q^2}{2 \cdot C}; \]
  \[ \mathcal{H}(q, p) = \frac{p^2}{2 \cdot L} + \frac{q^2}{2 \cdot C}; \]

2 Hamiltonian System: Model O-2 synchronous machine

The dynamic model of the synchronous machine is described as a set of equations of the form [8]:

\[ M \cdot \frac{d\omega}{dt} = P_m - P_0 \sin(\delta) \]
\[ \frac{d\delta}{dt} = \omega \]

To describe the previous system in terms of the kinetic and potential energies of the synchronous machine, the criterion of equal areas is used as proposed in [3], which consists in finding the points of equilibrium that are stable from the equalization of the electrical and mechanical powers of Figure 3, this criterion is actually a particular case of the Lyapunov theory for an energy function applied to an electric generator connected to an infinite busbar [6] [3]. Figure 3 shows that the system is stable at point a (there is no variation in power),
for points b and c of figure 3 a failure occurs due to a variation of the power in the state of failure, finally at points d, e, f and g a post failure state is present but the system does not lose stability because the mechanical power does not exceed the electrical power in these points.

The kinetic energy refers to the rotational energy of the axis of the machine, on the other hand, the potential energy is related to the operation of the machine connected to an infinite barrage and its variation of electrical power at the moment that a failure occurs in said bar, allows to calculate the energy gained or lost by the machine in different pre-fault, failure and post-failure states, for the analysis of this section it will be taken into account in the pre-fault state.

\[ L = T_{Kinetic \, energy} - V_{Potential \, energy} \]

\[ T_{Kinetic \, energy} = \frac{I \cdot \omega^2}{2} \]

\[ V_{Potential \, energy} = \frac{1}{M} \int_{\delta=\delta_0}^{\delta=\delta} (P_m - P_0 \cdot \sin(\delta)) \cdot d\delta = \frac{1}{M} [P_0 (\cos(\delta) - \cos(\delta_0)) + P_m (\delta - \delta_0)] \]

\[ \mathcal{L}(\delta, \omega) = \frac{I \cdot \omega^2}{2} - \frac{1}{M} [P_0 (\cos(\delta) - \cos(\delta_0)) + P_m (\delta - \delta_0)] \]

\[ \frac{\partial L(\delta, \omega)}{\partial \omega} = I \cdot \omega; \quad p = I \cdot \omega \]

\[ \frac{P}{I} = \omega \]

Now, use the Legendre transformation as
\[
g(\delta, p) = p \cdot \omega(p) - L(\delta, \omega(p))
\]

\[
g(\delta, p) = p \cdot p - \frac{I \cdot p^2}{2} + \frac{1}{M} [P_0 (\cos(\delta) - \cos(\delta_0)) + P_m (\delta - \delta_0)].
\]

Finally, we get

\[
\mathcal{L}(\delta, p) = \frac{p^2}{2 \cdot I} + \frac{1}{M} [P_0 (\cos(\delta) - \cos(\delta_0)) + P_m (\delta - \delta_0)].
\]

Hamiltonian formulation

\section{Conclusion}

When the respective analysis of a system is made by the energy method, whether Lagrangian or Hamiltonian, the mathematical development to arrive at the model of the system is more practical in comparison with the formulation from the point of view of physical laws such as laws of Newton (Mechanical Systems), Kirchhoff Laws (Electrical Systems).

The Legendre transformation can be seen as an operator that takes the functions of the dynamic system model to other functions based on the conjugated variables, and whose inverse is possible since the Legendre representation does not lose information from the original system.

Acknowledgements. We would like to thank the referee for his valuable suggestions that improved the presentation of this paper and our gratitude to the Department of Mathematics of the Universidad Técnologica de Pereira (Colombia), the GEDNOL Research Group and GIMAE (Grupo de Investigación en Matemática Aplicada y Educación).

\section{References}


Received: September 24, 2018; Published: October 30, 2018