On Numerical Simulation of the Black-Scholes Model Using the Non-linear Adomian Method

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Abstract

In this work the Adomian decomposition method (ADM) is used to solve the non-linear equation that represents the generalized model of Black-Scholes, that is to say that considers the volatility as a non-constant function. The efficiency of this method is illustrated by investigating the convergence results for this type of models.

Keywords: Adomian decomposition method, Black-Scholes, Volatility
1 Introduction

Under certain assumptions, Black and Scholes published in the early 1970s their article the pricing of options and corporate liabilities in which they presented the partial differential equation of second order parabolic type [1,2,3]:

$$\frac{\partial c}{\partial t} + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} \sigma^2 S_t^2 + \frac{\partial c}{\partial S_t} r S_t - rc = 0,$$

(1)

where $C(S,t)$ is the price of a European call option, $t$ the contract start time, $S_t$ the price of the asset, $\sigma$ the volatility and $r$ is the interest rate.

This work focuses special attention on the problem of non-constant volatility, since it is the opinion of the experts that the fundamental variable in the Black-Scholes equation [4,5]. In this paper we propose an alternative view to solve Eq.(1): Adomian Decomposition Method. The advantage of this method lies in its rapid convergence, which implies a smaller number of iterations, but mainly in that it does not require any discretization, linearization or perturbation techniques which could affect the solution of the model real.

2 Description of ADM

The Adomian decomposition method is applied to a general nonlinear equation in the form [3]

$$Lu + Ru + Nu = g$$

(2)

Here, the linear terms are decomposed into $L + R$ and the nonlinear terms are represented by $Nu$. Here, $L$ is the operator of the highest-ordered derivatives with respect to $t$ and $R$ is the remainder of the linear operator. Thus we get

$$Lu = -Ru - Nu + g$$

(3)

$L^{-1}$ is regarded as the inverse operator of $L$ and is defined by

$$L^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) dt \, dt$$

(4)

If $L$ is a second order operator, then $L^{-1}$ is defined by a two-fold indefinite integral

$$L^{-1}Lu = u(x,t) - u(x,0) - t \frac{\partial u(x,0)}{\partial t}$$

(5)

Now, operating on both sides of Eq.(3) by using $L^{-1}$ we obtain
\[ L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \]  

(6)

Therefore we have

\[ u(x, t) = u(x, 0) + t \frac{\partial u(x, 0)}{\partial t} + L^{-1}g - L^{-1}Ru - L^{-1}Nu \]  

(7)

The ADM represents the solution of Eq.(7) as a series

\[ u(x, t) = \sum_{n=0}^{+\infty} u_n(x, t) \]  

(8)

Here, the operator \( Nu \) (nonlinear) is decomposed as

\[ Nu = \sum_{n=0}^{+\infty} A_n \]  

(9)

Now, substituting (8) and (9) into (7) we obtain

\[ \sum_{n=0}^{+\infty} u_n(x, t) = u_0 - L^{-1}R \sum_{n=0}^{+\infty} u_n(x, t) - L^{-1} \sum_{n=0}^{+\infty} A_n \]  

(10)

where

\[ u_0 = u(x, 0) + t \frac{\partial u(x, 0)}{\partial t} + L^{-1}g \]  

(11)

Then, consequently we can obtain

\[ u_1 = -L^{-1}Ru_0 - L^{-1}A_0 \]
\[ u_2 = -L^{-1}Ru_1 - L^{-1}A_1 \]
\[ \vdots \]
\[ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n \]  

(12)

where \( u_n(x, t) \) will be determined recurrently, and \( A_n \) are the so-called polynomials (Adomian) of \( u_0, u_1, \ldots, u_n \) defined by

\[ A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ F \left( \sum_{i=0}^{\infty} \lambda_i u_i \right) \right], \quad n = 0, 1, 2, \ldots \]  

(13)

In this case we obtain
\[ A_0 = f(u_0) \]
\[ A_1 = u_1 f'(u_0) \]
\[ A_2 = u_2 f''(u_0) + \frac{1}{2!} u_1^2 f''(u_0) \]
\[ \vdots \]

(14)

Now, if we introduce the parameter \( \lambda \) conveniently, we can obtain that

\[ u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n \]  

(15)

where

\[ N(u(\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n \]  

(16)

Therefore, expanding by Taylor’s series at \( \lambda = 0 \) we have

\[
N(u(\lambda)) = \sum_{n=0}^{\infty} 1 \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N(u(\lambda)) \right] \lambda^n
= \sum_{n=0}^{\infty} 1 \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right] \lambda^n
\]

(17)

The Adomian’s polynomials \( A_n \) can be calculated using the recurrence equation [6]

\[
A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right] \bigg|_{\lambda=0}
\]

(18)

If we are working with systems of differential equations (or algebraic type alike), the nonlinear terms \( N \) can be of the form

\[ N = N(u_1, u_2, \ldots, u_k, \ldots) \]

where

\[ u_k = \sum_{n=0}^{\infty} u_{k,i} \]
Numerical simulation of the Black-Scholes model

3 Generalized B-S model and numerical result

**Theorem 3.1** The generalized Black-Scholes model with non-constant volatility

\[ \sigma^*(S,t,C_s,C_{ss}) = \frac{\sigma}{1 - \rho \lambda(S) C_{ss}}, \]

can be expressed as

\[
\begin{aligned}
C_t(S,t) + \frac{1}{2} \sigma^2 S^2 C_{ss}(S,t)(1 + 2 \rho SC_{ss}(S,t)) + r SC_s(S,t) - r C(S,t) &= 0 \\
C(S,T) &= f(S), \quad S \in [0, \infty],
\end{aligned}
\]

From Eq. (19), with

\[
\sigma = 0.15, \quad |\rho| = 0.011, \quad r = 0, \quad f(S) = 2S + 7\sqrt{S} + \frac{7}{2}
\]

we have the Adomian’s polynomials

\[
\begin{align*}
A_0 &= \frac{3.0625}{s^3} & C_0 &= 2S + 7\sqrt{S} + 3.5 \\
A_1 &= \frac{-0.01722t}{s^3} & C_1 &= -1.9687 \times 10^{-2} \sqrt{t} + 7.57968 \times 10^{-4}t \\
A_2 &= \frac{4.8449 \times 10^{-5}t^2}{s^3} & C_2 &= 2.7685 \times 10^{-5}\sqrt{t}^2 + 2.1317 \times 10^{-6}t^2 \\
A_3 &= \frac{-9.0843 \times 10^{-8}t^3}{s^3} & C_3 &= -2.5955 \times 10^{-8}\sqrt{t}^3 - 3.9971 \times 10^{-9}t^3 \\
A_4 &= \frac{1.2774 \times 10^{-10}t^4}{s^3} & C_4 &= 1.8249 \times 10^{-11}\sqrt{t}^4 + 5.6209 \times 10^{-12}t^4
\end{align*}
\]
Table 1. B-S model for $\sigma = 0.15, \quad |\rho| = 0.011, \quad r = 0, \quad f(S) = 2S + 7\sqrt{S} + \frac{7}{2}$

In this way, an approximate analytical solution has the form

$$C(S,t) = 2S + 7\sqrt{S} + 3.5 - 1.9687 \times 10^{-2} \sqrt{S} t + 7.57968 \times 10^{-4} t + 2.7685 \times 10^{-5} \sqrt{S} t^2 + 2.1317 \times 10^{-6} t^2 - 2.5955 \times 10^{-8} \sqrt{S} t^3 - 3.9971 \times 10^{-9} t^3 + 1.8249 \times 10^{-11} \sqrt{S} t^4 + 5.6209 \times 10^{-12} t^4.$$

Table 1 shows the price of a purchase option for an asset with the characteristics given by Eq. (20) and with a cost of between 0.1 and 3.5 monetary units over a period of 3 years. In addition, Figure 1 shows the curve of the
option price for $T = 0.5$ and $T = 0.3$ years.

In the simulation presented, we can see how the method converges rapidly, for example, we can see that the fourth term with the solution has coefficients of order $10^{-10}$.

4 Conclusion

The Adomian decomposition method yielded an efficient technique for solving the equation representing the Black-Scholes model, since it provides an approximate analytical solution for which numerical simulations show that it is it converges rapidly, so that the model and solution technique are two valuable tools for valorization of a European purchase option.

We study the generalized Black-Scholes model, that is, the one that considers volatility as a function that depends on the time, the price of the underlying and the premium of the option since, in the opinion of the experts, volatility is one of the main variables in the model since real markets do not obey linear behavior.

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References


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