2D Path Generation with a Five-Bar One DOF

Mechanism and Improved Non-Circular Elements

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Abstract

It is presented the reduction of the mobility in a planar five-bar mechanism by the integration of a non-circular gear connection. Analytical expressions for the gear relations were derived from the position angle equations. Angles were calculated from the two dimension kinematic task. Next, each of the five possible non-circular gear connections is evaluated. The presented methodology was used for planar path generation along ten points. Negative values were reached for the non-circular disc, leading to variation of the mechanism parameters under boundary conditions from design considerations, to obtain primitive curves for a feasible gear connection.

Keywords: path generation; planar 5R mechanism; non-circular gears; mobility

1 Introduction

A mechanism is a device for control or restriction of relative motion, transforming an input movement into a desired output movement of an element of interest or follower. The working tool is attached to the follower element to perform a desired kinematic task [1] by a machine. A transmission device type commonly found in mechanisms is the circular gear connection, used for both motion transmission and reducing the mobility of the mechanism [2]. When reducing mobility, less actuators are needed to control the position and orientation of the follower, thus saving money on assembly costs, maintenance and programming. These savings can be achieved by including non-circular gears into the mechanism. First attempts to reduce the mobility of 2D mechanisms considered circular gears, however, totality of path points was not fulfilled [3, 5]. Recently, non-circular gear connections are proposed
in motion generation along an interval [6-8]. Likewise, a methodology is available for integrating non-circular gears in closed-chain 2D mechanisms [8], with a connection evaluated at each mechanism link. However, [6] and [8] did not optimize the gears, in contrast to Mundo et al. [9] who optimized a five-link mechanism with the gear connection at the grounded link. This report aims the pitch curve synthesis of a non-circular gear transmission into a closed-chain five-bar 2D mechanism, reducing the mechanism's mobility to one. It is satisfied the path (x, f(x)) with the motion of point P, see Figure 1.A: L0, ..., L4 are link lengths, θ0, ..., θ4 are joint angles, θ0 and L0 are grounded angle and length, respectively. The position of point P must traverse a kinematic task (cutting, painting, welding). As a novelty, the non-circularity of the gear connection is evaluated and improved, by recursively considering as a connecting element each link of the mechanism.

2 Methodology

2.1 Kinematics of the Point Tool
The analysis begins with the reverse kinematics. Link angles are found depending on the location of the actuator given by path points (x, f(x)), see point P in Figure 1.A. The solution of the inverse kinematics is discussed below. Further information on the solution, see [2,9]. Known: Path of point P (x,f(x)); link lengths L0, L1, L2, L3, L4; ground angle θ0. Find: Link angles: θ1, θ2, θ3, θ4. Solution: Figure 1.B shows the vector loop of the mechanism, where \( \vec{L}_i \) is the vector of element i. Paths 1 and 2 are distinguished in Figure 1.B, leading to (1) and (2). Point P vector, \( \vec{P} \), is given by (3). Equations (2) and (3) generate x and y components for the path 1 in Figure 1.B. It is established the system (4) and (5), where \( c_i = \cos(\theta_i) \) and \( s_i = \sin(\theta_i) \). The system is solved for the joint angle \( \theta_2 \) to yield (6), where the function atan2 defined by [11]. The joint angle \( \theta_1 \) is found in a similar fashion, leading to the expression in (7). To obtain angles \( \theta_3 \) and \( \theta_4 \), components of (2) are separated to set the system (8) and (9). By following a similar procedure as for \( \theta_1 \) and \( \theta_2 \), expressions for \( \theta_3 \) and \( \theta_4 \) are given in (10) and (11), respectively, where T=x-L0c0, K=f(x)-L0s0.

\[
\begin{align*}
\vec{L}_1 + \vec{L}_2 - \vec{L}_3 - \vec{L}_4 - \vec{L}_0 &= \vec{0} \\
\vec{L}_1 + \vec{L}_2 &= \vec{P} = \vec{L}_0 + \vec{L}_4 + \vec{L}_3 \\
\vec{P} &= (x,f(x)) \\
L_1c_1 + L_2c_2 &= x \\
L_1s_1 + L_2s_2 &= f(x)
\end{align*}
\]

\[
\theta_2 = \text{atan2}
\left(\frac{f(x)}{\sqrt{x^2+f(x)^2}}, \frac{x}{\sqrt{x^2+f(x)^2}}\right)
\pm \cos\left(\frac{x^2+f(x)^2+L_2^2-L_1^2}{2L_2\sqrt{x^2+f(x)^2}}\right)
\]

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In the direct kinematics of the five-bar mechanism for the point P, the position angles are assumed to be known. Components of the tool are to be found. **Known:** Link lengths $L_0, L_1, L_2, L_3, L_4$; ground angle: $\theta_0$, joint angles: $\theta_1, \theta_2, \theta_3, \theta_4$. **Find:** Path of point P: $(x, f(x))$. **Solution:** Through path 1 in Figure 1.B, the components of the tool are determined by (12) and (13). Path 2 yields the components of the tool in (14) and (15). The values of (12) and (14) must be the same to ensure that the mechanism is a closed chain. Similarly, for (13) and (15).

\[
\theta_1 = \text{atan2} \left( \frac{f(x)}{\sqrt{x^2 + f(x)^2}}, \frac{x}{\sqrt{x^2 + f(x)^2}} \right) \pm a \cos \left( \frac{x^2 + f(x)^2 + L_1^2 - L_2^2}{2L_1 \sqrt{x^2 + f(x)^2}} \right) \tag{7}
\]

\[
\theta_3 = \text{atan2} \left( \frac{K}{\sqrt{T^2 + K^2}}, \frac{T}{\sqrt{T^2 + K^2}} \right) \pm \alpha \cos \left( \frac{T^2 + K^2 + L_3^2 - L_2^2}{2L_3 \sqrt{T^2 + K^2}} \right) \tag{10}
\]

\[
\theta_4 = \text{atan2} \left( \frac{K}{\sqrt{T^2 + K^2}}, \frac{T}{\sqrt{T^2 + K^2}} \right) \pm \alpha \cos \left( \frac{T^2 + K^2 + L_4^2 - L_3^2}{2L_4 \sqrt{T^2 + K^2}} \right) \tag{11}
\]

\[
x = L_1 c_1 + L_2 c_2 \tag{12}
\]

\[
f(x) = L_1 s_1 + L_2 s_2 \tag{13}
\]

\[
x = L_0 c_0 + L_4 c_4 + L_3 c_3 \tag{14}
\]

\[
f(x) = L_0 s_0 + L_4 s_4 + L_3 s_3 \tag{15}
\]
2.2 Gear Ratio

The expression to calculate the gear ratio, \( g \), when elements in mesh are not grounded, as in Figure 2.A, is defined as the quotient of the derivative of the absolute angles \( \theta_0 \) and \( \theta_i \) [6] as in (16). In this equation, \( g_j \) is the gear ratio for the connection of geared elements \( j-1 \) and \( j+1 \), connected by element \( j \), with distance between centers \( L_j \). The unitary ratio \( \frac{dx}{dx} \) is introduced to present the expression (16), allowing a direct relationship of the path of the tool with the profile of the connected gears.

The mobility, \( m \), of the mechanism in Figure 1.a is 2 by using the Grübler equation [2] (17), where \( n \) is the number of links, \( f_1 \) is the number of joints with one degree of freedom, \( f_2 \) is the number of joints with two degrees of freedom (\( n=5, f_1=5, f_2=0 \)). By finding the profile of the connecting non-circular elements, the mobility, \( m \), of the five-bar planar mechanism is set as 1 (now \( f_2=1 \)), thus needing only one actuator to perform the task.

\[
\frac{dx}{dx} = \frac{d\theta_0}{d\theta_i} = \frac{d(\theta_{j+1}-\theta_j)}{d(\theta_j-\theta_{j-1})} = \frac{d\theta_{j+1}}{d\theta_j} \frac{d\theta_j}{d\theta_{j-1}} \tag{16}
\]

\[
m = 3(n - 1) - 2f_1 - f_2 \tag{17}
\]

2.3 Completeness of the Non-Circular Disc

Gears design and manufacture require complete gear profile. The profile portion found from the design process is the working section (not necessarily is 360°), leading to the need of an additional section (nonworking) for manufacturing purposes, see Figure 2.B. Expression (16) is the used to find the working gear ratio, \( g_w \), see the continuous line in Figure 2.C, while \( g_{nw} \) labels the non-working section gear ratio. To reach \( 2\pi \) for the non-circular disc, the coefficients \( u_i \) of the fourth degree polynomial in (18) are found from the system in (20), which was set from the boundary conditions in Figure 2.C. The intermediate angle, \( \theta_{int} \) refers to the final value of the input angle, \( \theta_i \), for the working section. The integral in (19) must always be rational to complete the cycles [10]. The input angle for \( g_{nw} \) is in (21). The output non-working angle is obtained from (16) with \( g \) defined by (18) and (22) [6]. Derivatives of \( g \) are given by (23).

\[
g_{nw}(\theta_i) = u_4 \theta_i^4 + u_3 \theta_i^3 + u_2 \theta_i^2 + u_1 \theta_i + u_0 \tag{18}
\]

\[
2\pi = \int_0^{2\pi} g d\theta_i \tag{19}
\]

\[
u = \Omega^{-1} H \tag{20}
\]

2.4 Non Circular Pith Curve

The pitch curves profiles input \( r_i \) and output \( r_o \) are related in (24) with \( E \) as the distance between centers as shown in Figure 3. It is possible to relate the gear ratio \( g \) with the pitch curves by defining the quotient \( r_i \) and \( r_o \), as in (25), where \( r_0 \) and \( r_i \) are given by (26) and (27), respectively [6].
3 Results and Discussion

3.1 Kinematics Results

Consider the synthesis of a mechanism with input data given by: \( L_0=90 \text{mm}; L_1=70 \text{mm}; L_2=80 \text{mm}; L_3=60 \text{mm}; L_4=90 \text{mm}; \theta_0=0^\circ \), and the path of point P defined by: \( x=[82 84 86 88 90 92 94 96 98] \text{mm}, f(x)=[91 96 98 100 100 102 105 104 107] \text{mm} \). The solution of the kinematics to this case is presented in Figure 4.A, where four possible resulting solutions are visualized from the combination of the elbow up and the elbow down [2,11]. Figure 4.B shows g values for the working section in each combination shown in Figure 4.A. On its hand, Figure 5.A presents the output angle and gear ratio g versus input angle for the BAAB configuration which was found to present the best gear ratio when evaluated at the connecting link in element 4 of the mechanism. Figure 6.B shows the complete path of the gear ratio. Since negative values are found, optimization of the mechanism must be pursued. It is presented next.

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Figure 2. A Gear ratio for all movable links. Source: Authors, adapted from [8]-B
Working and non-working sections in a non-circular gear. Source: Authors, adapted from [6]- C Boundary conditions to obtain the complete profile of the non-circular gear. Source: Authors, adapted from [6]
\[ u = \begin{bmatrix} u_4 \\ u_3 \\ u_2 \\ u_1 \\ u_0 \end{bmatrix} \quad H = \begin{bmatrix} g_{w}(\theta_{\text{int}}) \\ g_{w}'(\theta_{\text{int}}) \\ g_{w}(0) \\ g_{w}'(0) \\ 2\pi - \int_{0}^{\theta_{\text{int}}} g_{w}(\theta_i) d\theta_i \end{bmatrix} \quad \Omega = \begin{bmatrix} \theta_{\text{int}}^i & \theta_{\text{int}}^i & \theta_{\text{int}}^i & \theta_{\text{int}}^i & \theta_{\text{int}}^i \\ 4\theta_{\text{int}}^i & 3\theta_{\text{int}}^i & 2\theta_{\text{int}}^i & 1 & 0 \\ 2(2\pi)^i & (2\pi)^i & (2\pi)^i & (2\pi)^i & 1 \\ 3(2\pi)^i & 2(2\pi)^i & 1 & 0 \\ (2\pi)^i - \theta_{\text{int}}^i & (2\pi)^i - \theta_{\text{int}}^i & (2\pi)^i - \theta_{\text{int}}^i & (2\pi)^i - \theta_{\text{int}}^i & 2 - 2\pi - \theta_{\text{int}}^i \end{bmatrix} \]

\[ \theta_{\text{inw}} = 2\pi - \theta_{\text{int}} \quad (21) \]

\[ \theta_{\text{onw}} = \sum_{i=0}^{4} \left[ \frac{\pi}{2} - \theta_{\text{int}}^i \right] \quad (22) \]

\[ g_i' = \frac{d\theta_i}{d\theta_i}, \quad g_i^l = \frac{\frac{d}{d\theta_i} (\theta_{\text{int}} - \theta_{\text{int}}^i)}{d(\theta_{\text{int}} - \theta_{\text{int}}^i)} \quad (23) \]

\[ r_0 + r_i = E \quad (24) \]

\[ g = \frac{r_i}{r_0} \quad (25) \]

\[ r_0 = \frac{E}{1+g} \quad (26) \]

\[ r_i = \frac{E}{1+g} \quad (27) \]

---

**Figure 3.** Rectangular coordinate systems for a pair of non-circular gears. Source: Authors, adapted from [6].

### 3.1 Optimization of Gear Relations

Since negative values in gear ratio in Figure 6. B are apparent, it follows the optimization process. To this end, link lengths and ground angle changes are evaluated. Changing from positive to negative values of the gear ratio imply an unfeasible gear connection that changes from external to internal connection [10]. The gear ratio, \( g^* \), is set as the objective function to optimize \( f_{\text{obj}} \) as in (28), with the specification of the boundary conditions given in (29). In the latter equation, condition a) ensures that the shortest link \( L_{\text{min}} \) must not be less than a minimum feasible condition \( L_{\text{Lim}} \), condition b) proposes that the average gear ratio, \( \tau_{\text{mean}} \), must be within acceptable values of non-circularity, set as \([0.5,1.5]\) [10]. Optimal design also requires link lengths be as short as possible.

\[ f_{\text{obj}} = g^* \quad (28) \]
a) \( L_{min} \geq L_{lim} \)

b) \( 0.5 \leq \tau_{mean} \leq 1.5 \)  

(29)

Table I shows the gear ratios obtained when modifying the input data, where \( g_{Li} \) is the new gear ratio obtained by modification of the length \( L_i \), and \( g_{\theta_0} \) represents the new gear ratio due to the variation of the ground angle \( \theta_0 \). Table I also presents an indicator, \( v \), to establish the “viability” of the results; showing “1” if they are optimal or “0” for unsatisfactory result. The modifications made one at a time are: \( L_0 = 70 \text{mm}, L_1 = 65 \text{mm}, L_2 = 75 \text{mm}, L_3 = 30 \text{mm}, L_4 = 72 \text{mm}, \theta_0 = 1 \text{ rad} \). Best results were obtained when changing \( L_3 \) from 60mm to 30mm. The resulting gear ratios for each case are illustrated in Figure 6.A. In Figure 6.B, the optimized path of the gear ratio is presented where \( g_w \) and \( g_{nw} \) are shown, in addition to the output angle. Figure 6.B the primitive curves of the non-circular bodies in mesh.

Figure 4. A. Four possible configurations of the mechanism for the prescribed path, in mm-B. Working gear ratio for each configuration. Source: Authors

Figure 5. A. Output angle and gear ratio \( g \) versus input angle. B. Complete gear ratio versus input angle. Source: Authors
Table I. Comparison $g$ values for changes in link lengths and ground angle

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4 Conclusions and Recommendations

The reduction of mobility in a five-bar planar mechanism considering the integration of a non-circular gear connection synthesized for a prescribed path generation was summarized in this work. Direct and inverse kinematics of the mechanism was carried out for a point of interest or location of the work tool. Then, equations for the gear ratio could be established by relating joint angles derivatives inside the gear ratio expressions as function of the path abscissa differential. This mathematical trick ensures that the gear primitive curve profiles are designed for that specific task. Next, five gear ratios were calculated, one per each element link. However, only one connection is necessary for solving the design problem. Reaching only positive values along the working profile was the basis for selecting the best gear ratio choice. Later, the disc of the non-circular gear is completed since it is not always required to describe a complete rotation of the non-circular gears to satisfy the kinematic task.

Figure 6. A. Modified gear ratio for $2\pi$ input angle-B. Working and nonworking input angle and gear ratio for $2\pi$ input angle-C. Working and non-working primitive curve profiles, in mm. Source: Authors
The methodology developed was illustrated for the case of the path generation for 9 points in the plane. The gear ratio chosen for synthesis corresponded to the fourth link. Negative values were reached for the non-working section, then optimization process was presented. Results were graphically validated by using Matlab. Finally, profile of the non-circular primitive curves were shown. It is recommended to further expand the number of path points and the synthesis of the teeth in the bodies.

References


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