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Blow-up Analysis for a Non-local Discrete Diffusion System

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Abstract

We made the blow-up analysis for the solutions of the numerical approximation of a non-local reaction-diffusion system in a bounded domain. Moreover, we obtain the blow-up rates for the solution. Finally, we show some numerical experiments which illustrate the results.

Subject Classification: 35K57, 35B40

Keywords: Nonlocal diffusion system; Discrete model; Blow-up rates; numerical experiments

1 Introduction

In recent years, non-local diffusion models have been used to model several problems of the pure and applied mathematics. Analytic studies and numerical approximations for this models have been made for a widely variety of them, see for instance the references [1], [2], [3], [4],[5], [6], [7], [8],[9], [10], [11].

The author in [5] studied the following nonlocal diffusion system with the Neumann boundary conditions

$$\begin{aligned} &u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + v^p(x,t), \quad (x,t) \in \Omega \times (0,T) \\ &v_t(x,t) = \int_\Omega J(x-y)(v(y,t)-v(x,t))dy + u^q(x,t), \quad (x,t) \in \Omega \times (0,T) \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \end{aligned} \tag{1}$$

with $p,\ q > 0$, $u_0(x),\ v_0(x) \in C(\overline{\Omega})$ nonnegative and nontrivial functions and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ a bounded connected and smooth domain. The existence of nonnegative solutions (u, v) was analyzes for (1), showing that the solution (u, v) is unique if $pq \geq 1$ or if at least one of the initials conditions is not zero for $pq < 1$. The globally existence was also studied. The author showed that if $pq > 1$ and $u_0,\ v_0$ are nonnegative and nontrivial functions the solution (u, v) blows-up in finite time T, and if $pq \leq 1$ the solution (u, v) exists globally. The blow-up phenomenon was also analyzed, as well as the blows-up rates. One of the most important results on the system (1) is that it shares important properties with the corresponding local diffusion coupled parabolic system with Neumann boundary conditions, see for instance [6], [7].

Recently, the authors in [8] studied the semi-discrete approximation of (1) in the interval $\Omega = [-L, L]$. With $N \in \mathbb{N}$ and $L > 0$ and for $h = \frac{L}{N} > 0$, they considered the uniform mesh on the interval $[-L, L]$, $P = \{-L = x_{-N}, \cdots, x_0, \cdots, X_N = L\} = \{x_i = hi,\ -N \leq i \leq N\}$ to obtain the following system for $-N \leq i \leq N$

$$
\begin{aligned}
u_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(u_j(t) - u_i(t)) + v_i^p(t) \\
v_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(v_j(t) - v_i(t)) + u_i^q(t) \\
u_i(0) &= u_0(x_i), \quad v_i(0) = v_0(x_i),
\end{aligned} \tag{2}
$$

whit $U(t) = (u_{-N}(t), \cdots, u_N(t))$, $V(t) = (v_{-N}(t), \cdots, v_N(t))$, $U_0 = U(0)$, $V_0 = V(0)$. Similar results to those obtained in [5] on the existence and uniqueness of solution to (2) was derived. In the same way, a comparison principle for solutions of (2) was given as well as, a result showing the consistence of the solutions using the approximation.

Our main objective in this paper is to continue the study of the problem proposed in [8]. More exactly, the blow-up phenomenon of the solutions of (2) will be studied. In the same way, we will obtain the blow-up rates for the solutions, in the case that existence of them is not globally. The rest of the paper is organized as follows: in Section 2 we study the globally existence or in other case the blow-up of the solutions of the (2). In section 3, we study the blow-up rates. Finally, in section 4, we show some numerical experiments with the aim to illustrate the results.

2 Globally Existence vs. Blow-up

In this section, we analyze the conditions on the exponent $p,\ q > 0$, for which the solutions of (2) blow-up in finite time or its are global. With this in mind, we begin saying that a solution (U,V) of (2) blows up in finite time if there exists a finite time $T_h > 0$ such that

$$\lim_{t\nearrow T_h} \left(\|U(t)\|_\infty + \|V(t)\|_\infty\right) = \lim_{t\nearrow T_h} \left(\max_k u_k(t) + \max_k v_k(t)\right) = \infty. \tag{3}$$

Remark 2.1. *Let $(u(t),v(t))$ a positive solution of the system*

$$u'(t) = v^p(t), \quad v'(t) = u^q(t) \text{ for } t > 0, \quad u(0) = a > 0,\ v(0) = b > 0. \tag{4}$$

Then (u,v) blows up in finite time if and only if $pq > 1$. (u,v) globally exists if and only if $pq \leq 1$. (See [5]).

We have the Theorem:

Theorem 2.1. *Let (U,V) a solution of (2) with initial datum $(U_0,V_0) > (0,0)$.*

1. *If $pq > 1$ them the solution $(U(t),V(t))$ of (2) blows up in finite time T_h*
2. *If $pq \leq 1$ them the solution $(U(t),V(t))$ of (2) globally exists.*

Proof. 1. First, we suppose there exists a constant $c > 0$ such that $u_i(0) \geq c > 0$ and $v_i(0) \geq c > 0$ for all $i = -N, \cdots, N$. Let $(w(t),z(t))$ the solution of (4) whit $w(0) = z(0) = c$ initial condition. Moreover, we have that $(w(t),z(t))$ is solution of (2), then by Comparison Principle [8], we have $(w(t),z(t)) \leq (u_i(t),v_i(t))$. As $pq > 1$ and by Remark 2.1 we have $(w(t),z(t))$ blows-up in finite time $\widetilde{T} > 0$, then $(u_i(t),v_i(t))$ blows-up in finite time $T_h < \widetilde{T}$.

Now, we suppose that $u_0 \neq 0$. As the solution (U,V) of (2) is nonnegative, by (2) we have

$$u_i'(t) \geq \sum_{j=-N}^{N} hJ(x_i - x_j)(u_j(t) - u_i(t)). \tag{5}$$

We consider the following problem

$$\underline{u}_i'(t) = \sum_{j=-N}^{N} hJ(x_i - x_j)(\underline{u}_j(t) - \underline{u}_i(t)), \qquad \underline{u}_i(0) = u_i(0). \tag{6}$$

From (5) we obtain $u_i(t)$ is a super-solution of (6), by Comparison Principle (see [8]) we have $\underline{u}_i(t) \leq u_i(t)$. Now, we assert that for $\delta > 0$, we obtain that $\underline{u}_i(\delta) \geq c > 0$ for all $i = -N, \cdots, N$. In fact, otherwise, there exists $k \in \{-N, \cdots, N\}$ such that $\underline{u}_k(\delta) = 0$ and

$$0 \geq \underline{u}_k'(\delta) = \sum_{j=-N}^{N} hJ(x_k - x_j)(\underline{u}_j(\delta) - \underline{u}_k(\delta)) \geq 0. \tag{7}$$

As J is supported in the unit ball $B(x_0, 1) = [-1, 1]$, then

$$\sum_{j=-N}^{N} hJ(x_k - x_j)\underline{u}_j(\delta) = 0, \quad in \quad [x_k - 1, x_k + 1].$$

Repeating the above analysis in any $x_k \in [-1, 1]$, we obtain

$$\sum_{j=-N}^{N} hJ(x_k - x_j)(\underline{u}_j(\delta)) = 0, \; for \; all \; x_k \in [-L, L],$$

since $[-L, L]$ is connected. By the conservation of the mass (see [8]), we have $0 = \sum_{i=-N}^{N} \underline{u}_i(\delta) = \sum_{i=-N}^{N} \underline{u}_i(0) = \sum_{i=-N}^{N} u_i(0) > 0$, which is a contradiction. So that, for $\delta > 0$, we obtain $u_i(\delta) \geq c > 0$ and $v_i(\delta) \geq c > 0$, which leads to the Case 1, therefore $(u_i(t), v_i(t))$ blows-up in finite time. Let's see that $v_i(\delta) \geq c > 0$. Suppose that $v_i(0) = 0$ in $[-L, L]$ and for $\delta > 0$, let $\delta/2 \leq t$ whit $u_i(t) \geq c > 0$. From (2), we have

$$v_i'(t) \geq \sum_{j=-N}^{N} hJ(x_i - x_j)(v_j(t) - v_i(t)) + c^q. \tag{8}$$

We consider the problem

$$\underline{v}_i'(t) = \sum_{j=-N}^{N} hJ(x_i - x_j)(\underline{v}_j(t) - \underline{v}_i(t)), \qquad \underline{v}_i(\delta/2) = 0. \tag{9}$$

Therefore $\underline{v}_i(t) = c^q\delta/2$ for $t > \delta/2$. Moreover, from (8), $v_i(t)$ is super-solution of (9) then by Comparison Principle [8], we have $\underline{v}_i(t) \leq v_i(t)$, then $v_i(t) \geq c^q\delta/2 > 0$.

2. First, we assume $pq < 1$. Let $(w(t), z(t))$ the solution of (4) whit $(w(0), z(0))$ initial condition such that $u_i(0) \leq w(0)$, $v_i(0) \leq z(0)$. As $pq < 1$, by Remark 2.1, we have $(w(t), z(t))$ exists for all $t > 0$. Moreover $(w_i(t), z_i(t))$ is solution of (2), then $u_i(t) \leq w(t)$ and $v_i(t) \leq z(t)$ for all $t > 0$.

Now, we assume without loss of generality that $pq = 1$ and $q \geq 1$. We have that for all $a > 0$, $(\overline{u}_i(t), \overline{v}_i(t)) = (ae^t, a^q e^{qt})$ is a super-solution of (2). Indeed, replacing in (2) we have that $ae^t \geq a^{pq}e^{pqt}$, $qa^q e^{qt} \geq a^q e^{qt}$ if $pq = 1$ and $q \geq 1$. Is suffices to choose $a \geq \max(\|u_0\|_\infty, \|v_0\|_\infty^{1/q})$. □

3 Blow-up rate

Next, we study the blow-up rate of the solutions of (2). First, we have the following Lemma

Lemma 3.1. *Let $(u_0(x), v_0(x))$ non-negative symmetric initial datum in $[-L, L]$, that is $u_0(-x) = u_0(x),\ v_0(-x) = v_0(x)$. Then, the solution (U, V) of (2) is symmetric, that is, it verifies $(u_{-i}(t), v_{-i}(t)) = (u_i(t), v_i(t))$.*

Proof. Let us define $(w_i(t), z_i(t)) = (u_{-i}(t), v_{-i}(t))$. Then $(w_i(t), z_i(t))$ satisfies

$$\begin{aligned} w_i'(t) = u_{-i}'(t) &= \sum_{j=-N}^{N} hJ(x_{-i} - x_j)(u_j(t) - u_{-i}(t)) + v_{-i}^p(t) \\ z_i'(t) = v_{-i}'(t) &= \sum_{j=-N}^{N} hJ(x_{-i} - x_j)(v_j(t) - v_{-i}(t)) + u_{-i}^q(t) \\ w_i(0) = u_{-i}(0), &\quad z_i(0) = v_{-i}(0). \end{aligned} \tag{10}$$

Since the partition $\{x_i = ih, i = -N, ..., N\}$ is symmetric and taking into account that J is symmetric and initial datum is symmetric, we have that the previous equation can be written as

$$\begin{aligned} w_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(w_j(t) - w_i(t)) + z_i^p(t) \\ z_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(z_j(t) - z_i(t)) + w_i^q(t) \\ w_i(0) &= u_i(0), \quad z_i(0) = v_i(0). \end{aligned} \tag{11}$$

By uniqueness of the solutions (2), we have $(w_i(t), z_i(t)) = (u_i(t), v_i(t))$, therefore $(u_{-i}(t), v_{-i}(t)) = (u_i(t), v_i(t))$. □

Theorem 3.1. *Let $pq > 1$ and $u_0(x),\ v_0(x) \in C([-L, L])$ be nonnegative and nontrivial functions. Let (U, V) the solution of (2) such that the maximum is reached at $x_0 = 0$. Then exists $C_1,\ C_2,\ C_3,\ C_4$ positive constants such that*

$$\begin{aligned} C_1(T-t)^{-(p+1)/(pq-1)} \le u_0(t) \le C_2(T-t)^{-(p+1)/(pq-1)} \\ C_3(T-t)^{-(q+1)/(pq-1)} \le v_0(t) \le C_4(T-t)^{-(q+1)/(pq-1)}. \end{aligned} \tag{12}$$

Proof. As $pq > 1$, we have (U, V) the solution of (2) blow-up in finite time T_h. Let $u_0(t) = \max_{-N \le i \le N} u_i(t)$ and $v_0(t) = \max_{-N \le i \le N} v_i(t)$. By (2), we have

$$\begin{aligned} u_0'(t) &= \sum_{j=-N}^{N} hJ(h(0-j))(u_j(t) - u_0(t)) + v_0^p(t) \le v_0^p(t) \\ v_0'(t) &= \sum_{j=-N}^{N} hJ(h(0-j))(v_j(t) - v_0(t)) + u_0^q(t) \le u_0^q(t) \end{aligned} \tag{13}$$

As $\displaystyle\sum_{j=-N}^{N} hJ(h(i-j)) \leq 1$, we have

$$u_0'(t) \geq -u_0(t) + v_0^p(t), \quad v_0'(t) \geq -v_0(t) + u_0^q(t). \tag{14}$$

Therefore, we have for all $0 < t < T_h$

$$-u_0(t) + v_0^p(t) \leq u_0'(t) \leq v_0^p(t) \tag{15}$$

and

$$-v_0(t) + u_0^q(t) \leq v_0'(t) \leq u_0^q(t). \tag{16}$$

Multiplying the second inequality of (15) by $u_0^q(t)$ and the first inequality of (16) by $v_0^p(t)$, we have

$$u_0'(t)u_0^q(t) \leq v_0'(t)v_0^p(t) + v_0^{p+1}(t),$$

which is equivalent to

$$\left(\frac{u_0^{q+1}(t)}{q+1}\right)' \leq \left(\frac{v_0^{p+1}(t)}{p+1}\right)' + v_0^{p+1}(t).$$

Multiplying the previous inequality by $(p+1)e^{(p+1)t}$ and integrating on $[0,t]$ with $t < T_h$, we have

$$\begin{aligned} u_0^q(t) &\leq \big((q+1)e^{(p+1)t}v_0^{p+1}(t) + C\big)^{q/(q+1)} \\ &\leq \big((q+1)e^{(p+1)T}v_0^{p+1}(t) + C\big)^{q/(q+1)} \\ &\leq C(v_0(t))^{(p+1)q/(q+1)}. \end{aligned} \tag{17}$$

Replacing the second inequality of (16) by the inequality (17), we have

$$v_0(t) \leq C(v_0(t))^{(p+1)q/(q+1)}.$$

Integrating the inequality from above on $[t, T_h)$, we obtain

$$v_0(t) \geq C_3(T_h - t)^{-\beta}, \ \textit{where} \ \ \beta = \frac{q+1}{pq-1}.$$

In analogous form we obtain

$$u_0(t) \geq C_1(T_h - t)^{-\alpha}, \ \textit{where} \ \ \alpha = \frac{p+1}{pq-1}.$$

Doing similar analysis to the one developed above, we obtain that exists constant $C > 0$ such that for $0 < t < T$

$$C(v_0(t))^{(p+1)q/(q+1)} \leq u_0^q(t). \tag{18}$$

Replacing the first inequality of (16) by the inequality (18) and as $pq > 1$ we have $(p+1)q/(q+1) > 1$ and

$$C(v(0,t))^{(p+1)q/(q+1)} \leq -v(0,t) + C(v(0,t))^{(p+1)q/(q+1)} \leq v_t(0,t).$$

Integrating the inequality from above on $[t, T_h)$, we obtain $v_0(t) \leq C_4(T_h - t)^{-\beta}$. In analogous form we obtain $u_0(t) \leq C_2(T_h - t)^{-\alpha}$. □

4 Numerical experiments

In this section, using Matlab (ode15s subroutine), some show some numerical experiments just to illustrate our general results. For that, we will take $L = 2,\ N = 50$ and $J(r) = \max\{0, 3/4(1 - r^2)\}$.

The Figure 1, show the evolution in time of the solution $u,\ v$ with a symmetric initial condition $u_0(x) = v_0(x) = 4 - x^2$ and $p = 1,\ q = 3/4$. We see that the solution is symmetric (Lemma 3.1) and exists globally (Theorem 2.1).

The Figure 2, show the evolution in time of a solution $u,\ v$ with a symmetric initial condition $u_0(x) = v_0(x) = 4 - x^2$ and $p = 5/4,\ q = 2$. We see that the solution is symmetric (Lemma 3.1) and blows up in finite time $T_h = 1.11$ (Theorem 2.1). The Figure 3, show the evolution of the logarithm of the maximum corresponding to solution $u(0,t),\ v(0,t)$ vs. $-a\ln(T - t)$, in dashed line. We see that for u the slope of the graph is approximately $3/2 = \alpha = \frac{p+1}{pq-1}$ and for v the slope of the graph is approximately $2 = \beta = \frac{q+1}{pq-1}$, the exponents that appears in the blow-up rates (Theorem 3.1)

The Figure 4, show the evolution in time of a solution $u,\ v$ with a no symmetric initial condition $u_0(x) = v_0(x) = \max\{0, 1 - |1 - x|\}$ and $p = 2/3,\ q = 1/2$. We see that the solution is no symmetric and exists globally (Theorem 2.1).

The Figure 5, show the evolution in time of the solution $u,\ v$ with a symmetric initial condition $u_0(x) = v_0(x) = \max\{0, 1 - |1 - x|\}$ and $p = 2,\ q = 3/2$. We see that the solution is no symmetric and blows up in finite time $T_h = 1.531$ (Theorem 2.1). The Figure 6, show the evolution of the logarithm of the maximum corresponding to solution $u(0,t),\ v(0,t)$ vs. $-a\ln(T - t)$, in dashed line. We see that for u the slope of the graph is approximately $3/2 = \alpha = \frac{p+1}{pq-1}$ and for v the slope of the graph is approximately $5/4 = \beta = \frac{q+1}{pq-1}$, the exponents that appears in the blow-up rates (Theorem 3.1).

5 Conclusion

We have studied the blow-up phenomenon for the solutions (U, V) of the nonlocal diffusion discrete system (2), with initial datum given as $(U_0, V_0) > (0, 0)$. We have obtained conditions under which the solutions blow-up in finite time or the solutions are globally. More exactly, we have obtained that if $pq > 1$ the solution blows up in finite time T_h and if $pq \leq 1$ then the solution exists globally. Moreover, the blow-up rate of the solutions of (2) were obtained. These obtained results can be compared with those obtained for the solutions of the continuous problem (1), which can be seen in [6]. Finally, some graphics corresponding to some numerical experiments were showed with the aim to confirm and illustrate the obtained results.

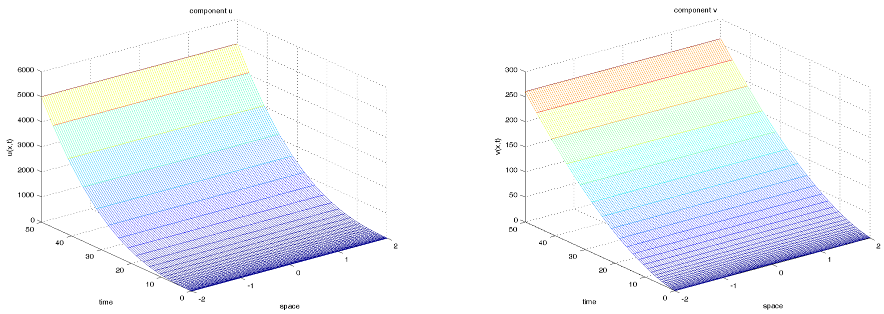

Figure 1: Symmetric case, $p = 1,\ q = 3/4$

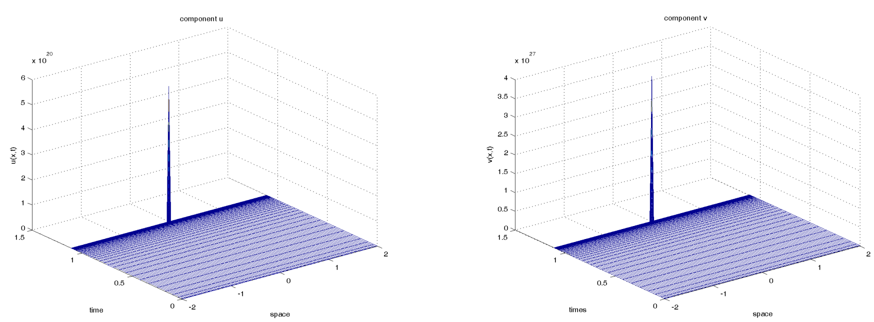

Figure 2: Symmetric case, $p = 5/4,\ q = 2$

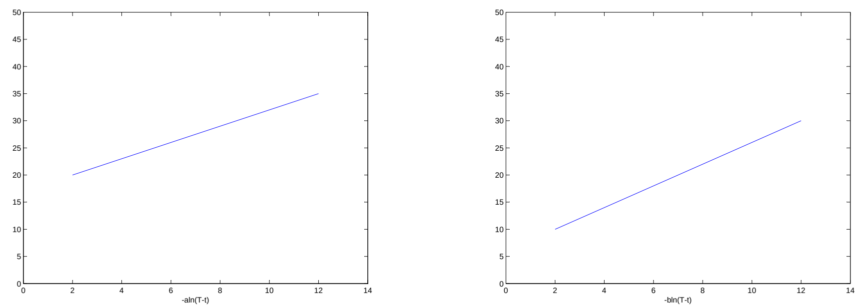

Figure 3: Blow-up rate, $\alpha = 3/2,\ \beta = 2$

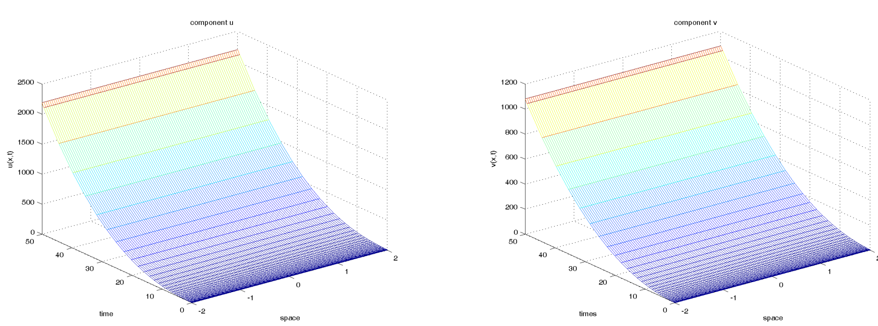

Figure 4: No symmetric case, $p = 2/3,\ q = 1/2$

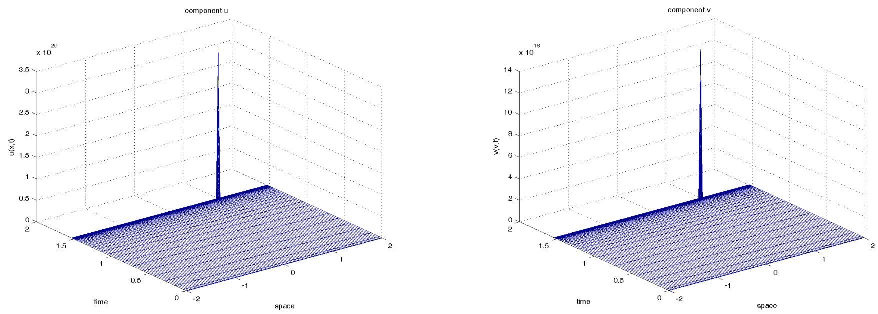

Figure 5: No symmetric case, $p = 2,\ q = 3/2$

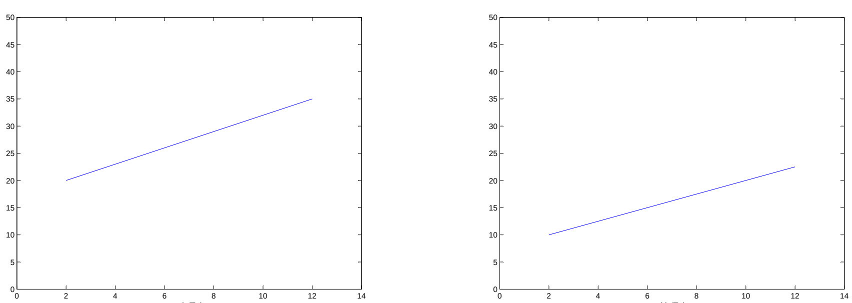

Figure 6: Blow-up rate, $\alpha = 3/2,\ \beta = 5/4$

References

[1] P. Bates, P. Fife, X. Ren and X. Wang, Travelling waves in a convolution model for phase transitions, *Arch. Rat. Mech. Anal.*, **138** (1997), 105-136. https://doi.org/10.1007/s002050050037

[2] M. Bogoya, A nonlocal nonlinear diffusion equation in higher space dimensions, *J. Math. Anal. Appl.*, **344** (2008), 601-615. https://doi.org/10.1016/j.jmaa.2008.02.067

[3] C. Cortázar, M. Elgueta and J. D. Rossi, A non-local diffusion equation whose solutions develop a free boundary, *Ann. Henri Poincaré*, **6** (2005), no. 2, 269-281. https://doi.org/10.1007/s00023-005-0206-z

[4] P. Fife, Some nonclassical trends in parabolic and parabolic-like evolutions, *Trends in Nonlinear Analysis*, Springer-Verlag, Berlin, 2003, 153-191. https://doi.org/10.1007/978-3-662-05281-5_3

[5] M. Bogoya, On non-local reaction-diffusion system in a bounded domain, *Boundary Value Problems*, **2018** (2018), no. 1, 1-18. https://doi.org/10.1186/s13661-018-0958-2

[6] M. Escobedo, M. A. Herrero, A semilinear parabolic system in a bounded domain, *Ann. Matematica Pura Appl.*, **165** (1993), 315-336. https://doi.org/10.1007/bf01765854

[7] C. V. Pao, On Nonlinear Reaction-Diffusion Systems, *J. Math. Anal. Appl.*, **87** (1982), 165- 108. https://doi.org/10.1016/0022-247x(82)90160-3

[8] M. Bogoya and C. A. Gómez, Numerical analysis for a non-local diffusion system, *Contemporary Engineering Sciences*, **11** (2018), no. 45, 2225 - 2236. https://doi.org/10.12988/ces.2018.85236

[9] G. Acosta, J. Fernádez Bonder, P. Groisman and J.D. Rossi, Simultaneous vs. non-simultaneous blow-up in numerical approximations of a parabolic system with nonlinear boundary conditions, *ESAIM: Math. Model. Numer. Anal.*, **36** (2002), no. 1, 55–68. https://doi.org/10.1051/m2an:2002003

[10] P. Groisman, F. Quirós and J.D. Rossi, Non-simultaneous blow-up in a numerical approximation of a parabolic system, *Comput. Appl. Math.*, **21** (2002), no. 3, 813-831.

[11] M. Bogoya and C. A. Gómez, Modelo discreto para una ecuación de difusión no local, *Revista Colombiana de Matemáticas*, **41** (2013), no. 1, 83-94.

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