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A Generalized Two-Mode KdV Equation: Exact Solutions

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Abstract

In this paper, we consider a generalization of the two-mode KdV equation (GTMKdV). The model have variable coefficients (depending on time t) from which, the classical two-mode KdV equation (TMKdV) as well as KdV-type equations can be derived as particular cases. We obtain exact solutions for the generalized model and we derive, new solutions for the particular cases. Periodic and soliton solutions are formally derived using the improved tanh-coth method. Based on the graphics of some solutions we show the several structure that we can obtain for them.

Keywords: Traveling wave solutions; Soliton solutions; periodic solutions; two-mode KdV equation (TMKdV); Improved tanh-coth method; generalized two-mode KdV equation (GTMKdV)

1 Introduction

This paper deal with the following generalized two-mode KdV equation (GTMKdV)

$$\begin{cases} u_{tt} + (c_1(t) + c_2(t))u_{xt} + c_1(t)c_2(t)u_{xx}+ \\ ((\alpha_1(t) + \alpha_2(t))\frac{\partial}{\partial t} + (\alpha_1(t)c_2(t) + \alpha_2(t)c_1(t))\frac{\partial}{\partial x})uu_x+ \\ ((\beta_1(t) + \beta_2(t))\frac{\partial}{\partial t} + (\beta_1(t)c_2(t) + \beta_2(t)c_1(t))\frac{\partial}{\partial x})u_{xxx} = 0. \end{cases} \tag{1}$$

In the case that $c_i(t) = c_i$, $\alpha_i(t) = \alpha_i$, $\beta_i(t) = \beta_i$, $i = 1, 2$ are constants and $\alpha_i \geq -1$, $\beta_i < 1$, $i = 1, 2$, the model (1) describes the propagation of two different wave modes in the same direction simultaneously with the same dispersion relation but, different phase velocities c_1 and c_2, different nonlinearity α_1, α_2 and different dispersion parameters β_1, β_2. In this last case $u = u(x,t)$ is the unknown function which represent the height of the water's free surface above a flat bottom, $x, t \in \mathbb{R}$ and the model is called two-mode KdV equation (TMKdV). Exact traveling wave solutions for TMKdV equations have been construct in the references [1][2][3][4]. The importance of the model (1) is that we can to derive two particular cases: the classical TMKdV equation [2][3][4] mentioned early and the following generalized KdV-type equation [2]

$$u_t + c_{11}(t)u_x + \alpha_{11}(t)uu_x + \beta_{11}(t)u_{xxx} = 0. \tag{2}$$

Clearly, from (2) we can to have the classical KdV equation [2]

$$u_t + 6uu_x + u_{xxx} = 0, \tag{3}$$

and the following important model used in dynamic of fluids

$$u_t + k_1 t^n uu_x + k_2 t^m u_{xxx} = 0. \tag{4}$$

As can be seen in the recent literature [5], the study of nonlinear differential equations with variable coefficients and forcing term, is a relevant area of investigations today by several reasons: First, they are a generalized models, from which, important particular cases can be derived. Second, they give us a new line of investigation in the area of nonlinear analysis. Finally, they have applications in some branch of sciences.

The main objective of this paper is to obtain exact solutions for the model described by (1) using the improved tanh-coth method [6]. From the obtained solutions, we show new exact solutions for the classical TMKdV equation. Clearly, solutions for (2) and (4) can be derived. We made the graphic of some solutions for some of the models mentioned early. The obtained solutions for the TMKdV equations are compared with those obtained, for instance in [4].

2 Traveling wave solutions for GTMKdV equation

With the aim to obtain exact traveling wave solutions for (1) we first consider the wave transformation

$$\begin{cases} u(x,t) = v(\xi), \\ \xi = x + \lambda t + \xi_0. \end{cases} \tag{5}$$

Here, ξ_0 arbitrary constant and λ the wave speed. Substituting (5) into (1), we have the equation

$$\begin{cases} \left[\lambda^2+\lambda(c_1(t)+c_2(t))+c_1(t)c_2(t)\right]v''(\xi)+ \\ \left[\lambda(\alpha_1(t)+\alpha_2(t))+(\alpha_1(t)c_2(t)+\alpha_2(t)c_1(t))\right](v'(\xi))^2+ \\ \left[\lambda(\alpha_1(t)+\alpha_2(t))+(\alpha_1(t)c_2(t)+\alpha_2(t)c_1(t))\right]v(\xi)v''(\xi)+ \\ \left[\lambda(\beta_1(t)+\beta_2(t))+(\beta_1(t)c_2(t)+\beta_2(t)c_1(t))\right]v''''(\xi)=0, \end{cases} \tag{6}$$

where $'$ is the ordinary differentiation $v'(\xi)=\frac{dv}{d\xi}$. Now, according with the improved tanh-coth method [6], we seek solutions for (6) in the form

$$v(\xi)=\sum_{i=0}^{M}a_i(t)\phi(\xi)^i+\sum_{i=M+1}^{2M}a_i(t)\phi(\xi)^{M-i}, \tag{7}$$

where M is a positive integer to be determinate later, $a_i(t)$ functions depending on t and $\phi(\xi)$ satisfying the Riccati equation

$$\phi'(\xi)=a(t)+b(t)\phi(\xi)+c(t)\phi(\xi)^2. \tag{8}$$

Now, substituting (7) into (6) and balancing $v''''(\xi)$ with $(v'(\xi))^2$ we obtain $M+4=2(M+1)$, so that, $M=2$. With this value, (7) reduces to

$$v(\xi)=a_0(t)+a_1(t)\phi(\xi)+a_2(t)\phi(\xi)^2+a_3(t)\phi(\xi)^{-1}+a_4(t)\phi(\xi)^{-2}. \tag{9}$$

Substituting (9) into (6) and taking into account (8) leads us to an algebraic system in the unknowns $a_i(t)$, $(i=0,\dots,4)$, $a(t)$, $b(t)$, $c(t)$ and $\lambda(t)$. Using the *Mathematica* software, and for sake of simplicity, we consider only the following two solutions:

First Case:

$$\begin{cases} a_0(t)= \\ \frac{-8a(t)\beta_1(t)c(t)\lambda-8a(t)\beta_2(t)c(t)\lambda-8a(t)\beta_2(t)c(t)c_1(t)-8a(t)\beta_1(t)c(t)c_2(t)-c_1(t)\lambda-c_2(t)\lambda-c_1(t)c_2(t)-\lambda^2}{\alpha_1(t)c_2(t)+\alpha_2(t)c_1(t)+\alpha_1(t)\lambda+\alpha_2(t)\lambda}, \\ a_2(t)=-\frac{12c^2(t)(\beta_1(t)c_2(t)+\beta_2(t)c_1(t)+\beta_1(t)\lambda+\beta_2(t)\lambda)}{\alpha_1(t)c_2(t)+\alpha_2(t)c_1(t)+\alpha_1(t)\lambda+\alpha_2(t)\lambda}, \\ a_4(t)=-\frac{12a^2(t)(\beta_2(t)(c_1(t)+\lambda)+\beta_1(t)(c_2(t)+\lambda))}{\alpha_2(t)(c_1(t)+\lambda)+\alpha_1(t)(c_2(t)+\lambda)}, \\ a_1(t)=a_3(t)=0, \quad b(t)=0. \end{cases} \tag{10}$$

With this set the values, the solution of (8) is given by [7]

$$\phi(\xi) = \frac{-\sqrt{-4a(t)c(t)}\tanh[\frac{1}{2}\sqrt{-4a(t)c(t)}\xi]}{2c(t)}, \quad -4a(t)c(t) \neq 0, \tag{11}$$

being $a(t)$ and $c(t)$ arbitrary functions depending on variable t. In accordance with(5) and (9) the respective solution for (1) take the form

$$u(x,t) = v(\xi) = a_0(t) + a_2(t)\phi(\xi)^2 + a_4(t)\phi(\xi)^{-2}, \tag{12}$$

where $a_0(t)$, $a_2(t)$ and $a_4(t)$ are the values given in (10), $\phi(\xi)$ given by (11) and $\xi = x + \lambda t + \xi_0$ with ξ_0, λ arbitrary constants.

Second Case:

$$\begin{cases} a_0(t) = \frac{1}{12c^2(t)(\alpha_1(t)\beta_2(t)-\alpha_2(t)\beta_1(t))(a_2(t)(\alpha_1(t)+\alpha_2(t))+12(\beta_1(t)+\beta_2(t))c^2(t))} \\ \begin{cases} -12a_2(t)\alpha_2(t)b^2(t)\beta_1^2(t)c^2(t) + 12a_2(t)\alpha_1(t)b^2(t)\beta_2^2(t)c^2(t)+ \\ 12a_2(t)\alpha_1(t)b^2(t)\beta_1(t)\beta_2(t)c^2(t) - 12a_2(t)\alpha_2(t)b^2(t)\beta_1(t)\beta_2(t)c^2(t) \\ -a_2^2(t)\alpha_2^2(t)b^2(t)\beta_1(t) - a_2^2(t)\alpha_1(t)\alpha_2(t)b^2(t)\beta_1(t) + a_2^2(t)\alpha_1^2(t)b^2(t)\beta_2(t)+ \\ a_2^2(t)\alpha_1(t)\alpha_2(t)b^2(t)\beta_2(t) - 12a_2(t)\alpha_2(t)\beta_1(t)c^2(t)c_1(t) \\ -12a_2(t)\alpha_1(t)\beta_2(t)c^2(t)c_1(t) + 12a_2(t)\alpha_2(t)\beta_1(t)c^2(t)c_2(t)+ \\ 12a_2(t)\alpha_1(t)\beta_2(t)c^2(t)c_2(t) - a_2^2(t)\alpha_1(t)\alpha_2(t)c_1(t) + a_2^2(t)\alpha_1(t)\alpha_2(t)c_2(t) \\ -144\beta_1(t)\beta_2(t)c^4(t)c_1(t) + 144\beta_1(t)\beta_2(t)c^4(t)c_2(t), \end{cases} \\ a_3(t) = a_4(t) = 0, \quad \alpha(t) = 0, \\ a_1(t) = \frac{a_2(t)b(t)}{c(t)}, \\ \lambda(t) = \frac{-a_2(t)\alpha_2(t)c_1(t)-a_2(t)\alpha_1(t)c_2(t)-12\beta_2(t)c^2(t)c_1(t)-12\beta_1(t)c^2(t)c_2(t)}{a_2(t)\alpha_1(t)+a_2(t)\alpha_2(t)+12\beta_1(t)c^2(t)+12\beta_2(t)c^2(t)}. \end{cases} \tag{13}$$

With respect to (13) then the solution for (8) take the form

$$\phi(\xi) = \frac{-\sqrt{b^2(t)}\tanh[\frac{1}{2}\sqrt{b^2(t)}\xi] - b(t)}{2c(t)}, \quad b(t) \neq 0, \tag{14}$$

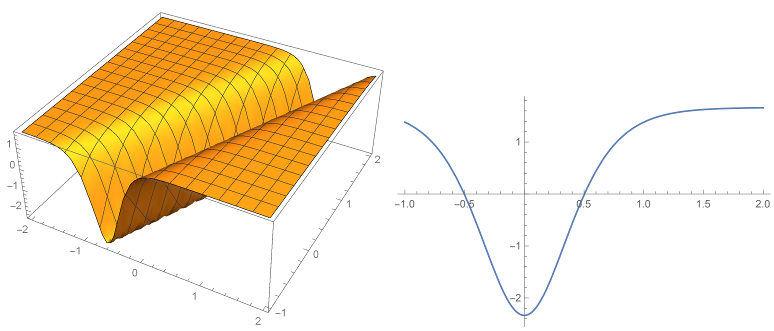
with $b(t)$ and $c(t)$ functions depending on variable t. Using (5) and (9) the solution for (1) is then given by

$$u(x,t) = v(\xi) = a_0(t) + a_1(t)\phi(\xi)^2 + a_2(t)\phi(\xi)^2, \tag{15}$$

with $a_0(t)$ and $a_1(t)$ the values given in (13), $\phi(\xi)$ given by (14) , $a_2(t)$ arbitrary function and $\xi = x + \lambda t + \xi_0$ with ξ_0 any constant and λ the value obtained in (13).

3 Results and Discussion

Inspired by the previous works on nonlinear partial differential equations with variable coefficients and some with forcing term [5], we have studied a generalization of the TMKdV equation. The new model have variable coefficients (depending on variable t). The importance of the model is that from it, we can to derive important particular cases as the KdV, modified KdV, and some equations as those studied in the mentioned reference. Clearly, the use of variable coefficients, give us some new structures on the solutions, important for those researches that investigate this type of nonlinear equations. With the aim to illustrate the structure of some of the solutions, we present some graphs corresponding to (15) only, considering two cases: constant coefficients and variable coefficients. On the other hand, as can be seen in [7], other type of solutions can be derived changing the value of the parameters. The figure u_1 is the soliton solution corresponding to following values: $b(t) = 4$, $c(t) = 1$, $c_1(t) = 1$, $c_2(t) = 2$, $\alpha_1(t) = 5$, $\alpha_2(t) = 2$, $\beta_1(t) = 3$, $\beta_2(t) = 2$, $a_2(t) = 1$. The figures u_{11} is the corresponding trajectories of u_1 at $t = 0$ for $x \in [-1, 2]$.



(a) u_1 (b) u_{11}

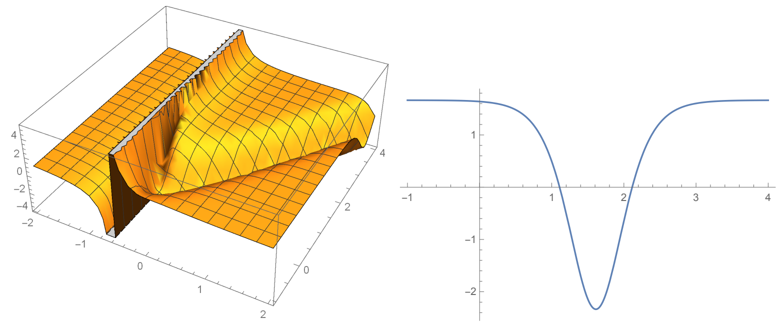
Figure 1: $u_1(x, t)$.

The figure u_2 is the soliton solution corresponding to values $\alpha_1(t) = 5\,t$, $\beta_1(t) = 3\,t$ and rest of values are the same that previous. The figures $u_{21}$is the corresponding trajectories of u_2 at $t = 1$ for $x \in [-1, 4]$.

Clearly, el use of variable coefficients, give us solutions with very different structures, however, the solutions presented in this work corresponding to the standard TMKdV are new, compared with those obtained for instance in [2][3][4].

4 Conclusion

Inspired by some works related with nonlinear partial differential equations with variable coefficients of forcing term, periodic and soliton solutions have



(a) u_2 (b) u_{21}

Figure 2: $u_2(x,t)$

been formally obtained for a generalization of the two-mode KdV equation (GTMKdV). Solutions for particular cases have been derived from it, an as special case, new periodic and soliton solutions have obtained for the classic TMKdV equation. We showed some graphic of the solutions with the aim to illustrate the several structure of solitons that can be considered using variable coefficients. The model studied, clearly is related with the standard KdV, so that it can have applications in naval engineering in the case of constant coefficients or in fluid mechanics, in the case of variable coefficients [5].

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