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Lyapunov Exponents Analysis and Phase Space Reconstruction to Chua's Circuit

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Abstract

The chaos in a system is a very important element for a complete analysis in its dynamics. This is an indicator that allows to quantify this phenomenon which can be studied using the exponents of Lyapunov [1]. Here we present the methodology of determining them from the dynamics of the system (differential equations). When the dynamics of the system are known, it is necessary to reconstruct the phase space from the time series that are arranged.

Keywords: Chua's circuit, Lyapunov exponents, chaos

1 Introduction

The Chua's circuit can be seen as a model system for chaos Many algorithms designed to investigate the different characteristics of systems in which chaos occurs, for example, to differentiate chaos of random noise or reconstruction of the general attractor which uses data generated in a circuit of Chua. It is fundamental to such studies that physical realization is accurately described by the theoretical model, since this could cause a difference, not because of a faulty algorithm, but because of the inadequacy of the original model.

Dynamic systems can be described by a set of ordinary differential equations, which when going into detail in terms of their modeling increases their complexity becoming non-linear. The analysis of linear dynamic systems in phase space has reduced behaviors; in terms of non-linear systems they contain a variety of scenarios, stationary or convergent solutions, periodic solutions, quasi-periodic or chaotic solutions. The chaos has been widely studied by mathematicians and engineers since its discovery by Edward Lorenz in 1970s [1]. The main characteristic of these systems is the extreme sensitivity to the initial conditions, that is, for nearby trajectories initially their distance in the state space diagram will increase exponentially, being quantified by the exponents of Lyapunov [2].

2 Estimation of the Lyapunov exponents

2.1 Reconstruction of phase space

In a dynamic system, it is normal that all its variables can not be measured simultaneously, in the worst case there is only one measurement (time series) $x(t)$, $\forall t = 1, 2, 3, \ldots$. The attractor in the multivariable space is necessary, which is known as reconstruction of the phase space, originally proposed by Ruelle [2]. This technique converts a scalar series $x(t)$ in a vector $\overrightarrow{v}(t)$ of dimension m (embedment dimension) using a delay time τ.

$$\overrightarrow{v}(t) = [v_1(t)\, v_2(t) \ldots\ v_m(t)]^T$$

$$\overrightarrow{v}(t) = [x(t)\ \ x(t+\tau) \ldots\ x(t+(m-1)\tau)]^T\ \in\ \mathbb{R}^m \tag{1}$$

Here, T is the transpose. The proper choice of delay time and embedment dimension is very important for the correct reconstruction of the attractor. There are methods to determine the delay time, such as the false neighboring neighbors. On the other hand, to determine the dimension of embedding of a

p-dimensional system is obtained according to the formulation of [3].

$$m \leq 2p + 1 \tag{2}$$

2.2 Lyapunov exponents

For an p-dimensional system described by $\dot{x}\left(t\right) = F(x,t)$, the $i-th$ exponent of Lyapunov(λ_i) is defined according to the variations of the trajectories from an initial point x_0 in t_0, these variations can be seen as the degeneration of the radius of a hyper sphere $\delta_0(t)$ in an ellipsoid of radio $\delta_i(t)$, i.e.,

$$\delta_i\left(t\right) = \delta_0 e^{\lambda_i (t-t_0)} \tag{3}$$

The classical procedure for the estimation of the exponents of Lyapunov was proposed by Wolf in 1985 [4], which consists in evaluating the local divergences from the application of the tangent plane associated with the system of state equations on n orthogonal vectors $\delta_{1x}\left(t\right), \delta_{2x}\left(t\right), \ldots, \delta_{nx}\left(t\right)$ defined initially by the identity matrix I_n. The variational expressions give rise to the tangent space, defined as:

$$\dot{\psi}\left(x,t\right) = J(x,t)\psi(x,t), \tag{4}$$

where $J(x,t)$ represents the Jacobian of $F(x,t)$. The system of equation of state together with the variational expressions are integrated from x_0, with $\psi\left(x_0\right) = I_n$ for a time T and thus obtain the divergent vectors transformed by the application of the tangent plane, that is, for the first exponent ${\delta_{1x}}^{(1)} = \psi(x,t){u_1}^{(0)}$ where ${u_1}^{(0)} = {\delta_{1x}}^{(1)} / \left\|\delta_{1x}^{1}\right\|$ and the superscript represents the current iteration of the calculation. By repeating this procedure of integration and normalization K times, the $i-th$ exponent of Lyapunov can be written in the following way:

$$\lambda_i = \lim_{K\to\infty} \frac{1}{KT} \sum_{k=1}^{K} ln \left\| {\delta_{ix}}^{(k)} \right\|, \tag{5}$$

in each iteration made the direction of the vectors $\delta_{1x}\left(t\right), \delta_{2x}\left(t\right), \ldots, \delta_{nx}\left(t\right)$ it must be aligned with the expansion direction of the system dynamics, for this reason the Gram-Schmidt method must be used, as shown below

$$
\begin{aligned}
v_1{}^{(k)} &= \delta_{1x}{}^{(k)} \\
u_1^{(k)} &= \frac{v_1{}^{(k)}}{\|v_1{}^{(k)}\|} \\
v_2{}^{(k)} &= \delta_{2x}{}^{(k)} - < \delta_{2x}{}^{(k)}, u_1{}^{(k)} > u_1{}^{(k)} \\
u_2{}^{(k)} &= \frac{v_2{}^{(k)}}{\|v_2{}^{(k)}\|} \\
&\vdots \\
v_n{}^{(k)} &= \delta_{nx}{}^{(k)} - < \delta_{nx}{}^{(k)}, u_1{}^{(k)} > u_1{}^{(k)} - \ldots - < \delta_{nx}{}^{(k)}, u_{n-1}{}^{(k)} > u_{n-1}{}^{(k)} \\
u_n{}^{(k)} &= \frac{v_n{}^{(k)}}{\|v_n{}^{(k)}\|}
\end{aligned} \tag{6}
$$

Once the exponents of Lyapunov have been calculated, the maximum number of them must be identified, since it is the one in charge of determining the presence of chaos or not in said dynamic system. For this, it is clear that if you have $\lambda_{max} > 0$, the initially close trajectories in the phase space will tend to separate, that is, the system presents a high sensitivity to the initial conditions.

3 Chua circuit dynamic analysis

The Chua's circuit is a simple electronic circuit that highlights a chaotic regime. It consists mainly of a combination of simple components equivalent to a perfect coil and a second dipole resulting from the parallel association of two negative resistances, called *Chua diode*. Since its discovery in 1983, the Chua circuit has served as the basis for the study of the chaos in electronic systems [5]. This circuit is composed of two capacitors (U_1 and U_2), an inductance (I_3), a resistance (R) and a non-linear resistance (I_4), in Fig.1 the canonical circuit can be observed.

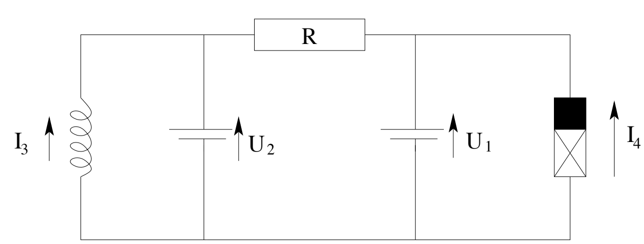

Figure 1: Chua's circuit.

Solving the circuit by state variables, the system is described by differential equations.

$$\begin{aligned} \dot{x_1} - \frac{1}{U_2 R}(x_2 - x_1) + f(x_1) = 0 \\ \dot{x_2} - \frac{1}{U_1 R}(x_1 - x_2) - x_3 = 0 \\ \dot{x_3} + \frac{1}{I_3} x_2 = 0. \end{aligned} \tag{7}$$

Here, $x_1 = V_{U_2},\ x_2 = V_{U_1}$ y $x_3 = I_3$. $f(x_1) = f(V_{U_2})$ is the function that characterizes non-linear resistance, and is defined as follows:

$$f(x_1) = m_1 x_1 + \frac{1}{2}(m_0 - m_1)(|x_1 + E| - |x_1 - E|), \tag{8}$$

where, E is a parameter dependent on the polarization voltage of the circuit. The mathematical model of the Chua's circuit can be normalized according to the proposal by Matsumoto [5], for greater ease in the simulation

$$\begin{aligned} \dot{x} - \alpha(y - h(x)) = 0 \\ \dot{y} - x + y - z = 0 \\ \dot{z} + \beta y = 0, \end{aligned} \tag{9}$$

being α, β system parameters with typical values are $\alpha = 9$, $\beta = 14.286$. And $h(x)$ it is represented in the form:

$$h(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x + 1| - |x - 1|) \tag{10}$$

The values of the constants m_0 and m_1 are assumed for this analysis with values $-1/7$ and $2/7$ respectively. By integrating the system (9) using the *Runge-Kutta45* algorithm, the time series is obtained.

Graphing the phase diagram between V_{U_2} and V_{U_1} shows that the resulting attractor is a limit cycle.

When the control parameter $\alpha = 9$ varies, a great change in the dynamics of the system is observed, becoming a chaotic system (strange attractor) as seen in the phase diagram of Fig.5.

4 Bifurcation analysis

Here, for all α, system (9) has exactly 3 equilibrium points: $(0, 0, 0)$, $(4, 0, -4)$ and $(-4, 0, 4)$. The linearizations of system (9) around $(0, 0, 0)$ and of the (external) equilibria are of the form $\mathbf{x}' = A\mathbf{x}$. The equilibrium $(0, 0, 0)$ is always

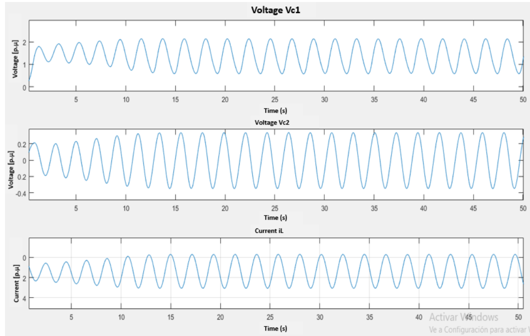

Figure 2: Time series V_{U_2}, V_{U_1} and iI_3 with $\alpha = 7.5$.

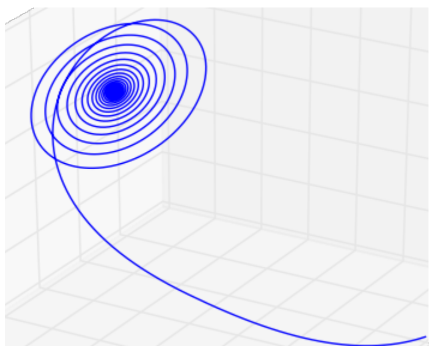

Figure 3: Limit cycle attractor $\alpha = 7.5$.

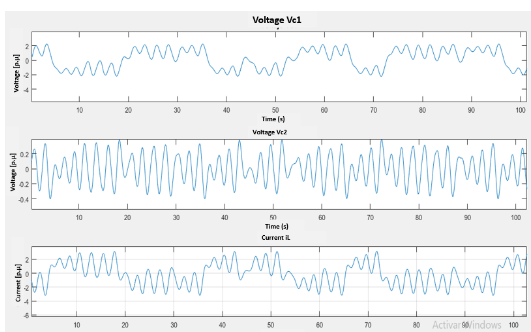

Figure 4: Time series $\alpha = 9$.

unstable because the matrix A has a positive real eigenvalue. Thus, for small values of α, the other two equilibria are stable because all eigenvalues have a strictly negative real part. Analyzing in depth the system, two eigenvalues are complex of the form $a(\alpha) \pm ib(\alpha)$ with $b(\alpha) > 0$. Thus, when α increases, the external equilibria are unstable since $a(\alpha)$ is positive for $\alpha \geq \alpha_h$. We know

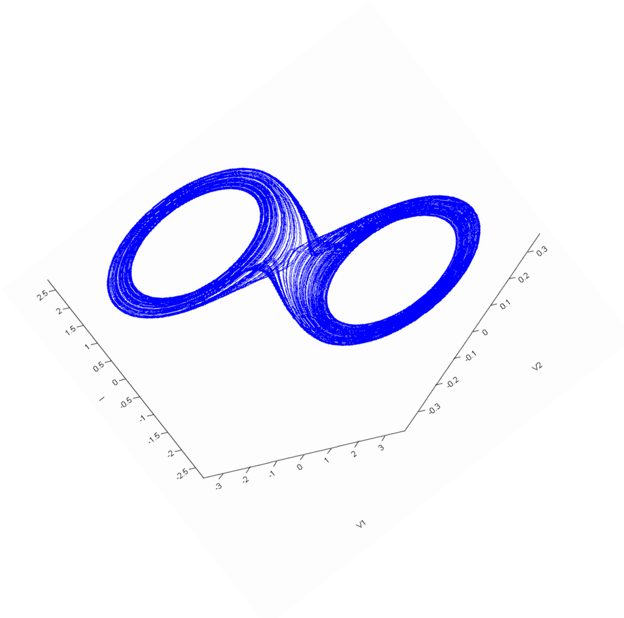

Figure 5: Strange attractor $\alpha = 9$.

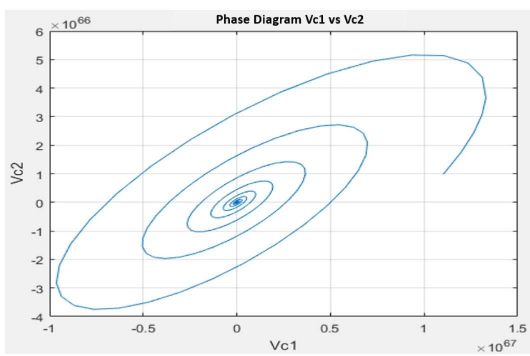

Figure 6: Divergence for $\alpha = 10$.

that this type of bifurcation is called *Hopf*.

Now, for very large values of α, we can observe an attractive chaos, in this case of *Lorentz*. Therefore, the typical trajectories are in a periodic orbit in a neighborhood of another periodic orbit.

5 Conclusion

This work on chaotic systems that we chose allowed us to highlight some of the main characteristics of chaos: sensitivity to initial conditions and deterministic character of chaos. It was possible to trace a strange attractor characteristic of the Chua circuit and take a look the diversity of the behavior of the chaotic systems. Among other things, it has confronted us with several experimental, mechanical, acquisition or signal processing, which forced us to adapt our protocols to our initial forecasts.

The characterization of the dynamics of any system is determined by the exponents of Lyapunov, which, depending on their value, allow a qualitative analysis, establishing that dynamic is converging to a fixed point of the state space, or if it is periodic (limit cycle and bull) or simply if the future states can not be determined from the initial states (strange chaotic-attractor system). Finally, based on the knowledge of the dynamics of a system, a control law can be proposed to stabilize the system around the desired operating point, that is why the phase space must be reconstructed when this dynamic is not known.

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