A Note on the Implementation of the Finite Element to Solve Differential Equations

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Abstract

This paper presents the mathematical analysis of the linear interpolation equations that form a basis around the finite element method for the formulation of a global function by parts that approximates the solution of a second order ordinary differential equation. The numerical solutions for the systematic calculation are compared with three elements and 26 elements supported on a computer with the exact solution that satisfies a second order ordinary differential equation [1]. It is also intended to take the reader with the demonstrations and obtain each of the equations described in this document in such a way that the results can be reproduced manually or with specialized software [5].
Keywords: Linear interpolation, finite element method, the potential and the Elastodynamic equation

1 Introduction

Theories around the approximate techniques to solve differential equations are widely accepted in different areas of knowledge as is usually done in engineering, mathematics, bioengineering sciences among others. Computational techniques such as interpolation of linear bases are known as the finite element method, which allows us to find an approximation of an ordinary or partial differential equation in one, two or three dimensions [3]. To solve the differential equations by means of the finite element method it is necessary to discretize the domain in uniformly spaced fragments depending on the space, that is, linear elements in one dimension, triangles or quadrilaterals in two dimensions or tetrahedral elements of 5 or 8 nodes depending on the desired precision, consequently to the partition of the domain fragments arises the need to find a set of linear equations that allow to build from the previously described bases, a global function that characterizes the solution of the differential equation and this implies a directly proportional relationship between the precision and the number of linear coefficients. On the other hand, the finite element method is proposed as a tool, which is described in the following sections as an alternative to numerically solve ordinary and partial differential equations that model physics and engineering phenomena. The advantage of this method over others is the ability to adapt the finite elements in the form of the domain of interest even though there are discontinuities [4].

2 Finite element method

The finite element method is a numerical method used to solve ordinary and partial differential equations. The ordinary and partial differential equations model phenomena found in different branches such as mathematics, physics and engineering. Some phenomena and models of differential equations are Maxwell’s equations that describe the behavior of electromagnetic variables in their differential form. Other differential equations are: The Laplace equation, the Poisson equation, the Helmholtz equation, the heat diffusion equation, the potential equation and the Elastodynamic equation among others [1]. The advantage of the finite element method over other methods is the partition (fragmentation of the solution space) of the domain into continuous elements that form the original domain, this allows to form in non-conventional domains a mesh that has the form of the domain, that is, If you want to model the behavior of the potential in a material such as a truncated cone, the mesh
generated by the elements (domain partitions) is adapted in said figure, and in general the mesh takes the form of the domain where you are interested to find the solution of a differential equation.

Once the domain $\Omega$ is divided into the elements, a function must be defined in a space $V$ in each element, these function is used locally on each element to form a global function in the domain $\Omega$. The domain partitions are made consecutively and the local interpolation function for each element must meet one of the following conditions [5]:

$$f(x_i) \in V, \quad \text{if}$$

- $f(x_i)$ it is decreasing from the element $i$ to the element $i+1$.
- $f(x_i)$ it is increasing from the element $i$ to the element $i+1$

The finite elements are formed by three characteristics $(i, V, L)$, the number of elements in which the domain $\Omega$ is divided, the function of linear interpolation around adjacent elements that belongs to the space $V$, $L$ the number of nodes that are equal $i+1$ elements. The domain is limited and is enclosed by boundary conditions. The domain divisions are a subset $\mathbb{R}^d = \{d = 1, 2, 3, \ldots\}$ of the domain where you are interested in solving the differential equation. The general solution is the summation of all the linear functions around each element in the following way:

$$V_h(x) = \sum_{i=1}^{L} b_i N_i(x),$$

where the coefficients represent the evaluated values in each node. The values satisfy the boundary conditions of the problem.

3 Application problem and results

We consider the following initial value problem that is modeled with the following ordinary second-order differential equation

$$-\frac{d^2 y}{dx^2} + y = \sin(\pi x)$$

subject to

$$y(0) = 0, \quad y(1) = 0$$

The previous differential equation can be solved exactly with the annular method and suppose a particular solution with a trigonometric form that satisfies the initial conditions and the differential equation. By means of the finite
element method, it is proposed to find a solution sufficiently close to the exact answer, dividing the interval into three and more finite elements to compare the results of both the exact solution and the approximate solutions.

<table>
<thead>
<tr>
<th>$\Omega_1$</th>
<th>$N_1(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_1^{(1)}(x) = 3\left(\frac{1}{3} - x\right)$</td>
<td>$0 \leq x \leq \frac{1}{3}$</td>
</tr>
<tr>
<td>$0$</td>
<td>in another case</td>
</tr>
</tbody>
</table>

Table 1: Linear interpolation function $N_1(x)$

<table>
<thead>
<tr>
<th>$\Omega_2$</th>
<th>$N_2(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_2^{(1)}(x) = 3x$</td>
<td>$0 \leq x \leq \frac{1}{3}$</td>
</tr>
<tr>
<td>$N_2^{(2)}(x) = 3\left(\frac{2}{3} - x\right)$</td>
<td>$\frac{1}{3} \leq x \leq \frac{2}{3}$</td>
</tr>
<tr>
<td>$0$</td>
<td>in another case</td>
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</tbody>
</table>

Table 2: Linear interpolation function $N_2(x)$

<table>
<thead>
<tr>
<th>$\Omega_3$</th>
<th>$N_3(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_3^{(2)}(x) = 3(x - \frac{1}{3})$</td>
<td>$\frac{1}{3} \leq x \leq \frac{2}{3}$</td>
</tr>
<tr>
<td>$N_3^{(3)}(x) = 3\left(1 - x\right)$</td>
<td>$\frac{2}{3} \leq x \leq 1$</td>
</tr>
<tr>
<td>$0$</td>
<td>in another case</td>
</tr>
</tbody>
</table>

Table 3: Linear interpolation function $N_3(x)$

Now, the non-linear function $f(x) = \sin(\pi x)$ is known from the differential equation $-\frac{d^2y}{dx^2} + y = \sin(\pi x)$ and the terms are calculated $f(x_i)$ where $x_i \in \Omega$. $x_1 = 0$, $x_2 = \frac{1}{3}$, $x_3 = \frac{2}{3}$, $x_4 = 1$, $y f(x_1) = f(x_4) = 0$, $f(x_2) = f(x_3) = \frac{\sqrt{3}}{2}$.

From equations of robustness matrix and load vector, the following system of equations is reached to find the coupling parameters $b_i$ and write a solution of the form $V_h(x) = \sum_{i=1}^{L} b_i N_i(x)$. Now, we obtain the function $V_h(x) = b_2 N_2(x) + b_3 N_3(x)$ with domain of finite elements as shown in tables 1, 2 and 3. The solution of the differential equation remains as a function by parts that is governed by the domains of the finite elements $N_2(x) + N_3(x)$. Therefore,

$$V_h(x) = \frac{5\sqrt{3}}{118} (N_2(x) + N_3(x))$$
Next, we compare graphically the results obtained in the previous literal with the exact solution proposed in [2] and the solution obtained by means of software with the exact solution for more finite elements in comparison with the previous literal.

\[ F(x) = \frac{\sin(\pi x)}{1 + \pi^2} \]

With the data of Fig.2 and Fig.3, the percentage error between the approximate solution and the exact solution proposed in [2] is calculated. In order to visualize how the variation of the approximation was in the domain, Fig.4 is plotted, and the error oscillates around 0.05%. On the other hand, it can be noted that the graph of Fig.2 follows the growth trend in the interval \( 0 \leq x \leq \frac{1}{3} \), remains constant in the interval \( \frac{1}{3} \leq x \leq \frac{2}{3} \) due to the sum of the linear basis functions \( N_2(x) + N_3(x) \), finally, the approximate solution decreases in the interval \( \frac{2}{3} \leq x \leq 1 \) how does the exact solution.

From the graphs of Fig.5 and Fig.6 it can be concluded that as the elements increase in what the domain is divided, the approximate response improves and the percentage error is reduced as shown in Fig.6.
4 Conclusion

It is necessary to make a preliminary analysis of the functions that form a linear basis in each element, that is, you guarantee that the functions decay from the element $i$ to $i+1$ and that it increases from the element $i$ to $i+1$ guarantee the integration of the compact support proposed in [1], [2] and [3]. It should be noted that in the bibliography cited in this document, each one of the integrals is solved under the assumption of compact support integration, but it is not clear which one or which were the functions that were actually used for that purpose, against part, which is developed in this document was precisely to bring the reader the systematic way of solving a second order ordinary differential equation with the finite element method and the construction of the global function that roughly describes the solution [2].
Figure 4: Percentage error

Figure 5: Approximate solution with 25 finite elements.

Figure 6: Percentage error between the exact solution and the approximate solution 26 finite elements.
As the divisions of the domain where you are interested in solving a differential equation increase, the amount of coefficient of the robustness matrix and the terms of the load vector increase since more images of the non-linear function of the equation are taken differential and consequently increase the linear basis functions in each element that belong to the domain. The finite element method is proposed as a computational tool to solve partial differential equations due to the advantage of adapting the elements to the figure that forms the domain of interest, that is, if it is required to solve the equation that describes the phenomenon of heat dissipation in a plate with a discontinuity, the elements in which the domain is fragmented form a mesh that adapts to the discontinuity and takes into account both the initial and border conditions in said domain as proposed in [1].

Acknowledgements. We would like to thank the referee for his valuable suggestions that improved the presentation of this paper and our gratitude to the Department of Mathematics of the Universidad Tecnológica de Pereira (Colombia) and the group GEDNOL.

References


Received: April 17, 2018; Published: July 6, 2018