Contemporary Engineering Sciences, Vol. 11, 2018, no. 45, 2225 - 2236
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ces.2018.85236

Numerical Analysis for a Non-local Diffusion System

Mauricio Bogoya and Cesar A. Gómez

Department of Mathematics
Universidad Nacional de Colombia, Bogotá, Colombia

Abstract

We study the numerical approximation for the following non-local reaction-diffusion system

$$\begin{aligned} u_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(u_j(t) - u_i(t)) + v_i^p(t) \\ v_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(v_j(t) - v_i(t)) + u_i^q(t) \\ u_i(0) &= u_0(x_i), \quad v_i(0) = v_0(x_i), \end{aligned}$$

in a bounded domain, where $p,\ q > 0$, $-N \leq i \leq N$ and $u_0(x),\ v_0(x) \in C(\overline{\Omega})$ are nonnegative and nontrivial functions and $\Omega \subset \mathbb{R}^N$ is a bounded, connected and smooth domain. We prove the existence and uniqueness of solution if $pq \geq 1$ or if one of the initials conditions is different from zero and $pq < 1$. We analyzed a comparison principle for the solutions. Finally, we study the convergence of the numerical scheme.

Keywords: Nonlocal diffusion system; Coupled system; Discrete model; convergence

1 Introduction

Equations of the form

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^n} J(x-y)u(y,t)dy - u(x,t), \tag{1}$$

and variations of it, have been widely used in the last decade to model diffusion processes, see for instance the references [1], [2], [3], [4]. As stated in [4], if $J : \mathbb{R}^n \to \mathbb{R}$ be a non-negative, smooth, symmetric ($J(-z) = J(z)$) and strictly decreasing function, with $\int_{\mathbb{R}^n} J(x)dx = 1$, J supported in the unit ball and if $u(x,t)$ is thought as a density at the point x at time t, and if $J(x-y)$ is thought as the probability distribution of jumping from location y to location x, then $(J * u)(x,t)$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^n} J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in absence of external sources, leads immediately to fact that the density u satisfies the equation (1). This equation is called non-local diffusion equation since the diffusion of the density u at point x and time t does not only depend on $u(x,t)$, but also on all the values of u in a neighborhood of x through the convolution term $J*u$. This equation shares many properties with the classical heat equation $u_t = \Delta u$: bounded stationary solutions are constant, a maximum principle holds for both of them and even if J is compactly supported, perturbations propagate with infinite speed, see [4]. However, there is no a regularizing effect in general [5].

Bogoya in [6] study the non-local reaction-diffusion system with Neumann boundary conditions

$$
\begin{aligned}
&u_t(x,t) = \int_\Omega J(x-y)(u(y,t) - u(x,t))dy + v^p(x,t), \quad (x,t) \in \Omega \times (0,T)\\
&v_t(x,t) = \int_\Omega J(x-y)(v(y,t) - v(x,t))dy + u^q(x,t), \quad (x,t) \in \Omega \times (0,T)\\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega,
\end{aligned}
\tag{2}
$$

with $p,\ q > 0$, $u_0(x),\ v_0(x) \in C(\overline{\Omega})$ nonnegative and nontrivial functions and $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) a bounded connected and smooth domain. How is it imposed that the diffusion takes place only in Ω, no individual may enter or leave the domain, that is known as Neumann boundary conditions.

In [6], is studied the existence of nonnegative solutions (u,v) for (2), showing the solution (u,v) is unique if $pq \geq 1$ or if at least one of the initials conditions is not zero for $pq < 1$. The globally existence is also studied. If $pq > 1$ and $u_0,\ v_0$ are nonnegative and nontrivial functions the solution (u,v) blows-up in finite time T, and if $pq \leq 1$ the solution (u,v) exists globally. Finally, it's analyzed the blows-up rates. The results obtained allow us to conclude the system (2) shares important properties with the corresponding local diffusion coupled parabolic system with Neumann boundary conditions, see [7], [8].

Our main objective in this paper is to study the semi-discrete approximation of (2). For that, we consider the one-dimensional case with $\Omega = [-L, L]$.

Let $N \in \mathbb{N}$ and $L > 0$. For $h = \frac{L}{N} > 0$, we consider a uniform mesh on the interval $[-L, L]$, $P = \{-L = x_{-N}, \cdots, x_0, \cdots, X_N = L\} = \{x_i = hi,\ -N \leq i \leq N\}$. Approximating the integrals in (2), we obtain the following system for $-N \leq i \leq N$

$$\begin{aligned} u_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(u_j(t) - u_i(t)) + v_i^p(t) \\ v_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(v_j(t) - v_i(t)) + u_i^q(t) \\ u_i(0) &= u_0(x_i), \quad v_i(0) = v_0(x_i). \end{aligned} \tag{3}$$

We denote $U(t) = (u_{-N}(t), \cdots, u_N(t))$, $V(t) = (v_{-N}(t), \cdots, v_N(t))$, $U_0 = U(0)$, $V_0 = V(0)$.

The semi-discrete approximation of the corresponding local diffusion coupled parabolic system with Neumann boundary conditions have been studied by many authors, see for instance [9], [10]. In this work, as in [11], we will study the semi-discrete approximation for (3), which shares some properties with the respective continuous problem (2) such as: the existence and uniqueness of solution if $pq \geq 1$, or if one of the initials conditions is different form zero and $pq < 1$. The paper is organized as follows: in Section 2 we study the existence and uniqueness of solutions for (3), the continuous dependence on initial datum is showed and a comparison principle is given. In section 3, we study the convergence of the numerical scheme.

2 Existence and uniqueness

We considered the problem

$$\begin{aligned} u_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(u_j(t) - u_i(t)) + f(v_i) \\ v_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(v_j(t) - v_i(t)) + g(u_i) \\ u_i(0) &= u_0(x_i) = u_{0,i}, \quad v_i(0) = v_0(x_i) = v_{0,i}, \end{aligned} \tag{4}$$

for $i = -N, ..., N$, where f, g are non-negatives locally Lipschtz functions.
As in the continuous case [6], existence and uniqueness of solutions for the system (4) will be a consequence of Banach fixed point theorem. Let $t_0 > 0$ and consider the Banach space $X_{t_0}^h = C([0, t_0]; l_h^1 \times l_h^1)$ with the norm

$$|||(U, V)||| = \max_{0 \leq t \leq t_0} \|(U(t), V(t))\|_I, \quad I = l_h^1 \times l_h^1$$

and

$$\|(U(t),V(t))\|_I = \|U(t)\|_{l_h^1} + \|V(t)\|_{l_h^1}, \quad where \quad \|U(t)\|_{l_h^1} = \sum_{i=-N}^{N} h|u_i|(t).$$

We also consider the closed subset

$$X^+ = \{(U,V) \in X_{t_0}^h : \ u_i, \ v_i \geq 0 \ for \ i = -N, ..., N\},$$

and the operator $\psi : X^+ \to X^+$ defined as $\psi_{(U_0,V_0)}(U,V) = (T_{U_0}(U), S_{V_0}(V))$,

$$\begin{aligned}(T_{U_0}^h(U(t)))_i &= \int_0^t \sum_{j=-N}^{N} hJ(h(i-j))(u_j(s)-u_i(s))ds + \int_0^t f(v_i(s))ds + u_i(0)\\ (S_{V_0}^h(V(t))_i &= \int_0^t \sum_{j=-N}^{N} hJ(h(i-j))(v_j(s)-v_i(s))ds + \int_0^t g(u_i(s))ds + v_i(0),\end{aligned} \tag{5}$$

for $i = -N, ..., N$.

Remark 2.1. *As* $\int_{\mathbb{R}} J(\zeta)d\zeta = 1$, *we have* $\sum_{j=-N}^{N} hJ(h(i-j)) \leq 1$.

Lemma 2.1. *Let f and g locally Lipschitz functions with Lipschitz's constants $K_1, K_2 > 0$ respectively, $(U_0,V_0),(W_0,Z_0) \in l_h^1 \times l_h^1$ non-negatives functions and $(U,V),(W,Z) \in X^+$. For every $h > 0$ there exists a positive constant $C = C(h,K_1,K_2)$ such that*

$$|||\psi_{(U_0,V_0)}(U,V) - \psi_{(W_0,Z_0)}(W,Z)||| \leq Ct_0|||(U,V)-(W,Z)||| + \|(U_0,V_0)-(W_0,Z_0)\|_I. \tag{6}$$

Proof. First, we see that the operator $\psi_{(U_0,V_0)}(U,V) = (T_{U_0}(U), S_{V_0}(V))$ is well defined. For any $0 < t_1 < t_2 \leq t_0$ we have

$$\begin{aligned}&\|(T_{U_0}^h(U(t_2))) - (T_{U_0}^h(U(t_1)))\|_{l_h^1}\\ &\quad \leq \int_{t_1}^{t_2} \sum_{i=-N}^{i=N} h \sum_{j=-N}^{j=N} hJ(h(i-j))|u_j(s)-u_i(s)|ds + \int_{t_1}^{t_2} \sum_{i=-N}^{i=N} h|f(v_i(s))|ds\\ &\quad \leq (C_1\|U\|_{l_h^1} + C_2)(t_2-t_1).\end{aligned} \tag{7}$$

Analogously we have

$$\|(S_{V_0}^h(V(t_2))) - (S_{V_0}^h(U(t_1)))\|_{l_h^1} \leq (C_1\|V\|_{l_h^1} + C_3)(t_2-t_1).$$

Therefore ψ is continuous.

Let $(U_0,V_0),\ (W_0,Z_0)\in l_h^1\times l_h^1$ non-negatives functions and $(U,V),\ (W,Z)\in X^+$. We have

$$\begin{aligned}
&\|(T^h_{U_0}(U(t)))-(T^h_{W_0}(W(t)))\|_{l_h^1}\\
&\quad\le \int_0^t\sum_{i=-N}^{i=N}h\sum_{j=-N}^{j=N}hJ(h(i-j))|u_j(s)-u_i(s)|ds\\
&\quad+\int_0^t\sum_{i=-N}^{i=N}h|f(v_i(s))-f(z_i(s))|ds+\sum_{i=-N}^{i=N}h|(u_i(0)-w_i(0)|\\
&\quad\le C_1\int_0^t\sum_{i=-N}^{i=N}h|u_j(s)-u_i(s)|ds\\
&\quad+K_1\int_0^t\sum_{i=-N}^{i=N}h|v_i(s)-z_i(s)|ds+\sum_{i=-N}^{i=N}h|(u_i(0)-w_i(0)|\\
&\quad\le\Big(C_1\|U-W\|_{l_h^1}+K_1\|V-Z\|_{l_h^1}\Big)t+\|U_0-W_0\|_{l_h^1}.
\end{aligned}\tag{8}$$

Analogously we have

$$\|(S^h_{V_0}(V(t)))-(T^h_{Z_0}(Z(t)))\|_{l_h^1}\le\Big(C_1\|V-Z\|_{l_h^1}+K_2\|U-W\|_{l_h^1}\Big)t+\|V_0-Z_0\|_{l_h^1}.$$

Therefore,

$$\begin{aligned}
&|||\psi_{(U_0,V_0)}(U,V)-\psi_{(W_0,Z_0)}(W,Z)|||\\
&=\max\nolimits_{0\le t\le t_0}\Big(\|(T^h_{U_0}(U(t)))-(T^h_{W_0}(W(t)))\|_{l_h^1}+\|(S^h_{V_0}(V(t)))-(T^h_{Z_0}(Z(t)))\|_{l_h^1}\Big)\\
&\le\max\nolimits_{0\le t\le t_0}\Big((C_1+K_2)\|U-W\|_{l_h^1}+(C_1+K_1)\|V-Z\|_{l_h^1}\Big)t+\|(U_0-W_0),(V_0-Z_0)\|_I\\
&\le Ct_0|||(U,V)-(W,Z)|||+\|(U_0,V_0)-(W_0,Z_0)\|_I,
\end{aligned}\tag{9}$$

where $C=\max\{C_1+K_1,\ C_1+K_2\}$. □

With the Lemma 2.1 is easy to prove existence and uniqueness of solutions for (4).

Theorem 2.1. *For every $h>0$, let f and g locally Lipschitz functions with Lipschitz's constants $K_1,K_2>0$ respectively. Let $(U_0,V_0)\in l_h^1\times l_h^1$ non-negatives functions, there exists a unique solution $(U,V)\in X^+$ for (4).*

Proof. Let $(U,V)\in X^+$. Taking $(U_0,V_0)=(W_0,Z_0)$ in Lemma 2.1 and choosing t_0 such that $Ct_0<1$, then $\psi_{(U_0,V_0)}(U,V)$ is a strict contraction in X^+, therefore there exists a unique fixed point of $\psi_{(U_0,V_0)}(U,V)$ in X^+ due to the Banach fixed point theorem, moreover, it's the unique solution to (4) in X^+. Arguing in the same way taking as initial datum $(U(t_0),V(t_0))\in l_h^1\times l_h^1$ it's possible to extend the solution up to some interval $[0,t_1)$, for certain $t_0<t_1$ if

$|||(U,V)||| < \infty$, we get a unique solution defined in $[0, 2t_0]$. In the same way, we can to continue for obtain $(U(t), V(t))$ solution of (4) in $[0, T]$, where $T > 0$ the maximal existence time of the solution. □

Now, we obtain the following results.

Corollary 2.1. *The solution (U, V) of (4) depend continuously on the initial data. In fact if (U, V) and (W, Z) are solutions to (4) with initial data (U_0, V_0) and (W_0, Z_0) respectively, then there exists a constant $\widetilde{C} = \widetilde{C}(t_0, h, K_1, K_2)$ such that*

$$|||(U, V) - (U, Z)||| \leq \widetilde{C}\|(U_0, V_0) - (W_0, Z_0)\|_I.$$

Corollary 2.2. *$(U, V) \in X^+$ is a solutions of (4), if and only if, for $i = -N, ..., N$*

$$\begin{aligned} u_i(t) &= \int_0^t \sum_{j=-N}^{N} hJ(h(i-j))(u_j(s) - u_i(s))ds + \int_0^t f(v_i(s))ds + u_i(0) \\ v_i(t) &= \int_0^t \sum_{j=-N}^{N} hJ(h(i-j))(v_j(s) - v_i(s))ds + \int_0^t g(u_i(s))ds + v_i(0). \end{aligned} \tag{10}$$

Corollary 2.3. *Let (U, V) the solution of (4). The total mass of the system, which is defined as*

$$m(t) = \sum_{j=-N}^{N} u_i(t), \quad n(t) = \sum_{j=-N}^{N} v_i(t),$$

satisfies

$$m(t) = \sum_{j=-N}^{N} u_i(0) + \int_0^t \sum_{j=-N}^{N} f(v_i(s))ds, \;\; n(t) = \sum_{j=-N}^{N} v_i(0) + \int_0^t \sum_{j=-N}^{N} g(u_i(s))ds.$$

Next we will apply the previous results to the solution of the problem (3).

Theorem 2.2. *Let $h > 0$ and $(U_0, V_0) \in l_h^1 \times l_h^1$ non-negatives functions. If $pq \geq 1$ then, there exists a unique solution $(U(t), V(t)) \in X^+$ of (3).*

Proof. Let $p,\ q \geq 1$. As $f(v) = v^p,\ g(u) = u^q$ are locally Lipschitz functions, then by Theorem 2.1, there exists a unique solution $(U, V) \in X^+$ of (3). □

Theorem 2.3. *Let $h > 0$ and $(U_0, V_0) \in l_h^1 \times l_h^1$ non-negatives functions. If $pq < 1$, then there exists a solution $(U(t), V(t)) \in X^+$ of (3).*

Proof. The existence of the solution of (3) is obtained by means of an approximation procedure as in [6]. We assume that $0 < p < 1 \leq q$, let $(f_n)_n$ a sequence of globally Lipschitz functions such that, for n fixed $f_n(s) = 0, \quad if \quad s \leq 0$, $f_n(s) = s^p \; if \; s \geq \frac{1}{2n}$, with f_n nondecreasing and $\lim_{n\to\infty} f_n(s) = s^p$ for $s \geq 0$. We considerer the ODEs system

$$\begin{aligned} u_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(u_j(t) - u_i(t)) + f_n(v_i(t)) \\ v_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(v_j(t) - v_i(t)) + u_i^q(t) \\ u_i(0) &= u_{0,i}, \quad v_i(0) = v_{0,i} + \tfrac{1}{n}. \end{aligned} \tag{11}$$

Arguing as Lemma (2.1) and Theorem (2.1), we have exists $((u_n)_i(t), (v_n)_i(t))$ solution of (10). As $(v_n)_i(0) > (v_m)_i(0)$ for $n < m$, we have $(u_m)_i(t)) < (u_n)_i(t) \, and \, (v_m)_i(t) < (v_n)_i(t)$, therefore $((u_n)_i(t))_n \, and \, ((v_n)_i(t))_n$ are nondecreasing and bounded sequence. Therefore letting $n \to \infty$ in (10), exist $(u_i(t), v_i(t))$ solution of (3). □

Remark 2.2. *Let's considerer the following ODE system*

$$\begin{aligned} &u'(t) = v^p(t), \quad v'(t) = u^q(t) \; for \; t > 0 \\ &u(0) = a \geq 0, \; v(0) = b \geq 0. \end{aligned} \tag{12}$$

If $pq < 1$, the solution of (12) is

$$\begin{aligned} u(t) &= \left(a^{(1-pq)/(p+1)} + C_1 t\right)^{(p+1)/(1-pq)}, \quad C_1 = \left(\tfrac{1-pq}{p+1}\right)\left(\tfrac{p+1}{q+1}\right)^{p/(p+1)} \\ v(t) &= \left(b^{(1-pq)/(q+1)} + C_2 t\right)^{(q+1)/(1-pq)}, \quad C_2 = \left(\tfrac{1-pq}{q+1}\right)\left(\tfrac{q+1}{p+1}\right)^{q/(q+1)}. \end{aligned} \tag{13}$$

Remark 2.3. *If $pq < 1$, then the solution of (3) is not unique. A solution of (3) with initial condition $(u_0, v_0) = (0, 0)$ is given by $(u_i(t), v_i(t)) = (0, 0)$ for all $t > 0$ and $i = -N, \cdots, N$. On the other hand, by (13) we have that $u(t) = (C_1 t)^{(p+1)/(1-pq)}, \;\; v(t) = (C_2 t)^{(q+1)/(1-pq)}$ it is also a positive solution of (3) with initial condition $(u_0, v_0) = (0, 0)$.*

We will use the notation $(a, b) \geq (c, d)$ to indicate that $a \geq c \; and \; b \geq d$.

Definition 2.1. *Let $\overline{U}, \overline{V} \in C^1([0, T); C(\overline{\Omega}))$. $(\overline{U}, \overline{V})$ is called a super-solution of (3) if*

$$\begin{aligned} \overline{u}_i'(t) &\geq \sum_{j=-N}^{N} hJ(x_i - x_j)(u_j(t) - u_i(t)) + \overline{v}_i^p \\ \overline{v}_i'(t) &\geq \sum_{j=-N}^{N} hJ(x_i - x_j)(v_j(t) - v_i(t)) + \overline{u}_i^q \\ \overline{u}_i(0) &\geq u_i(0), \quad \overline{v}_i(0) \geq v_i(0). \end{aligned} \tag{14}$$

Analogously $(\underline{U},\underline{V}) \in P_{t_0}$ *is called a sub-solution of (3) if it satisfies the opposite inequalities.*

Lemma 2.2. Comparison principle. *Let* $(\underline{U},\underline{V})$ *and* $(\overline{U},\overline{V})$ *be a sub-solution and super-solution of (3) respectively. If* $(\underline{U}(0),\underline{V}(0)) \le (\overline{U}(0),\overline{V}(0))$, *then* $(\underline{U}(t),\underline{V}(t)) \le (\overline{U}(t),\overline{V}(t))$ *for all* $t \ge 0$.

Proof. Let $(W(t),Z(t)) = (\overline{U}(t)-\underline{U}(t),\overline{V}(t)-\underline{V}(t))$ for $t \ge 0$. We can assume, by using an approximation argument, the strict inequalities in Definition 2.1 and $(W(0),Z(0) > (0,0)$. We have

$$
\begin{aligned}
w_i'(t) = \overline{u}_i'(t) - \underline{u}_i'(t) &\ge \sum_{j=N}^{N} hJ(x_i-x_j)(w_j(t)-w_i(t)) + \overline{v}_i^p(t) - \underline{v}_i^p(t) \\
&= \sum_{j=-N}^{N} hJ(x_i-x_j)(w_j(t)-w_i(t)) - w_i(t) + \frac{\overline{v}_i^p(t)-\underline{v}_i^p(t)}{\overline{v}_i(t)-\underline{v}_i(t)} z_i(t) \qquad (15) \\
\overline{z}_i'(t) &> \sum_{j=-N}^{N} hJ(x_i-x_j)(z_j(t)-z_i(t)) + \frac{\overline{u}_i^q(t)-\underline{u}_i^q(t)}{\overline{u}_i(t)-\underline{u}_i(t)} w_i(t).
\end{aligned}
$$

Now, let $\delta = min\{(W(t),Z(t)\}$ and suppose the conclusion of the Theorem is false. Thus, let t_0 be the first time such that $\min\{(W(t),Z(t)\} = \delta/2$. We can assume W attains the minimum. At that time (for instance t_0), there are a node $-N \le k \le N$ such that $w_k(t_0) = \delta/2$. But on the one hand $w_k'(t_0) \le 0$ and, on the other hand

$$
\begin{aligned}
\overline{w}_k'(t_0) &\ge \sum_{j=-N}^{N} hJ(x_k-x_j)(w_j(t_0)-w_k(t_0)) + \frac{\overline{v}_k^p(t_0)-\underline{v}_k^p(t_0)}{\overline{v}_k(t_0)-\underline{v}_k(t_0)} z_k(t_0) \\
&> \sum_{j\ne k} hJ(x_k-x_j)(w_j(t_0)-w_k(t_0)) + p\zeta_k^{p-1}(t_0) z_k(t_0) > 0,
\end{aligned} \qquad (16)
$$

where $\underline{v}_k(t_0) < \zeta_k(t_0) < \overline{v}_k(t_0)$, which is a contradiction. By Corollary 2.1 and an approximation argument, the result follows for general initial data. □

3 Convergence of the numerical scheme

In this section we prove result on uniform convergence for the numerical scheme (3). Throughout this section, we consider $0 < \tau < T$ fixed. We observe that for a function $\zeta \in C^1$

$$
\int_{x_j}^{x_j+h} \zeta(\tau)d\tau = h\zeta(x_j) + O(h^2), \quad \sum_{j=-N}^{N} O(h^2) = O(h).
$$

Let (u, v) a solution regular of (2), as $J \in C^1$, then $w_i(t) = u(x_i, t)$, $z_i(t) = v(x_i, t)$ verifies that for $t \in [0, T - \tau]$

$$\begin{aligned} w_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(w_j(t) - w_i(t)) + v_i^p(t) + \rho_{1,i}(h) \\ z_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(z_j(t) - z_i(t)) + u_i^q(t) + \rho_{2,i}(h), \end{aligned} \tag{17}$$

whit $\rho(h)$ a positive function such that $\max\{|\rho_{1,i}(h)|, |\rho_{2,i}(h)|\} \le \rho(h)$, $\max_i\{|w_i(0) - u_i(0)| + |z_i(0) - v_i(0)|\} \le \rho(h)$ and $\lim_{h\to 0} \rho(h) = 0$. The function ρ is called the modulus of consistency of the method.

Definition 3.1. *Let $(u(x,t), v(x,t))$ a regular solution of (2). We say that the scheme (3) is consistent if (17) holds.*

Theorem 3.1. *Let $(u(x,t), v(x,t))$ a solution regular of (2) and $(U(t), V(t))$ a solutions of (3). If the method is consistent, there exists a positive constant C, that does not depend on h, such that*

$$\max_{-N\le i\le N} \sup_{0\le t\le T-\tau} (|u(x_i, t) - u_i(t)| + |v(x_i, t) - v_i(t)|) \le C\rho(h).$$

Proof. Let T the maximal existence time and τ fixed such that $0 < \tau < T$. We defined the functions $\epsilon_i(t) = u(x_i, t) - u_i(t)$, $\varepsilon_i(t) = v(x_i, t) - v_i(t)$ for all $i = -N, \cdots, N$. As the method is consistent by (17) and (3) we have that

$$\begin{aligned} \epsilon_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(\epsilon_j(t) - \epsilon_i(t)) + v^p(x_i, t) - v_i^p(t) + \rho_{1,i}(h) \\ \varepsilon_i'(t) &= \sum_{j=-N}^{N} hJ(x_i - x_j)(\varepsilon_j(t) - \varepsilon_i(t)) + u^q(x_i, t) - u_i^q(t) + \rho_{2,i}(h). \end{aligned} \tag{18}$$

Let $t_0 = \max\{t : t < T - \tau, \max_i|\epsilon_i(t)| \le 1, \max_i|\varepsilon_i(t)| \le 1\}$. For $t \in [0, t_0]$ we have

$$\begin{aligned} \epsilon_i'(t) &\le \textstyle\sum_{j=-N}^{N} hJ(x_i - x_j)(\epsilon_j(t) - \epsilon_i(t)) + K_1\varepsilon_i(t) + \rho_{1,i}(h) \\ \varepsilon_i'(t) &\le \textstyle\sum_{j=-N}^{N} hJ(x_i - x_j)(\varepsilon_j(t) - \varepsilon_i(t)) + K_2\epsilon_i(t) + \rho_{2,i}(h), \end{aligned} \tag{19}$$

where $K_1 = (\|v\|_{L^\infty(\Omega\times[0,T-\tau])} + 1)^{p-1}$, $K_2 = (\|u\|_{L^\infty(\Omega\times[0,T-\tau])} + 1)^{q-1}, K = \max\{K_1, K_2\}$, $\mu = \|u\|_{L^\infty([-L,L]\times[0,T-\tau])} + 1$, $\nu = \|v\|_{L^\infty([-L,L]\times[0,T-\tau])} + 1$.

Let us now define the functions $(W(t), Z(t))$ for all $i = -N, \cdots, N$ as

$$w_i(t) = z_i(t) = e^{(2K+1)t}(\|\epsilon(0)\|_{L^\infty([-L,L])} + \rho(h)).$$

We have

$$\begin{aligned} w_i'(t) &= (2K+1)e^{(2K+1)t}(\|\epsilon(0)\|_{L^\infty([-L,L])} + \rho(h)) \\ &\geq \textstyle\sum_{j=-N}^{N} hJ(x_i - x_j)(w_j(t) - w_i(t)) + K(w_i(t) + z_i(t)) + \rho_{1,i}(h) \\ z_i'(t) &= (2K+1)e^{(K+1)t}(\|\epsilon(0)\|_{L^\infty([-L,L])} + \rho(h)) \\ &\geq \textstyle\sum_{j=-N}^{N} hJ(x_i - x_j)(z_j(t) - z_i(t)) + K(w_i(t) + z_i(t)) + \rho_{2,i}(h). \end{aligned} \tag{20}$$

Hence $(w_i(t), z_i(t))$ is a super-solution of the system

$$\begin{aligned} m_i'(t) &= \textstyle\sum_{j=-N}^{N} hJ(x_i - x_j)(m_j(t) - m_i(t)) + K(m_i(t) + n_i(t)) + \rho_{1,i}(h) \\ n_i'(t) &= \textstyle\sum_{j=-N}^{N} hJ(x_i - x_j)(n_j(t) - n_i(t)) + K(m_i(t) + n_i(t)) + \rho_{2,i}(h). \end{aligned} \tag{21}$$

Since the Comparison principle Lemma 2.2 holds for this system, we have

$$\max_i \epsilon_i(t) \leq e^{(2K+1)t}(\|\epsilon(0)\|_{L^\infty([-L,L])} + \rho(h)),$$

$$\max_i \varepsilon_i(t) \leq e^{(2K+1)t}(\|\epsilon(0)\|_{L^\infty([-L,L])} + \rho(h)).$$

Arguing as above whit $(-\epsilon_i(t), -\varepsilon_i(t))$ we obtain for $t \in [0, t_0]$

$$\max_i |\epsilon_i(t)| \leq e^{(2K+1)t}(\|\epsilon(0)\|_{L^\infty([-L,L])} + \rho(h)) \leq C\rho(h),$$

$$\max_i |\varepsilon_i(t)| \leq e^{(2K+1)t}(\|\epsilon(0)\|_{L^\infty([-L,L])} + \rho(h)) \leq C\rho(h).$$

As $\max_i\{|w_i(0)| + |z_i(0)|\} \leq \rho(h)$ we obtain for $t \in [0, t_0]$

$$\max_i |\epsilon_i(t)| \leq C\rho(h), \;\; \max_i |\varepsilon_i(t)| \leq C\rho(h).$$

Since $\rho(h) \to 0$ as $h \to 0$ and $|\epsilon_i(t)|, |\varepsilon_i(t)|$ for all $t \in [0, T-\tau]$ for all h small enough. Therefore $t_0 = T - \tau$ for all small h enough. □

4 Conclusion

A nonlocal and discrete systems have been studied. A result on existence and uniqueness of solutions was derived. We have proved the continuous dependence on the initial datums and a comparison principle was proved. Finally, the uniform convergence for the numerical scheme have been analyzed.

References

[1] P. Bates, P. Fife, X. Ren and X. Wang, Travelling waves in a convolution model for phase transitions, *Arch. Rat. Mech. Anal.*, **138** (1997), 105-136. https://doi.org/10.1007/s002050050037

[2] M. Bogoya. A nonlocal nonlinear diffusion equation in higher space dimensions, *J. Math. Anal. Appl.*, **344** (2008), 601-615. https://doi.org/10.1016/j.jmaa.2008.02.067

[3] C. Cortázar, M. Elgueta and J. D. Rossi, A non-local diffusion equation whose solutions develop a free boundary, *Ann. Henri Poincaré*, **6** (2005), no. 2, 269-281. https://doi.org/10.1007/s00023-005-0206-z

[4] P. Fife, Some nonclassical trends in parabolic and parabolic-like evolutions, Chapter in *Trends in Nonlinear Analysis*, Springer-Verlag, Berlin, 2003, 153-191. https://doi.org/10.1007/978-3-662-05281-5_3

[5] E. Chasseigne, M. Chaves and J. D. Rossi, Asymptotic behaviour for nonlocal diffusion equations, *J. Math. Pures Appl.*, **86** (2006), 271-291. https://doi.org/10.1016/j.matpur.2006.04.005

[6] M. Bogoya, On non-local reaction-diffusion system in a bounded domain, *Boundary Value Problems*, **2018** (2018), 38. https://doi.org/10.1186/s13661-018-0958-2

[7] M. Escobedo, M. A. Herrero, A semilinear parabolic system in a bounded domain, *Ann. Matematica Pura Appl.*, **165** (1993), 315-336. https://doi.org/10.1007/bf01765854

[8] C. V. Pao, On Nonlinear Reaction-Diffusion Systems, *J. Math. Anal. Appl.*, **87** (1982), 165- 108. https://doi.org/10.1016/0022-247x(82)90160-3

[9] G. Acosta, J. Fernádez Bonder, P. Groisman and J.D. Rossi, Simultaneous vs. non-simultaneous blow-up in numerical approximations of a parabolic system with nonlinear boundary conditions, *MSAIM: Math. Model. Numer. Anal.*, **36** (2002), no. 1, 55–68. https://doi.org/10.1051/m2an:2002003

[10] P. Groisman, F. Quirós and J.D. Rossi, Non-simultaneous blow-up in a numerical approximation of a parabolic system, *Comput. Appl. Math.*, **21** (2002), no. 3, 813-831.

[11] M. Bogoya and C. A. Gómez, Modelo discreto para una ecuación de difusión no local, *Revista Colombiana de Matemáticas*, **41** (2013), no. 1, 83-94.

Received: May 30, 2018; Published: June 20, 2018