Modeling the Dynamics of the Mosquito

*Anopheles calderoni* Transmitters of Malaria

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**Abstract**

Malaria is a life-threatening disease caused by parasites that are transmitted to humans through the bite of female mosquitoes of the genus *Anopheles*. In Colombia, it is a public health problem, 85% of the national territory has ideal conditions for the development of the mosquito. Particularly, in the Colombian Pacific Coast there is a high amount of *Anopheles*. For this reason, the department of Nariño is considered one of the areas with most cases of malaria in the country. In this work we formulate a mathematical model that describe growth dynamics of mosquitoes without explicitly considering the interaction with the human. The qualitative analysis reveals the existence of a forward bifurcation. The results suggest that in order to control spread of the mosquito, strategies that alter both its rate of growth and its death rate must be designed.

**Keywords:** Malaria, mosquito *Anopheles*, mathematical model.
1 Introduction

Malaria is a disease with a high mortality rate, it extends to all tropical and subtropical regions around the world. In 2015, the World Health Organization (WHO) reported 212 million new cases of malaria and 429,000 deaths due to the disease. The WHO goal is to eradicate malaria mainly in Asia and South America for the period 2016-2025. In South America, Brazil and Colombia reported the 68% of all the infected individual in the region [2]. In Colombia, this disease is present in 22 of its 33 states. In these states, there are between 18 and 24 million people who are susceptible to contracting the disease, in addition the height of the area corresponding to the states mentioned above is between 1500 and 1800 meters above sea level [1, 11]. Some research shows that the vector Anopheles calderoni is the main transmitter of malaria in some sub-regions of Colombia [14]. A study carried out in the Colombian Pacific Coast determined that the mosquito Anopheles calderoni is presented in 14 localities. The municipalities with the greatest presence of the mosquito are Boca de Prieta, Roberto Payan and Olaya Herrera. In the first, copies of An. calderoni were subjected to immunoenzymatic assays resulting positive for Plasmodium falciparum and Plasmodium vivax [14, 13]. The above motivates us to model the growth dynamics of the mosquito An. calderoni. To this end, we start with the parasite cycle (plasmodium) that produces the disease which consists of two phases, one sexual in the female mosquito and another asexual in the human. In the first, the susceptible insect (non-carrier) is loaded with plasmodium sex cells through the bite to an infected human. These cells form the zygote, which breaks into cells called sporocytes that move to the salivary glands, turning the female mosquito into a carrier. The second phase begins when the carrier mosquito bites a susceptible human loading human bloodstream with the sporocytes. The sporocytes reach the liver where they destroy the hepatocytes to finally invade the red blood cells [9]. Although the dynamics of malaria transmission involves both mosquito and human populations. In this work we are interested in describing principally the transformation dynamics of the carrier mosquito into the non-carrier mosquito without explicitly considering the interaction with humans. To this end, we define the transformation rate of non-carrier mosquitoes to carriers as a function that depends on time and it is equivalent to the force of infection in the classical models of malaria transmission [3, 12]. Montoya et al. [10] and Romero et al. [17] define the force of infection to mosquito as $\beta_V = \beta_v I_H / N_V$ where $\beta_v$ is the product between the contact rate and the transmission probability, under the assumption that the number of contacts is independent of the population size, $I_H$ is the infected human population and $N_V$ is the total mosquito population. In this case, both $I_H$ and $N_V$ depend on time. In consequence, $\beta_V$ depends on time; that is $\beta_V = \bar{\beta}(t)$.
2 Mathematical model

Let \( x(t) \) and \( y(t) \) populations of non carrier and carrier mosquitoes \textit{Anopheles calderoni} at time \( t \), respectively. We assume that the growth of non carrier mosquito follow a logistical regulation of total mosquito population \( x + y \) with carrying capacity denoted by \( k \) and intrinsic rate of growth \( \gamma \). On the other hand, non carrier and carrier mosquitoes die at per capita constat rate \( \mu \). Finally, the transformation rate of non-carrier mosquitoes to carriers, due to contact with a human infected with malaria is given by the function \( \beta(t) \). The above lead to the following system of nonlinear differential equations

\[
\begin{align*}
\frac{dx}{dt} &= \gamma(x + y) \left(1 - \frac{x + y}{k}\right) - (\beta(t) + \mu)x, \\
\frac{dy}{dt} &= \beta(t)x - \mu y.
\end{align*}
\] (1)

Adding the two equations of (1) we obtain

\[
\frac{d(x + y)}{dt} = \gamma(x + y) \left(1 - \frac{x + y}{k}\right) - \mu(x + y).
\] (2)

From the equation (2) we obtain de following inequality

\[
\frac{d(x + y)}{dt} \leq \gamma(x + y) \left(1 - \frac{x + y}{k}\right),
\]

where \( x(t) + y(t) \geq 0 \) for all \( t \geq 0 \). The solution of above logistic inequality satisfies that \( 0 \leq x(t) + y(t) \leq k \) for all \( t \geq 0 \). The following lemma summarizes the previous result.

**Lemma 2.1** The set

\[
\Omega = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq k\}
\] (3)

is positively invariant for the system (1). That is, all solutions starting in \( \Omega \) remain there for all \( t \geq 0 \).

See [5, 8, 10, 16] for a proof of Lemma 3.

3 Constant transformation rate

In this section we make the qualitative analysis of the model (1) considering the transformation rate, \( \beta \), constant. The equilibrium solutions of the system (1) are given by the solutions of following algebraic system

\[
\gamma(x + y) \left(1 - \frac{x + y}{k}\right) - (\beta + \mu)x = 0, \beta x - \mu y = 0,
\]
which are
\[ E_0 = (0, 0), \quad E_1 = \left( \frac{k\mu(\Phi - 1)}{(\beta + \mu)\Phi}, \frac{k\beta(\Phi - 1)}{(\beta + \mu)\Phi} \right), \]
where \( \Phi = \frac{\gamma}{\mu} \). The following proposition resume the above result.

**Proposition 3.1** If \( \Phi \leq 1 \), then \( E_0 \) is the only equilibrium solution of (1) in \( \Omega \) defined in (3). If \( \Phi > 1 \), in addition to \( E_0 \) there exists the coexistence equilibrium \( E_1 \) in \( \Omega \).

In order to analyze the stability of \( E_0 \) and \( E_1 \), we start by calculating the Jacobian matrix associated with the linearization of the system (1) at the generic equilibrium point \( E \) given by
\[ J(E) = \begin{pmatrix} 
\gamma \left( 1 - \frac{2(x+y)}{k} \right) - (\beta + \mu) & \gamma \left( 1 - \frac{2(S+I)}{k} \right) \\
-\mu & -\mu 
\end{pmatrix}. \]

From (4) we verify that the eigenvalues of the Jacobian of system (1) evaluated at \( E_0 \) \( (J(E_0)) \) are given by \( \lambda_1 = - (\beta + \mu) \) and \( \lambda_2 = \mu (\Phi - 1) \). We observe that \( \lambda_1 < 0 \) while the sign of \( \lambda_2 \) depends of the value of \( \Phi \), from above we obtain the following result.

**Proposition 3.2** If \( \Phi < 1 \) then \( E_0 \) is locally asymptotically stable, if \( \Phi = 1 \) then \( E_0 \) is a non-hyperbolic equilibrium point, and if \( \Phi > 1 \) then \( E_0 \) is unstable.

Calculating the central manifold of \( E_0 \) and applying Theorem of Center Manifold [15] we obtain the following result.

**Proposition 3.3** If \( \Phi = 1 \), then \( E_0 \) is stable.

Since \( V(x, y) = x + y \) is a Lyaponov function to \( E_0 \) [6, 7], then applying the Direct Method of Lyapunov and the LaSalle Therem is verified the following result.

**Proposition 3.4** If \( \Phi \leq 1 \), then \( E_0 \) is globally asymptotically stable in \( \Omega \).

Following the same ideas, we obtain that the eigenvalues of \( J(E_1) \) are given by \( \lambda_1 = - (\beta + \mu) \) and \( \lambda_2 = \mu (1 - \Phi) \). In consequence, \( E_1 \) is locally asymptotically stable when \( \Phi > 1 \). The following proposition resume the above result

**Proposition 3.5** \( E_1 \) is locally asymptotically stable in \( \Omega \).

Finally, from the the stability test (Theorem 5) in [4] we obtain the following result

**Lemma 3.1** \( E_1 \) is globally asymptotically stable in \( \Omega \).

The qualitative analysis reveals the existence of a forward bifurcation, in which mosquito-free equilibrium, \( E_0 \), loses its stability to \( \Phi = 1 \) and a transcritical bifurcation takes place.
4 Numerical simulations

In this section we present numerical simulations and graphs illustrating the population growth of carrier and non carrier mosquitoes. The value of parameters $\mu = 0.0039 \text{ day}^{-1}$, $\gamma = 0.0558 \text{ day}^{-1}$ and $k = 5000$ were taken from [16], for this values $\Phi = 14.3$ which implies that 14.3 new mosquitoes arises from a mosquito. Figure 1 shows the growth dynamics of mosquitoes with transformation rate of 1% ($\beta = 0.01$). In this case the initial population consists of 1000 non-carrier mosquitoes and zero carrier mosquitoes. Due to the magnitude of the threshold value, in the first days both populations grow rapidly. However, after 50 days the non-carrier population decreases until it stabilizes above its initial population while the population of carriers grows until it stabilizes.

![Graph showing growth of mosquitoes](image)

Figure 1: Growth curves of non-carrier and carrier mosquitoes when the transformation rate is 1%, growth rate is 5.58% and natural death rate is 0.39%.

5 Discussion

Nariño State located in the southwest of Colombia is considered to be at high risk for malaria transmission with an average annual parasitic index of ten and 9,010 malaria cases per year between 2004 and 2013 with 81.5% of cases due to *Plasmodium falciparum*, 18.3% due to *Plasmodium vivax*, and less than 1% due to *Plasmodium malariae* [18]. The study made by Orejuela et al. in [14] showed that *An. calderoni* is a primary malaria vector in the southwest
of Colombia. Above motivates us to formulate a mathematical model that tries to describe the transformation dynamics of non-carrier mosquito to carrier mosquito. In this sense, we are interested only in the dynamics of vectors. However, since humans also participate in this process, we implicitly incorporate contact between mosquitoes and humans through a function that depends on time. In this work, we perform the qualitative analysis when the transformation rate is constant. The results are determined in terms of the parameter \( \Phi \) which is interpreted as the number of new carrier mosquitoes produced from a carrier mosquito during its lifetime. The analysis of the model revealed the existence of two globally stable stationary states (the mosquito-free equilibrium \( E_0 \) and the mosquito-coexistence equilibrium \( E_1 \)) whose dynamics is governed by a forward bifurcation; that is, if \( \Phi \leq 1 \) then \( E_0 \) is stable. On the other hand, if \( \Phi > 1 \) the equilibrium \( E_0 \) loses its stability and bifurcates in the stable equilibrium \( E_1 \). Above suggests that when the carrier mosquitoes are not able to reproduce new mosquitoes or at most they produce one mosquito during their lifetime, the total mosquito population can be controlled with tendency to be cleared. However, if the carrier mosquitoes are capable of producing more than one mosquito during their lifetime the mosquito population will persist endemically in the region. Additionally, since \( \Phi \) is the quotient between the intrinsic rate of growth \( \gamma \) and the natural death rate \( \mu \), if one wishes to control the population growth of mosquitoes then the intrinsic rate of growth should always be kept below the rate of natural death. In the studies made by Orejuela et al. [14] and Naranjo et al. [11] are concluded that in 2014 the highest abundance of \( An. \) calderoni in the region was recorded during the rainy season in the first 6 months of the year where peak abundance corresponded to the highest rainy period in the southwest of Colombia in San Andrés in Tumaco. Through the results of the model it is interpreted that in the rainy season the intrinsic rate of growth moves away above the death rate, generating an increase in value of \( \Phi \) and therefore a growth in the mosquito population.

6 Conclusion

Although our model is quite simple, it predicts in terms of the threshold \( \Phi \), when the population of \( An. \) calderoni is cleared or persist. In addition, the dynamics described by the model coincides with studies carried out in the southwest of Colombia where this species is present. Corroborating the suggestions made in these works. In order to try to control the proliferation of the mosquito, strategies that alter both its growth rate and its death rate could be designed. Finally, in a future work we hope to present an optimal control problem that considers different control strategies and also the impact of migration on the growth of the mosquito population.
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