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Periodic and Soliton Solutions for a Generalized Two-Mode KdV-Burger's Type Equation

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Abstract

A generalized version for the two-mode KdV-Burger's type equation is studied from the point of view of its traveling wave solutions. The model include variable coefficients depending on the time from which several classical versions such as the two-mode KdV equations, the two-mode KdV-Burger's type equation with constant coefficients, the KdV equation, the Burger's equation and the standard KdV-Burger's equation can be obtain as particular cases. The improved tanh-coth method is used to obtain traveling wave solutions which include periodic and soliton type solutions and therefore, solutions for the mentioned equations can be obtain again.

Keywords: Periodic solution; soliton solutions; two-mode KdV-Burger's equation; Improved tanh-coth method; variable coefficients

1 Introduction

The standard Korteweg-de Vries-Burger's equation (KdV-B) is given by equation

$$u_t + k_1 u u_x + k_2 u_{xx} + k_3 u_{xxx} = 0, \tag{1}$$

where $u = u(x,t)$ is the unknown function, k_1, k_2, k_3 constants which can be functions depending on t (see for instance [1]). The equation (1) is used in several branch of sciences (see for instance [2] and references therein). In the case $k_2 = 0$, we obtain the classical KdV equation

$$u_t + k_1 u u_x + k_3 u_{xxx} = 0, \tag{2}$$

where the coefficients can be as the previous model. Finally, if in (1) we take $k_3 = 0$ we obtain the standard Burger's equation which have widely physical applications

$$u_t + k_1 u u_x + k_2 u_{xx} = 0, \tag{3}$$

where, as in the previous models, the coefficients can be variables depending on t or constants.

The main objective in this work, is obtain periodic and soliton solutions for the following generalization of the two-mode KdV-Burger's type equation (containing variable coefficients depending on the time)

$$\begin{cases} u_{tt} + (c_1(t) + c_2(t))u_{xt} + c_1(t)c_2(t)u_{xx} + \\ ((\alpha_1(t) + \alpha_2(t))\frac{\partial}{\partial t} + (\alpha_1(t)c_2(t) + \alpha_2(t)c_1(t))\frac{\partial}{\partial x})uu_x + \\ ((\beta_1(t) + \beta_2(t))\frac{\partial}{\partial t} + (\beta_1(t)c_2(t) + \beta_2(t)c_1(t))\frac{\partial}{\partial x})u_{xxx} + \\ ((\sigma_1(t) + \sigma_2(t))\frac{\partial}{\partial t} + (\sigma_1(t)c_2(t) + \sigma_2(t)c_1(t))\frac{\partial}{\partial x})u_{xx} = 0. \end{cases} \tag{4}$$

In the case that all coefficients are constants and $\sigma_1(t) = \sigma_2(t) = 0$, (4) reduces to the two-mode KdV equation studied in [3], which contain others cases of KdV type equations. A major description of (4) can be found in [4] and references therein. In this case, $u(x,t)$ is the unknown function, c_1 and c_2 the phase velocities, α_1, α_2 the nonlinearities, β_1, β_2 the dispersion parameters and σ_1 and σ_2 the dissipation parameters . Clearly, from this generalized model we can obtain not only (1), (2) and (3) but many other KdV type models.

2 Exact solutions for Eq. (4)

We will use the improved tanh-coth method, which description can be found in [5]. So that, we assume that (4) have solutions of the form

$$\begin{cases} u(x,t) = v(\xi), \\ \xi = x + \lambda t + \xi_0, \end{cases} \tag{5}$$

where ξ_0 is an arbitrary constant. Using (5) then (4) reduces to following ordinary differential equations in the unknowns $v(\xi)$

$$\begin{cases} \left[\lambda^2+\lambda(c_1(t)+c_2(t))+c_1(t)c_2(t)\right]v''(\xi)+ \\ \left[\lambda(\alpha_1(t)+\alpha_2(t))+(\alpha_1(t)c_2(t)+\alpha_2(t)c_1(t))\right](v'(\xi))^2+ \\ \left[\lambda(\alpha_1(t)+\alpha_2(t))+(\alpha_1(t)c_2(t)+\alpha_2(t)c_1(t))\right]v(\xi)v''(\xi)+ \\ \left[\lambda(\beta_1(t)+\beta_2(t))+(\beta_1(t)c_2(t)+\beta_2(t)c_1(t))\right]v''''(\xi)+ \\ \left[\lambda(\sigma_1(t)+\sigma_2(t))+(\sigma_1(t)c_2(t)+\sigma_2(t)c_1(t))\right]v'''(\xi)=0. \end{cases} \tag{6}$$

Here, $'$ is the ordinary differentiation with respect to ξ, $v'(\xi)=\frac{dv}{d\xi}$. Now, we seek solutions for (6) in the form

$$v(\xi)=\sum_{i=0}^{M}a_i(t)\phi(\xi)^i+\sum_{i=M+1}^{2M}a_i(t)\phi(\xi)^{M-i}, \tag{7}$$

being $a_i(t)$ to be determined and $\phi(\xi)$ satisfying the Riccati equation [6]

$$\phi'(\xi)=c(t)\phi^2(\xi)+b(t)\phi(\xi)+a(t), \tag{8}$$

where $a(t)$, $b(t)$ and $c(t)$ can be determinate later.

Substituting (7) into (6) and balancing the highest nonlinear term with the highest linear term we can obtain M. In this case, we use v'''' and $(v')^2$ se that we have $M+4=2(M+1)$ and therefore $M=2$. With that value, (7) converts to

$$v(\xi)=a_0(t)+a_1(t)\phi(\xi)+a_2(t)\phi(\xi)^2+a_3(t)\phi(\xi)^{-1}+a_4(t)\phi(\xi)^{-2}. \tag{9}$$

Substituting (9) into (6) again and using (8) we obtain an algebraic system in the unknown $a(t)$, $b(t)$, $c(t)$, $a_i(t)$, $i=1,\ldots,4$, λ. For the sake of simplicity, we omit the system. Solving it with aid of the *Mathematica* software we obtain a lot of solutions, however, we consider only the following two, which are the most general of alls:

First case:

$$\begin{cases} a_0(t)=\frac{\beta_1(t)\beta_2(t)(c_1(t)-c_2(t))}{(\beta_1(t)+\beta_2(t))(\alpha_2(t)\beta_1(t)-\alpha_1(t)\beta_2(t))}, \\ a_1(t)=-\frac{2(\beta_1(t)c(t)\sigma_2(t)-\beta_2(t)c(t)\sigma_1(t))}{\alpha_2(t)\beta_1(t)-\alpha_1(t)\beta_2(t)}, \\ a_3(t)=\frac{2(a(t)\beta_1(t)\sigma_2(t)-a(t)\beta_2(t)\sigma_1(t))}{\alpha_2(t)\beta_1(t)-\alpha_1(t)\beta_2(t)}, \\ b(t)=0, \quad a_2(t)=a_4(t)=0, \\ \lambda(t)=\frac{-\beta_2(t)c_1(t)-\beta_1(t)c_2(t)}{\beta_1(t)+\beta_2(t)}. \end{cases} \tag{10}$$

Taking into account that the solutions of (8) is given by (see for instance [6])

$$\phi(\xi) = \begin{cases} \frac{-\sqrt{b^2(t)-4a(t)c(t)}\tanh[\frac{1}{2}\sqrt{b^2(t)-4a(t)c(t)}\xi]-b(t)}{2c(t)}, \\ b^2(t) - 4a(t)c(t) \neq 0, \quad c(t) \neq 0, \end{cases} \tag{11}$$

then, respect to (10), (11) reduces to

$$\phi(\xi) = \frac{-\sqrt{-4a(t)c(t)}\tanh[\frac{1}{2}\sqrt{-4a(t)c(t)}\xi]}{2c(t)}, \quad c(t) \neq 0, \tag{12}$$

being $a(t)$, $c(t)$ arbitrary functions depending on variable t. Finally, according with (5) and (9) the respective solution for (4) is given as

$$u(x,t) = v(\xi) = a_0(t) + a_1(t)\phi(\xi)^1 + a_3(t)\phi(\xi)^{-1}, \tag{13}$$

whit $a_0(t)$, $a_1(t)$, $a_3(t)$ the values in (10), $\xi = x + (\frac{-\beta_2(t)c_1(t)-\beta_1(t)c_2(t)}{\beta_1(t)+\beta_2(t)})t + \xi_0$ and $\phi(\xi)$ as in (12).

Second case:

$$\begin{cases} a(t) = 0, \quad a_1(t) = a_2(t) = a_4(t) = 0, \\ \lambda(t) = \frac{-\alpha_2(t)c_1(t)-\alpha_1(t)c_2(t)}{\alpha_1(t)+\alpha_2(t)}, \\ b(t) = \\ \frac{-\sqrt{\left(\alpha_1^2\sigma_2-\alpha_2\alpha_1\sigma_1+\alpha_2\alpha_1\sigma_2-\alpha_2^2\sigma_1\right)^2-4\left(\alpha_1^2(-\beta_2)+\alpha_2\alpha_1\beta_1-\alpha_2\alpha_1\beta_2+\alpha_2^2\beta_1\right)(\alpha_1\alpha_2c_1-\alpha_1\alpha_2c_2)}}{2\left(\alpha_1^2(-\beta_2)+\alpha_2\alpha_1\beta_1-\alpha_2\alpha_1\beta_2+\alpha_2^2\beta_1\right)} \\ +\frac{\alpha_1^2(-\sigma_2)+\alpha_2\alpha_1\sigma_1-\alpha_2\alpha_1\sigma_2+\alpha_2^2\sigma_1}{2\left(\alpha_1^2(-\beta_2)+\alpha_2\alpha_1\beta_1-\alpha_2\alpha_1\beta_2+\alpha_2^2\beta_1\right)}, \end{cases} \tag{14}$$

Remark: in the expression for $b(t)$, we have $\alpha_i = \alpha_i(t)$, $\beta_i = \beta_i(t)$, $\sigma_i = \sigma_i(t)$, $c_i = c_i(t)$ and $i = 1, 2$. With this values, (11) reduces to

$$\phi(\xi) = \frac{-\sqrt{b^2(t)}\tanh[\frac{1}{2}\sqrt{b^2(t)}\xi] - b(t)}{2c(t)}, \quad c(t) \neq 0. \tag{15}$$

being $c(t)$ an arbitrary functions and $b(t)$ given in (14). Therefore, the solution for (4) is then

$$u(x,t) = v(\xi) = a_0(t) + a_3(t)\phi(\xi)^{-1} \tag{16}$$

where $a_0(t)$, $a_3(t)$ arbitrary functions, $\phi(\xi)$ given by (15) and

$$\xi = x + (\frac{-\alpha_2(t)c_1(t) - \alpha_1(t)c_2(t)}{\alpha_1(t) + \alpha_2(t)})t + \xi_0$$

.

3 Results and Discussion

Clearly, the model studied here is more general that those studied in [4] and the used method give us new solutions. A variety of models can be derived from that analyzed here, therefore, the respective solutions can be obtaind from the solutions derived here, in particular, solutions for these equationes mentioned in the introduction. From the mathematical point of view, the results are interesting, due to the generality of the model. On the other hand, the variable coefficients (depending on the time) give us solutions with a variety of structures as can be seen in the Figure1 and Figure2. The obtained solutions are soliton type, however, as can be seen in [6] periodic and rational solutions can be considered. For the sake of simplicity, we omit here this type of solutions, which can be obtain varying the parameter ξ_0 in (5) and depending on the sign of $b(t)^2 - 4a(t)c(t)$.

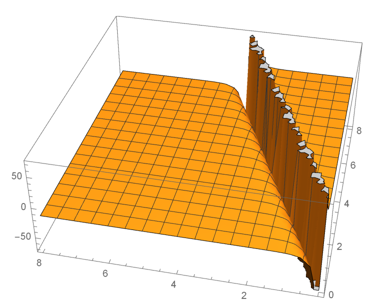

Figure 1: constant coefficients

The Figure1 is the soliton solution corresponding to (13) for the following values: $\alpha_1(t) = 1$, $\alpha_2(t) = 0.8$, $\beta_1(t) = 0.8$, $\beta_2(t) = 0.6$, $\sigma_1(t) = 0.7$, $\sigma_2(t) = 0.3$, $c_1(t) = 0.5$, $c_2(t) = 0.4$, $a(t) = -1$, $c(t) = 1$ and $(x, t) \in [0, 8] \times [0, 8]$.

The figure2 is the soliton solution corresponding to (13) for the following values which include variable coefficients: $\alpha_1(t) = t$, $\alpha_2(t) = 0.8$, $\beta_1(t) = 0.8t^2$, $\beta_2(t) = 0.6$, $\sigma_1(t) = 0.7$, $\sigma_2(t) = 0.3$, $c_1(t) = 0.5t^3$, $c_2(t) = 0.4$, $a(t) = -1$, $c(t) = 1$ and $(x, t) \in [-8, 8] \times [-8, 8]$

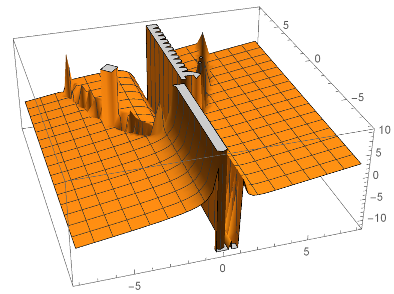

Figure 2: variable coefficients

4 Conclusion

Periodic and soliton solutions have been obtained for a generalized two-mode KdV-Burger's type equation. The considered model include a variety of important models, whose solutions can be obtained as particular cases. If we consider the constants coefficients, the solutions derived for (4) using the tanh-coth method are new in the literature. In the case of variable coefficients, clearly the solutions are new. In particular, solutions for these models analyzed for instance in [1] which include variable coefficients can be derived as particular cases of those derived here for (4).

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