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Exact Solutions for a Fifth-Order Two-Mode KdV Equation with Variable Coefficients

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Abstract

We obtain exact traveling wave solutions for a fifth-order two-mode Korteweg-de Vries equation with variable coefficients (which are depending on time t). The classical fifth-order two-mode KdV equation and other models such as the two mode KdV of third order equation are derived as particular cases. Using the improved tanh-coth method, we obtain periodic and soliton solutions formally. Showing the graphs of some of the solutions, we compare the type of solutions that can be obtain in the two cases: constant and variable coefficients.

Keywords: Exact solutions; soliton solutions; fifth-order two-mode KdV equation (fTMKdV); Improved tanh-coth method; variable coefficients

1 Introduction

We investigate the following fifth-order two-mode Korteweg-de Vries equation (fTMKdV) with coefficients depending on the time t

$$
\begin{cases}
u_{tt} + (c_1(t) + c_2(t))u_{xt} + c_1(t)c_2(t)u_{xx} + \\
((\alpha_1(t) + \alpha_2(t))\frac{\partial}{\partial t} + (\alpha_1(t)c_2(t) + \alpha_2(t)c_1(t))\frac{\partial}{\partial x})uu_x + \\
((\beta_1(t) + \beta_2(t))\frac{\partial}{\partial t} + (\beta_1(t)c_2(t) + \beta_2(t)c_1(t))\frac{\partial}{\partial x})u_{xxx} + \\
((\gamma_1(t) + \gamma_2(t))\frac{\partial}{\partial t} + (\gamma_1(t)c_2(t) + \gamma_2(t)c_1(t))\frac{\partial}{\partial x})u_{xxxxx} = 0.
\end{cases}
\tag{1}
$$

In the case that all coefficients are constants, (1) reduces to the classical fifth-order two-mode KdV equation [1], which describes the propagation of two different wave modes in the same direction simultaneously with the same dispersion relation but, different phase velocities c_1 and c_2, different nonlinearity α_1, α_2, different dispersion parameters β_1, β_2, γ_1 and γ_2. Exact solutions for that classical model have been derived by the authors in reference [1]. It is clear from the model studied here we can obtain the simple fifth-order KdV equation with variable coefficients [2][3] and other important models as the classical two-mode KdV equations of third-order, from which other models can be derived as can be seen in [4].

The models with variable coefficients are having importance in many branch of sciences: First, they give us generalized models, from which we can derive the classical models [4]; recently, some phenomena are modeled by means of this type of equations (see for instance [5]); finally, this type of models give us a new line of investigation in the area of nonlinear analysis. In this order of ideas, our main objective in this work is to obtain exact solutions for (1) using the improved tanh-coth method [6], which is one of the most generalized and easy to handle methods used for this type of computational works. .

2 Exact solutions for (1)

First, we give a brief description of the improved tanh-coth method for solving nonlinear partial differential equations with variable coefficients. Let the nonlinear partial differential equation in the variables x and t and variable coefficients depending only on the variable t

$$P(u, u_x, , u_t, u_{xt}, u_{xx}, \ldots) = 0. \tag{2}$$

The transformation

$$\begin{cases} u(x,t) = v(\xi), \\ \xi = x + \lambda t + \xi_0, \end{cases} \tag{3}$$

reduce (2) to an ordinary differential equations in the unknowns $v(\xi)$

$$P_1(v, v', v'', \ldots) = 0. \tag{4}$$

According with the method [6], we seek solution for (4) using the following expansion

$$v(\xi) = \sum_{i=0}^{M} a_i(t)\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i(t)\phi(\xi)^{M-i}, \tag{5}$$

where M is a positive integer that will be determined later by balancing method and $\phi = \phi(\xi)$ satisfying the Riccati equation

$$\phi'(\xi) = c(t)\phi^2(\xi) + b(t)\phi(\xi) + a(t), \tag{6}$$

whose solution is given by [7]

$$\phi(\xi) = \begin{cases} \frac{-\sqrt{b^2(t)-4a(t)c(t)}\tanh[\frac{1}{2}\sqrt{b^2(t)-4a(t)c(t)}\xi]-b(t)}{2c(t)}, \\ b^2(t) - 4a(t)c(t) \neq 0, \quad c(t) \neq 0. \end{cases} \tag{7}$$

Substituting (5) into (4) and after balancing the linear term of highest order with the highest order nonlinear term, we obtain M, after which, using (6) we obtain an algebraic system for $a(t)$, $b(t)$, $c(t)$, λ, $a_i(t)$, $\ldots$, a_{2M}, because all coefficients of $\phi(\xi)^i$ $(i = 1, 2, \ldots)$ have to vanish. Solving the algebraic system, and reversing the steps used, finally we obtain exact solutions for (1).

According with the previous steps, applying (3) to (1), we have

$$\begin{cases} \left[\lambda^2 + \lambda(c_1(t) + c_2(t)) + c_1(t)c_2(t)\right]v''(\xi) + \\ \left[\lambda(\alpha_1(t) + \alpha_2(t)) + (\alpha_1(t)c_2(t) + \alpha_2(t)c_1(t))\right](v'(\xi))^2 + \\ \left[\lambda(\alpha_1(t) + \alpha_2(t)) + (\alpha_1(t)c_2(t) + \alpha_2(t)c_1(t))\right]v(\xi)v''(\xi) + \\ \left[\lambda(\beta_1(t) + \beta_2(t)) + (\beta_1(t)c_2(t) + \beta_2(t)c_1(t))\right]v''''(\xi) + \\ \left[\lambda(\gamma_1(t) + \gamma_2(t)) + (\gamma_1(t)c_2(t) + \gamma_2(t)c_1(t))\right]v''''''(\xi) = 0, \end{cases} \tag{8}$$

where $'$ is the ordinary differentiation with respect to ξ, $v'(\xi) = \frac{dv}{d\xi}$. Now, we seek solutions for (8) taking into account (5). Balancing $v''''''(\xi)$ with $(v'(\xi))^2$ we obtain $M + 6 = 2(M + 1)$, so that, $M = 4$. With this value, (5) reduces to

$$v(\xi) = \sum_{i=0}^{4} a_i(t)\phi(\xi)^i + \sum_{i=5}^{8} a_i(t)\phi(\xi)^{4-i}. \tag{9}$$

Again, the substitution of (9) into (8) and taking into account (6) we obtain an algebraic system in the unknowns mentioned previously (with $M = 4$) in the description of the method. We solve it using the *Mathematica* software. For sake of the simplicity, we consider only the following solution:

$$
\begin{cases}
a_1(t) = a_2(t) = a_3(t) = a_4(t) = a_6(t) = a_7(t) = a_8(t) = 0, \\
a(t) = 0, \\
b(t) = \dfrac{\sqrt{\frac{-\sqrt{(\alpha_1+\alpha_2)\left((\alpha_1+\alpha_2)(\alpha_2\beta_1-\alpha_1\beta_2)^2+4\alpha_1\alpha_2(c_1-c_2)(\alpha_1\gamma_2-\alpha_2\gamma_1)\right)}+\alpha_1^2(-\beta_2)+\alpha_2\alpha_1(\beta_1-\beta_2)+\alpha_2^2\beta_1}{(\alpha_1+\alpha_2)(\alpha_1\gamma_2-\alpha_2\gamma_1)}}}{\sqrt{2}} \\
\lambda(t) = \frac{-\alpha_1(t)\gamma_2(t)-\alpha_2(t)\gamma_1(t)}{\alpha_1(t)+\alpha_2(t)}.
\end{cases}
\tag{10}
$$

Remark: in the expression for $b(t)$, we have $\alpha_i = \alpha_i(t)$, $\beta_i = \beta_i(t)$, $\gamma_i = \gamma_i(t)$, $c_i = c_i(t)$ and $i = 1, 2$.

Respect to this set of values, (7) take the form

$$
\phi(\xi) = \frac{-\sqrt{b(t)^2}\tanh[\frac{1}{2}\sqrt{b(t)^2}\xi] - b(t)}{2c(t)}, \quad c(t) \neq 0, \tag{11}
$$

being $c(t)$ arbitrary function depending on variable t, $b(t)$ given by the expression in (10). Therefore, (9) converts to

$$
v(\xi) = a_0(t) + a_5(t)\phi(\xi)^{-1}, \tag{12}
$$

where $a_0(t)$, $a_5(t)$ arbitrary functions depending on variable t, $\phi(\xi)$ given by (11). Finally, taking into account (3), the respective solution for (1) reduces to

$$
u(x,t) = v(\xi), \tag{13}
$$

where $v(\xi)$ is given by (12), and $\xi = x + (\frac{-\alpha_1(t)\gamma_2(t)-\alpha_2(t)\gamma_1(t)}{\alpha_1(t)+\alpha_2(t)})t + \xi_0$, with ξ_0 constant.

3 Results and Discussion

The model considered here is more general that those studied in the reference [1]. Clearly, several classical models associated with the Korteweg-de Vries equations can be derived, for instance the studied in [4], which have important particular cases. On the hand, the use of variable coefficients, give us new waves type soliton or periodic. This fact, can be help us to a best understanding on the classical models which have constant coefficients. Moreover, in the last decade, several models with variable coefficients, have been used in applications in several branch of the sciences (see for instance [5]). In the following graphics, we show the solutions (13) in the two cases: variable and constants coefficients, however, by the sake of simplicity, we consider only solutions type soliton, however, as can be see in (7), periodic solutions can be considered.

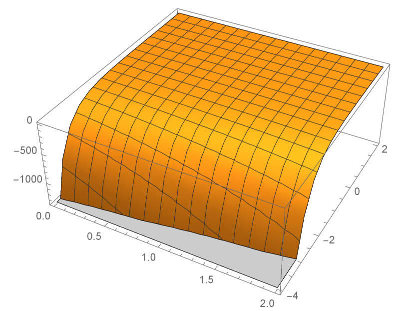

Figure 1: constant coefficients

The figure1 is the soliton solution corresponding to (13) for the following values: $\alpha_1(t) = 0.8$, $\alpha_2(t) = 0.6$, $\beta_1(t) = 0.3$, $\beta_2(t) = 0.6$, $\gamma_1(t) = 0.8$, $\gamma_2(t) = 0.5$, $c_1(t) = 0.6$, $c_2(t) = 0.3$, $a_0(t) = 5$, $a_5(t) = 1$, $c(t) = 2$ and $(x, t) \in [-4, 2] \times [0, 2]$.

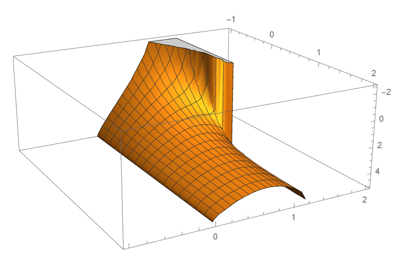

Figure 2: variable coefficients

The figure2 is the soliton solution corresponding to (13) for the following values which include variable coefficients: $\alpha_1(t) = 0.8\,t$, $\alpha_2(t) = 0.6$, $\beta_1(t) = 0.3\,t^2$, $\beta_2(t) = 0.6$, $\gamma_1(t) = 0.8$, $\gamma_2(t) = 0.5$, $c_1(t) = 0.6\,t$, $c_2(t) = 0.3$, $a_0(t) = 5$, $a_5(t) = 1$, $c(t) = 2\,t$ and $(x, t) \in [-1, 2] \times [-1, 2]$.

4 Conclusion

Exact solutions for a fifth-order two-mode Korteweg-de Vries equation with variable coefficients have been derived using the improved tanh-coth method.

The model is more general that the classical fifth-order two-mode KdV equation model considered by the authors in [1], therefore, from the solutions obtained here, we can obtain new solutions for that model. In the same way, solutions for the classical fifth-order KdV equation [2][3] can be again derived. The use of variable coefficients (depending on the time) give us many new structures of the solutions which can be used to obtain a best understanding on the models studied in the case of constants coefficients, so that, the results obtained here, are an important aport to line of investigation on the non-lineal analysis.

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