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On Elastic Rod Equation with Forcing Term: Traveling Wave Solutions

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Abstract

The improved tanh-coth method is employed to construct exact traveling wave solutions for a generalized elastic rod equation which include a forcing term. Periodic and soliton solutions are formally derived, from which, solutions for particular cases of the model can be obtained. Two particular cases, derived from the used method here, are formally analyzed. We can show that the obtained solutions have several interesting structures.

Keywords: Improved tanh-coth method, traveling wave solutions, forcing term, elastic rod equation

1 Introduction

Zhuang et al. [1], have obtained the following rod equation

$$u_{tt} - c_0^2 u_{xx} - c_0^2 a_n n (u_x)^{n-1} u_{xx} - \frac{\nu^2 J_p}{s} u_{ttxx} = 0, \tag{1}$$

where s is the cross-section area of the rod, J_p is the polar moment of inertia, ν the Poisson ratio, $c_0^2 = \frac{E}{\rho}$ is the square of the linear elastic longitudinal wave velocity, E is the modulus of elasticity, ρ is the density of the rod, a_n material constants of the rod, n is an integer. The case $a_n < 0$ is associated with soft-nonlinear materials (as the majority of metals) while the case $a_n > 0$ correspond to hard-nonlinear materials such as rubbers polymers and some metals [1]. The existence and other studies relative to solitary wave solutions of (1) have been considered in the case $n = 2$ and $n = 3$ by the authors in the references [1],[2],[3]. More recently, traveling waves solutions for (1) have been considered in [4],[5].

In this paper, we will use the improved tanh-coth method [6] for construct traveling wave solutions for the following nonlinear partial differential equation

$$u_{tt} - c_0^2 u_{xx} - c_0^2 a_n n (u_x)^{n-1} u_{xx} - \frac{\nu^2 J_p}{s} u_{ttxx} = F(t), \tag{2}$$

where $F(t)$ is a forcing term. Clearly, when $F(t) = 0$ the model reduce to (1), so that, from the results obtained here we can obtain new exact solutions for (1). Moreover, as can be seen in [1], when $a_n = 0$, under adequate assumptions (1) can be converted to classical wave equation in a elastic thin rod

$$u_{tt} - c_0^2 u_{xx} = 0.$$

Note that, in this last case, a forcing term can be considered again, so that, (2) is a good new model to be studied. By simplicity, we consider only the case $n = 2$ and $n = 3$, values derived from the used method here. The general case in not easy to handle by the technique used by us. Finally, we mentioned that, the relevance of the study of (2) is associated with the following facts: First, is a generalized model as we mentioned early, mathematically speaking this a very important; second, due to the use of a forcing term (depending on the variable t) we can obtain solutions with several structures, which, from the physical point of view, can help us to understand in a better way the phenomena described by the model (1). Finally, el use of a forcing term, as well as variable coefficients (as in [7][8][9]) allow us to have to a new line of investigation about of nonlinear models.

2 Traveling wave solutions for Eq. (2)

Assume that (2) has solutions as follows

$$\begin{cases} u(x,t) = v(\xi) + \int\int F(t)dt, \\ \xi = x + \lambda t + \xi_0. \end{cases} \tag{3}$$

Here, ξ_0 arbitrary constant and λ in known as the wave speed, and $'$ is the ordinary differentiation. Now, substituting (3) into (2) we obtain the differential equation

$$(\lambda^2 - c_0^2)v''(\xi) - c_0^2 a_n[(v'(\xi))^n]' - \frac{\nu^2 J_p}{s}v''''(\xi) = 0. \tag{4}$$

Integrating (4) with respect to variable ξ, we have

$$(\lambda^2 - c_0^2)v'(\xi) - c_0^2 a_n(v'(\xi))^n - \frac{\nu^2 J_p}{s}v'''(\xi) + k = 0, \tag{5}$$

being k the integration constant. Now, setting

$$w(\xi) = v'(\xi), \tag{6}$$

then (5) reduces to

$$(\lambda^2 - c_0^2)w(\xi) - c_0^2 a_n(w(\xi))^n - \frac{\nu^2 J_p}{s}w''(\xi) + k = 0. \tag{7}$$

Then, taking into account the improved tanh-coth method [6], we seek solutions to (7) by using the expansion

$$w(\xi) = \sum_{i=0}^{M} a_i\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i\phi(\xi)^{M-i}, \tag{8}$$

where M is a positive integer to be determinate, a_i constants and $\phi(\xi)$ satisfying the Riccati equation

$$\phi'(\xi) = \alpha + \beta\phi(\xi) + \gamma\phi(\xi)^2, \tag{9}$$

with solution [10]:

$$\phi(\xi) = \begin{cases} \frac{1}{\gamma}(-\frac{1}{\xi+\xi_0} - \frac{\beta}{2}), & \beta^2 - 4\alpha\gamma = 0, \\ \\ \frac{-\sqrt{\beta^2-4\alpha\gamma}\tanh[\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\xi]-\beta}{2\gamma}, & \beta^2 - 4\alpha\gamma \neq 0. \end{cases} \tag{10}$$

Substituting (8) into (7) and balancing $w''(\xi)$ with $(w(\xi))^n$ we have $M + 2 = nM$, or in equivalent form $M(n-1) = 2$, so that, the method allow us to consider only the cases $n = 2$ or $n = 3$. If $n = 2$, we have $M = 2$ and if $n = 3$ then $M = 1$.

2.1 First case

We consider the first case

$$n = 2, \;\; M = 2.$$

With this values, (8) reduces to

$$w(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2 + a_3\phi(\xi)^{-1} + a_4\phi(\xi)^{-2}. \tag{11}$$

Now, substituting (11) into (7) (with $n = 2$) leads us to an algebraic system in the unknowns a_i, $(i = 0, \ldots, 4)$, α, β, γ, λ and k. Using the *Mathematica* software, we obtain a lot of solutions of the resultant system, however, for sake of simplicity we consider only the following (the most general of alls):

$$\begin{cases} a_0 = \frac{\pm\sqrt{-4c_0^2a_nk+16\alpha^2\gamma^2(\frac{\nu^2J_p}{s})^2-8\alpha\beta^2\gamma(\frac{\nu^2J_p}{s})^2+\beta^4(\frac{\nu^2J_p}{s})^2}-8\alpha\gamma(\frac{\nu^2J_p}{s})-(\frac{\nu^2J_p}{s})\beta^2}{2c_0^2a_n}, \\ a_1 = -\frac{6\beta\gamma(\frac{\nu^2J_p}{s})}{c_0^2a_n}, \;\; a_2 = -\frac{6\gamma^2(\frac{\nu^2J_p}{s})}{c_0^2a_n}, \\ a_3 = a_4 = 0, \\ \lambda = \sqrt{c_0^2 \pm \sqrt{-4c_0^2a_nk + 16\alpha^2\gamma^2(\frac{\nu^2J_p}{s})^2 - 8\alpha\beta^2\gamma(\frac{\nu^2J_p}{s})^2 + \beta^4(\frac{\nu^2J_p}{s})^2}} \end{cases} \tag{12}$$

$$\begin{cases} a_0 = \frac{\pm\sqrt{-4c_0^2a_nk+16\alpha^2\gamma^2(\frac{\nu^2J_p}{s})^2-8\alpha\beta^2\gamma(\frac{\nu^2J_p}{s})^2+\beta^4(\frac{\nu^2J_p}{s})^2}-8\alpha\gamma(\frac{\nu^2J_p}{s})-(\frac{\nu^2J_p}{s})\beta^2}{2c_0^2a_n}, \\ a_1 = -\frac{6\beta\gamma(\frac{\nu^2J_p}{s})}{c_0^2a_n}, \;\; a_2 = -\frac{6\gamma^2(\frac{\nu^2J_p}{s})}{c_0^2a_n}, \\ a_3 = a_4 = 0, \\ \lambda = -\sqrt{c_0^2 \pm \sqrt{-4c_0^2a_nk + 16\alpha^2\gamma^2(\frac{\nu^2J_p}{s})^2 - 8\alpha\beta^2\gamma(\frac{\nu^2J_p}{s})^2 + \beta^4(\frac{\nu^2J_p}{s})^2}} \end{cases} \tag{13}$$

$$\begin{cases} a_0 = \frac{\pm\sqrt{-4c_0^2a_nk+16\alpha^2\gamma^2(\frac{\nu^2J_p}{s})^2-8\alpha\beta^2\gamma(\frac{\nu^2J_p}{s})^2+\beta^4(\frac{\nu^2J_p}{s})^2}-8\alpha\gamma(\frac{\nu^2J_p}{s})-(\frac{\nu^2J_p}{s})\beta^2}{2c_0^2a_n}, \\ a_1 = a_2 = 0, \\ a_3 = -\frac{6\alpha\beta(\frac{\nu^2J_p}{s})}{c_0^2a_n}, \;\; a_4 = -\frac{6\alpha^2(\frac{\nu^2J_p}{s})}{c_0^2a_n}, \\ \lambda = \sqrt{c_0^2 \pm \sqrt{-4c_0^2a_nk + 16\alpha^2\gamma^2(\frac{\nu^2J_p}{s})^2 - 8\alpha\beta^2\gamma(\frac{\nu^2J_p}{s})^2 + \beta^4(\frac{\nu^2J_p}{s})^2}} \end{cases} \tag{14}$$

$$\begin{cases} a_0 = \frac{\pm\sqrt{-4c_0^2a_nk+16\alpha^2\gamma^2(\frac{\nu^2 J_p}{s})^2-8\alpha\beta^2\gamma(\frac{\nu^2 J_p}{s})^2+\beta^4(\frac{\nu^2 J_p}{s})^2}-8\alpha\gamma(\frac{\nu^2 J_p}{s})-(\frac{\nu^2 J_p}{s})\beta^2}{2c_0^2a_n}, \\ a_1 = a_2 = 0, \\ a_3 = -\frac{6\alpha\beta(\frac{\nu^2 J_p}{s})}{c_0^2a_n}, \quad a_4 = -\frac{6\alpha^2(\frac{\nu^2 J_p}{s})}{c_0^2a_n}, \\ \lambda = -\sqrt{c_0^2 \pm \sqrt{-4c_0^2a_nk + 16\alpha^2\gamma^2(\frac{\nu^2 J_p}{s})^2 - 8\alpha\beta^2\gamma(\frac{\nu^2 J_p}{s})^2 + \beta^4(\frac{\nu^2 J_p}{s})^2}} \end{cases} \tag{15}$$

With respect to (12) the solution for (9) is then given by (10), with α, β and γ arbitrary constants. So that, taking into account (11) the respective solution to (7) reduces to

$$w(x,t) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2, \tag{16}$$

where $\phi(\xi)$ is given by (10), a_0, a_1 and a_2 the values given by (12). Finally, by (6) and (3) the solution for (2) have the form

$$u(x,t) = \int w(\xi)d\xi + \int\int F(t)dt, \tag{17}$$

where

$$\xi = x + (\sqrt{c_0^2 \pm \sqrt{-4c_0^2a_nk + 16\alpha^2\gamma^2(\frac{\nu^2 J_p}{s})^2 - 8\alpha\beta^2\gamma(\frac{\nu^2 J_p}{s})^2 + \beta^4(\frac{\nu^2 J_p}{s})^2}})t + \xi_0,$$

with ξ_0 arbitrary constant. A similar solutions can be obtained for (2) by using the values given in (13). Now, following the same steps used with (12) we obtain the following solution for (2) using the values in (14)

$$u(x,t) = \int w(\xi)d\xi + \int\int F(t)dt, \tag{18}$$

where

$$w(x,t) = a_0 + a_3\phi(\xi)^{-1} + a_4\phi(\xi)^{-2},$$

and a_0, a_3 and a_4 given by (14), $\phi(\xi)$ the solution of (7) (the same of the previous case) and

$$\xi = x + (\sqrt{c_0^2 \pm \sqrt{-4c_0^2a_nk + 16\alpha^2\gamma^2(\frac{\nu^2 J_p}{s})^2 - 8\alpha\beta^2\gamma(\frac{\nu^2 J_p}{s})^2 + \beta^4(\frac{\nu^2 J_p}{s})^2}}t + \xi_0.$$

As before, similar solution is obtained using (15).

2.2 Second Case

We consider the case

$$n = 3, \quad M = 1.$$

With this values, (8) reduces to

$$w(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^{-1}. \tag{19}$$

We substitute (19) into (7) (with $n = 3$). As in the previous case, we obtain an algebraic system in the unknowns a_i, $(i = 0, \dots, 2)$, α, β, γ, λ and k. Using the *Mathematica* software, we obtain a lot of solutions of the resultant system, however, we consider only the following

$$\begin{cases} a_0 = \frac{k}{2a_1a_2c_0^2a_n}, \quad \alpha = -\frac{ia_2\sqrt{c_0^2}\sqrt{a_n}}{\sqrt{2}\sqrt{(\frac{\nu^2 J_p}{s})}}, \quad \beta = -\frac{ik}{\sqrt{2}a_1a_2\sqrt{c_0^2}\sqrt{a_n}\sqrt{(\frac{\nu^2 J_p}{s})}}, \\ \gamma = -\frac{ia_1\sqrt{c_0^2}\sqrt{a_n}}{\sqrt{2}\sqrt{(\frac{\nu^2 J_p}{s})}}, \quad \lambda = \pm\frac{\sqrt{8a_1^3a_2^3c_0^2a_n^2 + 4a_1^2a_2^2c_0^2a_n + k^2}}{2a_1a_2\sqrt{c_0^2}\sqrt{a_n}}. \end{cases} \tag{20}$$

Other solution of the algebraic system is obtained, changing the sign of α, β and γ. The other values are the same, so that, for sake of simplicity, we omit here.

With the values given in (20), (10) reduces to (for $\beta^2 - 4\alpha\gamma \neq 0$)

$$\begin{cases} \phi(\xi) = \\ -\frac{k + ia_1a_2\sqrt{c_0^2}\sqrt{a_n}\sqrt{(\frac{\nu^2 J_p}{s})}\sqrt{\frac{4a_1a_2c_0^2a_n}{(\frac{\nu^2 J_p}{s})} - \frac{k^2}{a_1^2a_2^2c_0^2a_n(\frac{\nu^2 J_p}{s})}}\tanh\left(\frac{1}{2}x\sqrt{\frac{2a_1a_2c_0^2a_n}{(\frac{\nu^2 J_p}{s})} - \frac{k^2}{2a_1^2a_2^2c_0^2a_n(\frac{\nu^2 J_p}{s})}}\right)}{2a_1^2a_2c_0^2a_n}. \end{cases} \tag{21}$$

In this order of ideas (19) reduces to

$$w(\xi) = \frac{k}{2a_1a_2c_0^2a_n} + a_1\phi(\xi) + a_2\phi(\xi)^{-1}, \tag{22}$$

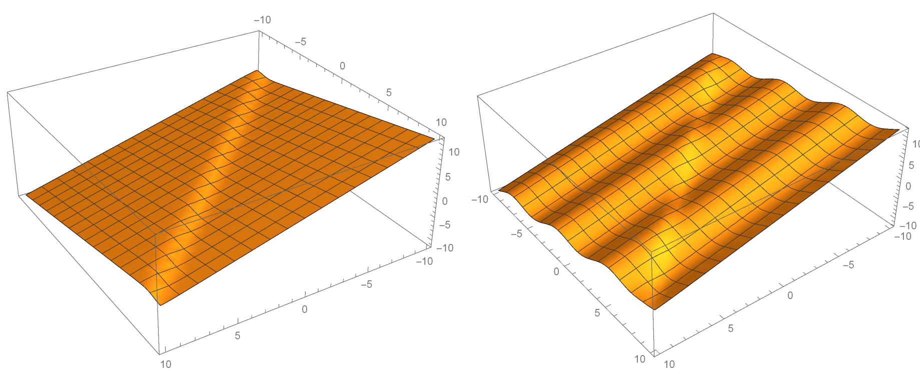
where $\phi(\xi)$ is given by (21), a_1 and a_2 arbitrary constants, and $\xi = x + (\pm\frac{\sqrt{8a_1^3a_2^3c_0^2a_n^2 + 4a_1^2a_2^2c_0^2a_n + k^2}}{2a_1a_2\sqrt{c_0^2}\sqrt{a_n}})t + \xi_0$. Therefore, as in the first case, taking into account (6) and (3), the solution for (2) take the form

$$u(x,t) = \int w(\xi)d\xi + \int\int F(t)dt, \tag{23}$$

being $w(\xi)$ as in (22).

3 Results and Discussion

We have considered solutions for (2) of the type soliton. However, it is clear that varying the sign of $\beta^2(t) - 4\alpha(t)\gamma(t)$ and the value of ξ_0, we can obtain other type of solutions, particulary, periodic solutions. It can be see in a better way, seeing the reference [10]. In the reduction made to obtain (5), some authors (as in [5]) taken the integration constant as zero from which, the solution obtained by them loss generality [11]. This is not the case in this work.



(a) u_1 (b) u_2

Figure 1: $u(x,t)$

The figure u_1 is the solution (17) for the values: $c_0^2 = 9$, $a_n = 1$, $k = -1$, $\xi_0 = 0$, $\beta = 3$, $\gamma = 1$, $\alpha = 1$, $F(t) = 0$, and $(x,t) \in [-10,10] \times [-10,10]$. The Figure u_2 correspond to same values, but now $F(t) = \sin t$.

4 Conclusion

The improved tanh-coth have been used to solve an elastic rod equation with forcing term. Periodic and soliton solutions was formally derived, from which, solutions for the homogeneous case are obtain (see fig. u_1). The solutions here derived are new, and several structures for them can be considered using the forcing term. It clear that the model considered here can be used in applications of physics and engineering, and it can be compared with non-homogeneous models studied in classical books of differential equations. The use of a forcing term in nonlinear models such as the studied here, open us a new line of investigation in the nonlinear analysis.

References

[1] Wei Zhuang and Yang Guiltong, The propagation of solitary wave in a nonlinear elastic rod, *Applied Mathematics and Mechanics* **7** (1986), no. 7, 615-626. https://doi.org/10.1007/bf01895973

[2] W. Zhuang and S.Y. Zhang, The strain solitary wave in a nonlinear elastic rod, *Acta Mech. Sin.*, **3** (1987), no. 1, 62-72. https://doi.org/10.1007/bf02486784

[3] W.S. Duan, J.B. Zhao, Solitary waves in a cuadratic nonlinear elastic rod, *Chaos Solitons and Fract.* **11** (2000), 1265-1267. https://doi.org/10.1016/s0960-0779(99)00014-4

[4] J. Li, Y. Zhang, Exact traveling eave solutions in a nonlinear elastic rod equation, *Appl. Math. and Comp.*, **202** (2008), 504-510. https://doi.org/10.1016/j.amc.2008.02.027

[5] M.M. Kabir, Exact travelng wave solutions for nonlinear elastic rod equation *Journal of King Saud University-Science*, (2017), in Press https://doi.org/10.1016/j.jksus.2017.08.010

[6] Cesar A. Gomez S, A.H. Salas , The Cole-Hopf transformation and improved tanh-coth method applied to new integrable system (KdV6), *Appl. Math. and Comp.*, **204** (2008), 957-962. https://doi.org/10.1016/j.amc.2008.08.006

[7] R.M. Miura, The Korteweg-de Vries equations and generalizations. I. A remarkable explicit nonlinear transformation, *J. Math. Physics*, **9** (1968), 1202-1204. https://doi.org/10.1063/1.1664700

[8] N. Nirmala, M.J. Vedan and B.V. Baby, Auto-Bäcklund transformation, Lax Pairs Painlevé property or a variable coefficient Korteweg-de Vries equation. I, *J. Math. Physics*, **27** (1986), 2640-2646. https://doi.org/10.1063/1.527282

[9] Z. Liu, C. Yang, The applications of bifurvation method to a higher-order KdV equation, *J. Math. Anal. Appl.*, **275** (2012), 1-12. https://doi.org/10.1016/s0022-247x(02)00210-x

[10] C. A. Gomez S and A. Salas, Special symmetries to standard Riccati equations and applications, *Appl. Math. and Comp.*, **216** (2010), no. 10, 3089-3096. https://doi.org/10.1016/j.amc.2010.04.039

[11] N.A. Kudryashov, Seven common errors in finding exact solutions of nonlinear differential equations, *Commun. Nonlinear Sci. Num. Simul.*, **14** (2009), 3507-3529. https://doi.org/10.1016/j.cnsns.2009.01.023

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