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Solution of the Short Pulse Equation Using Lattice-Boltzmann and the tanh Method

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Abstract

In this work we solved the nonlinear one-dimensional short pulse equation using lattice-Boltzmann method and a $d1q3$ velocity scheme. Also, we apply the Tanh solitary wave method, so that, we find several families of solutions.

Keywords: Short Pulse equation, lattice-Boltzmann, tanh method

1 Introduction

The short pulse equation (SPEq) is a nonlinear theory describes electromagnetic ultra short pulses in optical linear and nonlinear media[1]-[3]. The solution of many differential equations has been skillfully addressed, e.g., using lattice-Boltzmann, [4]-[9], and also the so-called solitary wave methods, among them one of the most popular, the so-called Tanh [8].
We present the paper as follows. In section (2), we presents the lattice-Boltzmann model applied to the SPEq. In addition, section (3) gives the moments of the particle distribution. Also, in section (5), we provide the equilibrium distribution function. Besides, in Section (6), we use the Tanh method, [10], working to accomplish solitary wave solutions of the SPEq. Lastly, in section (7), we give results and conclusions.

2 The lattice Boltzmann model

The lattice-Boltzmann equation, [5]-[6], is:

$$f_i(x+v_x\delta t, t+\delta t) - f_i(x,t) = -\frac{1}{\tau}\left[f_i(x,t) - f_i^{eq}(x,v_x)\right] + \Xi_i(x,t) \qquad (1)$$

Which is given in the B.G.K. approximation, [9], and $\Xi(x,t)$ is the source term [7]-[8]. Expanding in a Taylor series at second order the left-hand side of eq.(1), and doing a perturbative expansion in the spatial and time derivatives, and the particle distribution function, as:

$$\frac{\partial}{\partial x} = \epsilon\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial t} = \epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2}, \quad f_i = f_i^0 + \epsilon f_i^1 + \epsilon^2 f_i^2 \qquad (2)$$

Using eqs. (2) in eq. (1), and assuming the source term as $\Xi_i(x,t) = \epsilon^2\phi_i$, [7], we have:

$$\begin{aligned} -\frac{1}{\tau}(f_i^0 + \epsilon f_i^1 + \epsilon^2 f_i^2 - f^{eq}) &= \delta t((\epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2}) + v_{x,i}[\epsilon\frac{\partial}{\partial x_1}])(f_i^0 \qquad (3) \\ +\epsilon f_i^1 + \epsilon^2 f_i^2) + \frac{\delta t^2}{2}[(\epsilon\frac{\partial}{\partial t_1} &+ \epsilon^2\frac{\partial}{\partial t_2}) + v_{x,i}[\epsilon\frac{\partial}{\partial x_1}]]^2(f_i^0 + \epsilon f_i^1 + \epsilon^2 f_i^2) + \epsilon^2\phi_i \end{aligned}$$

The terms at order ϵ in eq. (3) and assuming $f_i^0 = f^{eq}$, we have:

$$-\frac{1}{\tau}\left[\epsilon f_i^1\right] = \epsilon\delta t\left[\frac{\partial}{\partial t_1} + v_{x,i}\frac{\partial}{\partial x_1}\right] f_i^0 \qquad (4)$$

The terms at order ϵ^2 in eq (3), we get:

$$\begin{aligned} -\frac{1}{\tau}[\epsilon^2 f_i^2] \;\; &= \;\; \epsilon^2[\delta t\frac{\partial}{\partial t_2} + \frac{\delta t^2}{2}(\frac{\partial}{\partial t_1} + v_{x,i}\frac{\partial}{\partial x_1})^2]f_i^0 \qquad (5) \\ &\quad +\epsilon\delta t\left[\left(\frac{\partial}{\partial t_1}\right) + v_{x,i}\frac{\partial}{\partial x_1}\right]\left(\epsilon f_i^1\right) + \epsilon^2\phi_i \end{aligned}$$

Using eq. (4) and eq. (5), we have

$$-\frac{1}{\tau\delta t}[\epsilon f_i^1 + \epsilon^2 f_i^2] = [(\frac{\partial}{\partial t_1}) + v_{x,i}\frac{\partial}{\partial x_1}][(\epsilon f_i^0) + \epsilon^2 f_i^1(1 - \frac{1}{2\tau})] + \epsilon^2\frac{\partial}{\partial t_2}f_i^0 \qquad (6)$$

Figure 1: The lattice velocity scheme $d1q3$.

3 Moments of the distribution function

The moments of the distributions are defined as:

$$-\frac{\partial \rho}{\partial x} = \sum_i \left(f_i^0\right), \quad \frac{1}{6}\frac{\partial \rho^3}{\partial x} = \sum_i \left(v_{x,i} f_i^0\right) \tag{7}$$

$$\Pi^0_{\alpha,\beta} = \sum_i v_{i,\alpha} v_{i,\beta} f_i^0 = \delta_{\alpha,\beta}\lambda\rho \tag{8}$$

$$\sum_i \left(f_i^k\right) = 0, k > 0; \quad \sum_i \left(v_{x,i} f_i^k\right) = 0, k > 0 \tag{9}$$

4 The Short Pulse equation

Summing on i in eq. (6) and using eq. (7) and assuming $\left(\frac{\partial}{\partial x} v_{x,i}\right) = 0$, and $\epsilon \sum_i \phi_i = (b+1)\lambda\rho = \Lambda\Phi$, [8]. Where b is the dimension of the discretized velocity space. Then, we get:

$$\frac{\partial^2 \rho}{\partial x \partial t} = \frac{1}{6}\frac{\partial^2(\rho^3)}{\partial x^2} + \Lambda\Phi \tag{10}$$

If we consider

$$\Lambda\Phi = \rho \tag{11}$$

Therefore, eq. (12) is

$$\frac{\partial^2 \rho}{\partial x \partial t} = \frac{1}{6}\frac{\partial^2(\rho)^3}{\partial x^2} + \rho \tag{12}$$

5 Distribution function $d1q3$

We use the $d1q3$ velocity scheme considering $e_\alpha = c\{0, 1, -1\}$ [5]-[6], as:

$$f_{i,\alpha,\beta}^{(eq)} = \left\{ \begin{array}{ccc} -\frac{\lambda}{c^2}\rho - \frac{\partial\rho}{\partial x} & \rightarrow & i=0 \\ \frac{1}{2c}\frac{\partial\rho^3}{\partial x} + \frac{\lambda}{2c^2}\rho & \rightarrow & i=1 \\ -\frac{1}{2c}\frac{\partial\rho^3}{\partial x} + \frac{\lambda}{2c^2}\rho & \rightarrow & i=2 \end{array} \right\} \tag{13}$$

The derivative $\frac{\partial\rho}{\partial x}$ is taken with the backward difference discretization scheme

$$\frac{\partial\rho(x,t)^3}{\partial x} \rightarrow 3\rho(x,t)^2\frac{\rho(x,t)-\rho(x-\Delta x,t)}{\Delta x} \tag{14}$$

6 The tanh method

We start using the next trasformation:

$$u = x - kt \tag{15}$$

$$\frac{\partial}{\partial t} = -k\frac{d}{du}; \quad \frac{\partial}{\partial x} = \frac{d}{du}; \quad \frac{\partial^2}{\partial x^2} = \frac{d^2}{du^2}; \quad \frac{\partial^2}{\partial t^2} = k^2\frac{d^2}{du^2} \tag{16}$$

Then, eq. (**??**)

$$k\frac{d^2\rho}{du^2} + \rho + \frac{1}{6}\frac{\partial^2(\rho)^3}{\partial u^2} = 0 \tag{17}$$

Now, we balance the highest-order linear derivative with the highest order nonlinear terms in eq. (26). We get:

$$\frac{\partial^2(\rho)^3}{\partial u^2} \rightarrow \rho \rightarrow 3m+2 = m \rightarrow m = -1 \tag{18}$$

Then, we do the next transformation:

$$\rho = (v)^{-1} \tag{19}$$

Then

$$\frac{d\rho}{du} = \frac{d\rho}{dv}\frac{dv}{du} = -v^{-2}\frac{dv}{du}; \quad \frac{d^2\rho}{du^2} = 2v^{-3}(\frac{dv}{du})^2 - v^{-2}\frac{d^2v}{du^2} \tag{20}$$

$$\frac{d\rho^3}{du} = \frac{d\rho^3}{dv}\frac{dv}{du} = -3v^{-4}\frac{dv}{du}; \quad \frac{d^2\rho^3}{du^2} = 12v^{-5}(\frac{dv}{du})^2 - 3v^{-4}\frac{d^2v}{du^2} \tag{21}$$

Replacing in eq. (17), and multiplying by v^5, we get

$$8(kv^2+1)(\frac{dv}{du})^2-(4kv^3+v)\frac{d^2v}{du^2}+4v^4=0 \tag{22}$$

Now, we introduce a new independent variable [10]:

$$Y(x,t)=\tanh{(u)} \tag{23}$$

Then, the first and second derivatives of u, are:

$$\frac{d}{du}=(1-Y^2)\frac{d}{dY};\quad \frac{d^2}{du^2}=-2Y(1-Y^2)\frac{d}{dY}+(1-Y^2)^2\frac{d^2}{dY^2} \tag{24}$$

The solutions are postulated as:

$$v(\xi)=\sum_{i=1}^{q}a_iY^i \tag{25}$$

Then, replacing in eq. (22)

$$\begin{aligned}&8(kv^2+1)((1-Y^2)\frac{dv}{dY})^2+4v^4\\&-(4kv^3+v)(-2Y(1-Y^2)\frac{dv}{dY}+(1-Y^2)^2\frac{d^2v}{dY^2})=0\end{aligned} \tag{26}$$

Now, we balance the highest-order linear derivative with the highest order nonlinear terms in eq. (26). We get:

$$Y^4(\frac{dv}{dY})^2\to v^4\to 2m+2=4m\to m=1 \tag{27}$$

$$v=a_0+a_1Y,\to\frac{dv}{dY}=a_1,\to\frac{d^2v}{dY^2}=0 \tag{28}$$

we have in eq. (26):

$$\begin{aligned}&8(k(a_0+a_1Y)^2+1)((1-Y^2)a_1)^2+4(a_0+a_1Y)^4\\&-(4k(a_0+a_1Y)^3+(a_0+a_1Y))(-2Y(1-Y^2)a_1)=0\end{aligned} \tag{29}$$

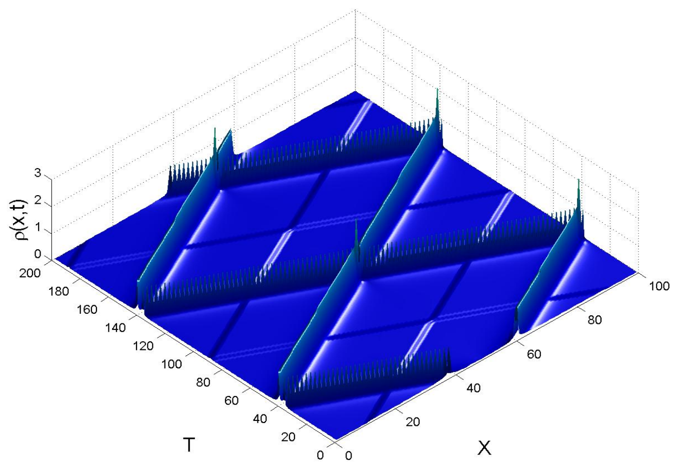

Figure 2: The spatiotemporal, LB, evolution of $\rho(x,t)$ using a $d1q3$ lattice velocity, for two initial profiles given by eq. (32).

Doing some algebra

$$a_0 = 0, \;\; a_{1_{1,2}} = \pm\sqrt{\frac{5}{4-2k}}, \;\; a_{1_{3,4}} = \pm i\sqrt{\frac{1}{8-4k}} \tag{30}$$

$$a_{1_{5,6}} = \pm\sqrt{\frac{7}{4k}}, \;\; a_{1_{7,8}} = \pm i\sqrt{\frac{1}{8k}} \tag{31}$$

Then, we find 8 families of solutions.

7 Conclusions

This work presents the LB and tanh methods applied to the one-dimensional short pulse equation. In fig. (2), we show the temporal evolution of two initial profile given by eq. (32). The solution is:

$$\rho_i = \tanh^{-1}\left(a_{1,i}\tanh\left(x - kt\right)\right) \tag{32}$$

We find 8 families of solutions for the SPEq using tanh method [10]. The extension to higher dimensions is straighforward.

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